

Some new applications of the Stanley-Macdonald Pieri Rules

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Dedicated to Richard Stanley for his 70th birthday

Abstract

In a seminal paper [22] Richard Stanley derived Pieri rules for the Jack symmetric function basis. These rules were extended by Macdonald to his now famous symmetric function basis. The original form of these rules had a forbidding complexity that made them difficult to use in explicit calculations. In the early 90's it was discovered [10] that, due to massive cancellations, the dual rule, which expresses skewing by e_1 the modified Macdonald polynomial $\tilde{H}_\mu[X; q, t]$, can be given a very simple combinatorial form in terms of corner weights of the Ferrers' diagram of μ . A similar formula was later obtained by the last named author for the multiplication of $\tilde{H}_\mu[X; q, t]$ by e_1 , but never published. In the years that followed we have seen some truly remarkable uses of these two Pieri rules in establishing highly non trivial combinatorial results in the Theory of Macdonald polynomials. This theory has recently been spectacularly enriched by various Algebraic Geometrical results in the works of Hikita [15], Schiffmann [19], Schiffmann-Vasserot [20], [21], A. Negut [18] and Gorsky-Negut [12]. This development opens up the challenging task of deriving their results by purely Algebraic Combinatorial methods. Substantial progress in this task was obtained in [3]. In this paper we present the progress obtained by means of Pieri rules.

I. Introduction

In this paper, as in [3], the main actors are the family of symmetric function operators D_k and D_k^* introduced in [6], whose action on a symmetric function $F[X]$ may be written on the form

$$\begin{aligned} a) \quad D_k F[X] &= F[X + \frac{M}{z}] \sum_{i \geq 0} (-z)^i e_i[X] \Big|_{z^k} & (\text{with } M = (1-t)(1-q)) \\ b) \quad D_k^* F[X] &= F[X - \frac{\tilde{M}}{z}] \sum_{i \geq 0} z^i h_i[X] \Big|_{z^k} & (\tilde{M} = (1-1/t)(1-1/q)) \end{aligned} \quad \text{I.1}$$

where expressions are given in plethystic notation which we shall review in Section 1.

We will focus here, as in [3], on the algebra $\mathcal{A}$ of symmetric function operators generated by the family $\{D_k\}_{k \geq 0}$. It was shown in [3] that $\mathcal{A}$ is bi-graded by assigning the generator D_k bi-degree $(1, k)$. Its connection to the Algebraic Geometrical developments is that $\mathcal{A}$ is a concrete realization of a portion of the Elliptic Hall Algebra studied by Schiffmann and Vasserot in [19], [20] and [21].

In particular $\mathcal{A}$ contains a distinguished family of operators $\{Q_{u,v}\}_{u,v \geq 0}$ of bi-degree given by their index that can be shown to play a central role in the connection between Macdonald Polynomials and Parking Functions (see [4]). For nonnegative, and co-prime index pair (m, n) the construction of the operators $Q_{m,n}$ is quite simple. We write

$$Split(m, n) = (a, b) + (c, d) \quad \text{if and only if} \quad \begin{cases} a) & (m, n) = (a, b) + (c, d) \\ b) & \det \begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1 \end{cases} \quad \text{I.2}$$

Geometrically this simply says that (a, b) is the lattice point closest to the segment $(0, 0) \rightarrow (m, n)$ and $(0, 0), \rightarrow (a, b), \rightarrow (m, n)$ is the counter-clockwise order of the vertices of a non-trivial triangle. These conditions force also the pairs (a, b) and (c, d) to be co-prime and we can recursively define

$$Q_{m,n} = \frac{1}{M} [Q_{c,d}, Q_{a,b}] = \frac{1}{M} (Q_{c,d} Q_{a,b} - Q_{a,b} Q_{c,d}) \quad \text{I.3}$$

with base cases

$$a) \quad Q_{1,0} = D_0 \quad \text{and} \quad b) \quad Q_{0,1} = -e_1 \quad \text{I.4}$$

where for a symmetric function f we let “ $\underline{f}$ ” denote the operator, “*multiplication by f .*”

Now from the definition in I.1 a) it easily follows that

$$D_k = \frac{1}{M}[D_{k-1}, e_1] \quad (\text{for all } k \geq 1) \quad \text{I.5}$$

and I.3 then yields

$$Q_{1,k} = D_k. \quad \text{I.6}$$

Thus for m, n positive integers we may replace the recursion in I.3 by

$$Q_{m,n} = \begin{cases} \frac{1}{M}[Q_{c,d}, Q_{a,b}] & \text{if } m > 1, \\ D_n & \text{if } m = 1. \end{cases} \quad \text{I.7}$$

This implies that $Q_{m,n} \in \mathcal{A}$ for all (m, n) positive and co-prime.

The definition of the operators $Q_{u,v}$ for non co-prime pairs (u, v) and the proof of some of their remarkable properties was carried out in [3] by means of two basic tools.

To be more explicit we need some auxiliary material. Firstly, we will write $(u, v) = (km, kn)$ with (m, n) a co-prime pair and $k = \gcd(u, v)$. Next note that from $Split(m, n) = (a, b) + (c, d)$ we derive that

$$(u, v) = (im + a, in + b) + (jm + c, jn + d) \quad (\text{for all } i + j = k - 1) \quad \text{I.8}$$

This given, in [3] it is shown that we may set

$$Q_{u,v} = \frac{1}{M}[Q_{jm+c, jn+d}, Q_{im+a, in+b}] \quad (\text{for any } i + j = k - 1) \quad \text{I.9}$$

by proving that all the operators on the right hand side are the same. Since both pairs $(im + a, in + b)$ and $(jm + c, jn + d)$ always turn out to be co-prime, this shows that $Q_{u,v}$ is well defined, and also shows that $Q_{u,v} \in \mathcal{A}$ for any $u, v \geq 1$.

The equality of the operators occurring on the right hand side of I.9 was obtained in [3] by two steps. In the first step I.9 is established for $(m, n) = (1, 1)$ and in the second step I.9 is derived in full generality from this special case by means of a natural action of $SL_2[\mathbf{Z}]$ on $\mathcal{A}$ which preserves all the identities satisfied by the operators D_k . In particular a proof is given in [3] that for the generators $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ we have

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} Q_{m,n} = Q_{am+cn, bm+dn}.$$

This given, it is first shown in [3] that the case $(m, n) = (1, 1)$ of I.9 is a consequence of the commutator identity

$$(D_a D_b^* - D_b^* D_a) P[X] = M \frac{(qt)^{-a}}{qt-1} h_{a+b} [X(1-tq)] P[X] \quad (\text{for } a + b > 0) \quad \text{I.10}$$

In fact, by setting

$$D_k = Q_{1,k} \quad \text{and} \quad D_k^* = -(qt)^{k-1} Q_{-1,k} \quad \text{I.11}$$

I.10 becomes the operator identity

$$\frac{1}{M}[Q_{-1,b}, Q_{1,a}] = \frac{qt}{qt-1} h_{a+b} [X(1/qt-1)]. \quad \text{I.12}$$

Next it is shown in [3] that for all co-prime (m, n) we have

$$\nabla Q_{m,n} \nabla^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} Q_{m,n} = Q_{m+n,n} \quad \text{I.13}$$

where ∇ is the operator, introduced in [1] with eigen-functions the modified Macdonald basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$. Thus conjugating I.12 by ∇ , gives

$$\frac{1}{M}[Q_{b-1,b}, Q_{a+1,a}] = \frac{qt}{qt-1} \nabla h_{a+b}[X(1/qt-1)] \nabla^{-1} \quad (\text{for } a+b > 0). \quad \text{I.14}$$

Since for $(m, n) = (1, 1)$ we have $(a, b) = (1, 0)$ and $(c, d) = (0, 1)$ we see that I.14 implies that for $(m, n) = (1, 1)$ all the operators on the right hand side of I.9 are identical and we may thus set

$$Q_{k,k} = \frac{qt}{qt-1} \nabla h_k[X(1/qt-1)] \nabla^{-1}. \quad \text{I.15}$$

The final step is obtained by showing that the identities in I.9 are simply images of I.14 and I.15 by the $SL_2[Z]$ action.

In this paper, using the symmetric function tools created in the 90's in the study of Macdonald polynomials, most particularly in [8], [6], [2], we develop a parallel variety of identities by letting the operators $Q_{k,-1}$ and $Q_{k,1}$ play the role that $Q_{1,k}$ and $Q_{-1,k}$ play in [3]. The fact that for

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

we have

$$UQ_{1,k} = Q_{k,-1}, \quad UQ_{-1,k} = Q_{k,1}$$

guided us to a number of surprising discoveries. In particular it turns out that an identity that was discovered in 2008 in the research that yielded the results in [11] may be viewed as an image by U of the identity in I.12.

To state our results we need to recall some notational conventions. To begin, we will identify partitions with their French Ferrers diagrams. Next, for a cell $c \in \mu$ we let $l_\mu(c)$, $a_\mu(c)$, $l'_\mu(c)$ and $a'_\mu(c)$ denote the “*leg*”, “*arm*”, “*coleg*” and “*coarm*”, of c in μ (as in [17]). Because we are using French notation, these parameters count the number of cells of μ that are respectively strictly North, East, South and West of c in μ . We then set

$$B_\mu(q, t) = \sum_{c \in \mu} t^{l'_\mu(c)} q^{a'_\mu(c)}, \quad \Pi_\mu(q, t) = \prod_{c \in \mu}^{(0,0)} (1 - t^{l'_\mu(c)} q^{a'_\mu(c)}) \quad \text{I.16}$$

where the superscript $(0, 0)$ in the product is to avoid the vanishing factor. In addition we set

$$T_\mu = \prod_{c \in \mu} t^{l'_\mu(c)} t^{a'_\mu(c)}, \quad w_\mu(q, t) = \prod_{c \in \mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}), \quad \text{I.17}$$

The modified Macdonald polynomials $\{\tilde{H}_\mu[X; q, t]\}_\mu$ we work with here form the unique symmetric function basis that is (in dominance order) upper triangularly related to the modified Schur basis $\{s_\lambda[\frac{X}{t-1}]\}_\lambda$ and satisfies the orthogonality condition

$$\langle \tilde{H}_\lambda, \tilde{H}_\mu \rangle_* = \chi(\lambda = \mu) w_\mu(q, t), \quad \text{I.18}$$

where $\langle , \rangle_*$ is the deformation

$$\langle p_\lambda, p_\mu \rangle_* = (-1)^{|\mu|-l(\mu)} \prod_{i=1}^{l(\mu)} (1-t^{\mu_i})(1-q^{\mu_i}) z_\mu \chi(\lambda = \mu) \quad \text{I.19}$$

of the Hall scalar product $\langle p_\lambda, p_\mu \rangle = z_\mu \chi(\lambda = \mu)$. We will call $\langle, \rangle_*$ the “*star scalar product*”.

Let us also recall that the operator ∇ is defined by setting

$$\nabla \tilde{H}_\mu[X; q, t] = T_\mu \tilde{H}_\mu[X; q, t]. \quad \text{I.20}$$

It is also shown in [6] that

$$D_0 \tilde{H}_\mu[X; q, t] = (1 - MB_\mu(q, t)) \tilde{H}_\mu[X; q, t]. \quad \text{I.21}$$

Both ∇ and D_0 are special cases of a commuting family of operators defined in [6] by setting for a symmetric function F

$$\Delta_F \tilde{H}_\mu[X; q, t] = F[B_\mu(q, t)] \tilde{H}_\mu[X; q, t]. \quad \text{I.22}$$

Finally, the family of operators $Q_{m,n}$ with m, n co-prime is extended to the fourth lattice quadrant by setting

$$Q_{m,-n} = Q_{m,n}^{\perp*} \quad \text{I.23}$$

where the symbol “ $\perp^*$ ” denotes the operation of taking the adjoint of an operator with respect to the star scalar product.

This given, our first result can be stated as follows.

Theorem I.1

For all $m \geq 1$ we have

$$Q_{m,0} = \frac{qt}{qt-1} \Delta_{h_m}[(MX-1)(1/qt-1)] \quad \text{I.24}$$

Notice that, by I.21, this identity may be viewed as the extension to $m > 1$ of the equality $Q_{1,0} = D_0$. Notice further that since $Q_{0,m} = \frac{qt}{qt-1} \underline{h}_m[X(1/qt-1)]$ and the collection $\{\prod_{i=1}^{l(\lambda)} Q_{\lambda_i,0}\}_\lambda$ may be taken as a basis for the family of symmetric function multiplication operators, we can derive from Theorem I.1 and the identity $UQ_{0,m} = Q_{m,0}$ the following truly remarkable result.

Theorem I.2

The action of the 2×2 matrix U on a symmetric function operator $\underline{F}$ may be expressed by the identity

$$U \underline{F}[X] = \Delta_{F[(MX-1)]} \quad \text{I.25}$$

Another significant fact that emerges from our findings here is that while viewing the operators $Q_{m,n}$ as non-commutative polynomials in the family $Q_{1,n} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n Q_{0,1} = D_n$ expresses their action by constant term formulas, our present way of viewing the operators $Q_{m,n}$ as non-commutative polynomials in the family $Q_{m,1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^m Q_{0,1} = \nabla^m Q_{0,1} \nabla^{-m}$ expresses their action by standard tableaux expansions.

To be more precise we need notation. Let $\mathcal{ST}_n$ be the set of all standard tableaux with labels $1, 2, \dots, n$ and $ST(\mu)$ be all the standard tableaux of shape μ . For a given $T \in \mathcal{ST}_n$, we set $w_T(k) = q^{j-1}t^{i-1}$, if the label k of T is in the i -th row and the j -th column. This given, our simplest result in this context may be stated as follows.

Theorem I.3

The operators $Q_{u,v}$ may be written as a linear combination of the family $\{Q_{a_n,1} \cdots Q_{a_2,1} Q_{a_1,1}\}_{a_i \geq 0}$, whose action on $(-1)^n$ may be expressed in the form

$$\frac{1}{M^n} Q_{a_n,1} \cdots Q_{a_2,1} Q_{a_1,1} (-1)^n = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{1 - w_T(k)qt}{1 - qt} \prod_{1 \leq h < k \leq n} f\left(\frac{w_T(h)}{w_T(k)}\right) \quad \text{I.26}$$

where for convenience we have set $f(u) = \frac{(1-u/t)(1-u/q)}{(1-u)(1-u/qt)}$.

We should mention that a similar tableau expansion was first obtained in [9] and in fact our proof of I.26 follows very closely the arguments used in [9]. This is precisely where the Stanley-Macdonald Pieri rules play a crucial role. We must also mention that an entirely analogous standard tableaux expansion for this particular action of the $Q_{m,n}$ operators (for (m, n) co-prime) was given in [12]. Nevertheless, a distinguishing feature of I.26 is that it is a closed formula. By contrast, the expansions in [12] as well as in [9] are really algorithms, in that the substitutions $x_k \rightarrow 1/w_T(k)$ used there are to be carried out iteratively (as we will see in section 3) due to the fact that the kernel to which these substitutions are performed has denominators that vanish under these substitutions.

Here the crucial tool that makes all this possible is the following identity which may be viewed as the image by U of the identity in I.12.

Theorem I.4

For all $a, b \geq 0$ we have

$$\frac{1}{M} [Q_{a,1}, Q_{b,-1}] = \frac{qt}{qt-1} \Delta_{h_{a+b}[(MX-1)(1/qt-1)]}. \quad \text{I.27}$$

Our presentation is divided into 3 sections. In the first section we will review some of the identities and definitions that we need in the present development which were introduced or proved elsewhere. In the second section we give the proof of Theorem I.1 and complete our treatment of the algebra $\mathcal{A}$ as generated by the operators $Q_{m,1}$. In the third section we give the heretofore unpublished combinatorial argument that derives, from the Stanley-Macdonald Pieri rules, “corner weights” expressions for the coefficients $d_{\mu,\nu}$ in the expansion

$$e_1 \tilde{H}_\nu[X; q, t] = \sum_{\mu \leftarrow \nu} d_{\mu\nu} \tilde{H}_\mu[X; q, t]. \quad \text{I.28}$$

This done, in this section we prove two versions of our standard tableaux expansion formulas one of which is I.26. We terminate this section and the paper by pointing out that a recent result of Bergeron-Haiman [5] shows that the Pieri coefficients $d_{\mu\nu}$ and $c_{\mu,\nu}$ are not as limited as may appear on the surface.

1. Preliminaries

The space of symmetric polynomials with coefficients in $\mathbf{Q}[q, t]$ will be denoted Λ . The subspace of homogeneous symmetric polynomials of degree m will be denoted Λ^m . We will seldom work with symmetric polynomials expressed in terms of variables but rather express them in terms of one of the classical symmetric function bases $\{m_\lambda\}_\lambda$, $\{p_\lambda\}_\lambda$, $\{h_\lambda\}_\lambda$, $\{e_\lambda\}_\lambda$ and $\{s_\lambda\}_\lambda$ (Schur).

We recall that the fundamental involution ω may be defined by setting for the power basis

$$\omega p_\lambda = (-1)^{n-k} p_\lambda = (-1)^{|\lambda| - l(\lambda)} p_\lambda \quad \text{I.1}$$

where for any vector $v = (v_1, v_2, \dots, v_k)$ we set $|v| = \sum_{i=1}^k v_i$ and $l(v) = k$.

In dealing with symmetric function identities, specially with those arising in the Theory of Macdonald Polynomials, we find it convenient and often indispensable to use plethystic notation. This device has a straightforward definition which can be verbatim implemented in MAPLE or MATHEMATICA for computer experimentation. We simply set for any expression $E = E(t_1, t_2, \dots)$ and any power symmetric function p_k

$$p_k[E] = E(t_1^k, t_2^k, \dots). \quad 1.2$$

This given, for any symmetric function F we set

$$F[E] = Q_F(p_1, p_2, \dots) \Big|_{p_k \rightarrow E(t_1^k, t_2^k, \dots)} \quad 1.3$$

where Q_F is the polynomial yielding the expansion of F in terms of the power basis.

A paradoxical but necessary property of plethystic substitutions is that 1.3 requires

$$p_k[-E] = -p_k[E]. \quad 1.4$$

This notwithstanding, we will still need to carry out ordinary changes of signs. To distinguish it from the “*plethystic*” minus sign, we will carry out the “*ordinary*” sign change by means of a new variable ϵ which outside of the plethystic bracket is simply replaced by -1 . For instance, these conventions give for $X_k = x_1 + x_2 + \dots + x_n$

$$p_k[-\epsilon X_n] = -\epsilon^k \sum_{i=1}^n x_i^k = (-1)^{k-1} \sum_{i=1}^n x_i^k$$

Thus for any symmetric function $F \in \Lambda$ and any expression E we have

$$\omega F[E] = F[-\epsilon E] \quad 1.5$$

In particular, if $F \in \Lambda^{=k}$ we may also rewrite this as

$$F[-E] = (-1)^k \omega F[E]. \quad 1.6$$

The formal power series

$$\Omega = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right)$$

combined with plethystic substitutions provides a powerful way of dealing with the many generating functions occurring in our manipulations. In fact, for any given expression E we will set

$$\Omega[E] = \exp\left(\sum_{k \geq 1} \frac{p_k[E]}{k}\right)$$

and since for any two expressions A, B 1.3 gives

$$p_k[A + B] = p_k[A] + p_k[B] \quad 1.7$$

We derive from this the fundamental formula

$$\Omega[A + B] = \Omega[A] \Omega[B] \quad 1.8$$

In particular for $A = \sum_{i=1}^n a_i$ and $B = \sum_{j=1}^m b_j$ we also get

$$\Omega[z(A - B)] = \frac{\prod_{j=1}^m (1 - b_j z)}{\prod_{i=1}^n (1 - a_i z)} \quad 1.9$$

Clearly, for any two expressions A, B we can view $\Omega[z(A - B)]$ as the generating functions of the homogeneous symmetric functions plethystically evaluated at $A - B$

$$\Omega[z(A - B)] = \sum_{m \geq 1} z^m h_m[A - B]$$

In particular, by equating coefficients of z^m on both sides of 1.9, we get (using 1.6)

$$h_m[A - B] = \sum_{r=0}^m h_{m-r}[A] h_r[-B] = \sum_{r=0}^m h_{m-r}[A] (-1)^r e_r[B]$$

In particular it follows from this that

$$h_m[(1-t)(1-q)] = \begin{cases} (1-t)(1-q) \sum_{i=0}^{m-1} (qt)^i & \text{if } m > 0 \\ 1 & \text{if } m = 0 \end{cases} \quad 1.10$$

The following facts (proved in [6]) will play a basic role here

Proposition 1.1

D_k and D_k^* are $*$ -adjoint to $(-1)^k D_{-k}$ and $(-qt)^k D_{-k}^*$ respectively. Moreover they are related to the modified Macdonald polynomials $\tilde{H}_\mu[X; q, t]$ and ∇ by the identities

$$\begin{array}{ll} (i) & D_0 \tilde{H}_\mu = -D_\mu(q, t) \tilde{H}_\mu \\ (ii) & D_k \underline{e}_1 - \underline{e}_1 D_k = M D_{k+1} \\ (iii) & \nabla \underline{e}_1 \nabla^{-1} = -D_1 \\ (iv) & \nabla^{-1} e_1^\perp \nabla = \frac{1}{M} D_{-1} \\ (v) & D_k e_1^\perp - e_1^\perp D_k = D_{k-1} \end{array} \quad \begin{array}{ll} (i)^* & D_0^* \tilde{H}_\mu = -D_\mu(1/q, 1/t) \tilde{H}_\mu \\ (ii)^* & D_k^* \underline{e}_1 - \underline{e}_1 D_k^* = -\widetilde{M} D_{k+1}^* \\ (iii)^* & \nabla D_1^* \nabla^{-1} = \underline{e}_1 \\ (iv)^* & \nabla^{-1} D_{-1}^* \nabla = -\widetilde{M} e_1^\perp \\ (v)^* & D_k^* e_1^\perp - e_1^\perp D_k^* = -D_{k-1}^* \end{array} \quad 1.11$$

with $e_1^\perp$ the Hall scalar product adjoint of multiplication by e_1 , $\widetilde{M} = (1 - 1/t)(1 - 1/q)$ and

$$D_\mu(q, t) = M B_\mu(q, t) - 1 \quad 1.12$$

As in [3] our starting point are the identifications

$$a) \quad Q_{0,1} = -\underline{e}_1, \quad b) \quad Q_{1,0} = D_0, \quad 1.13$$

Thus it follows from 1.11 (ii), (iii) and the definition in I.3 that

$$a) \quad Q_{1,k} = D_k \quad b) \quad Q_{1,1} = \nabla Q_{0,1} \nabla^{-1} \quad 1.14$$

Now it is shown in [3] that the definition in I.3 combined with 1.13 b) implies the following fundamental identity.

Proposition 1.2

For any co-prime pair m, n we have

$$Q_{m+n,n} = \nabla Q_{m,n} \nabla^{-1}.$$

In particular it follows that we also have

$$Q_{m,1} = \nabla^m Q_{0,1} \nabla^{-m} = -\nabla^m e_1 \nabla^{-m}. \quad 1.15$$

For notational convenience, here and after we may use the symbol “ T_m ” to represent “ $-\nabla^m e_1 \nabla^{-m}$ ”. The following basic fact is also an immediate consequence of the definition in I.3

Theorem 1.1

For any co-prime pair m, n we may set

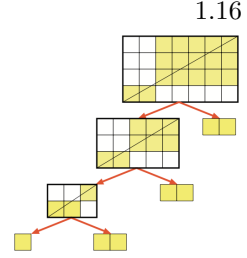
$$Q_{m,n} = \begin{cases} \frac{1}{M}[Q_{c,d}, Q_{a,b}] & \text{if } n > 1 \text{ and } \text{Split}(m, n) = (a, b) + (c, d) \\ T_m & \text{if } n = 1. \end{cases} \quad 1.16$$

For instance, since we have (see adjacent figure)

$$\text{Split}(7, 4) = (2, 1) + (5, 3), \quad \text{Split}(5, 3) = (2, 1) + (3, 2), \quad \text{Split}(3, 2) = (2, 1) + (1, 1).$$

The definition in 1.16 gives

$$Q_{7,4} = \frac{1}{M}[Q_{5,3}, T_2], \quad Q_{5,3} = \frac{1}{M}[Q_{3,2}, T_2], \quad Q_{3,2} = \frac{1}{M}[T_1, T_2].$$



Thus

$$Q_{7,4} = \frac{1}{M^3} [[[T_1, T_2], T_2], T_2] = \frac{1}{M^3} (T_1 T_2^3 - 3T_2 T_1 T_2^2 + 3T_2^2 T_1 T_2 - T_2^3 T_1)$$

We should mention that it follows from the Stanley-Macdonald Pieri rules that to compute the action of an operator $Q_{m,n}$ we only need its “symbol” $\Xi_{m,n}$. This is the polynomial in $x_1, x_2, \dots, x_n$ that is obtained by replacing, in each monomial, the i^{th} factor T_{a_i} by $x_i^{a_i}$. For instance for $Q_{7,4}$ this gives

$$\Xi_{m,n}[x_1, x_2, x_3, x_4] = \frac{1}{M} (x_1 x_2^2 x_3^2 x_4^2 - 3x_1^2 x_2 x_3^2 x_4^2 + 3x_1^2 x_2^2 x_3 x_4^2 + x_1 x_2^2 x_3^2 x_4^2)$$

In the general case, denoting by $\mathcal{S}_k$ the operation of making the replacements $x_i \rightarrow x_{i+k}$ in a polynomial in the x_i variables, we can construct $\Xi_{m,n}[x_1, x_2, \dots, x_n]$ by the recursion

$$\Xi_{m,n} = \begin{cases} \frac{1}{M} (\Xi_{c,d} \mathcal{S}_d \Xi_{a,b} - \Xi_{a,b} \mathcal{S}_b \Xi_{c,d}) & \text{if } n > 1 \text{ and } \text{Split}(m, n) = (a, b) + (c, d) \\ x_1^m & \text{if } n = 1. \end{cases} \quad 1.17$$

This given, as a corollary of Theorem I.3 we will obtain that

$$Q_{m,n}(-1)^n = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \sum_{T \in ST(\mu)} \Xi[w_T(1), w_T(2), \dots, w_T(n)] \prod_{k=2}^n \frac{1 - w_T(k)qt}{1 - qt} \prod_{1 \leq h < k \leq n} f\left(\frac{w_T(h)}{w_T(k)}\right) \quad 1.18$$

Our first task will be to establish the identities that permit our definition of the operators $Q_{u,v}$ when u, v are not co-prime. This was carried out in [3] by viewing $\mathcal{A}$ as generated by the family $\{Q_{1,n}\}_{n \geq 0}$. In the present development, (due to Theorem 1.1), we view $\mathcal{A}$ as generated by the family $\{Q_{m,1}\}_{m \geq 0}$. It turns out that an identity established in 2008 in the research that led to results in [11] turns out to provide the basic ingredient needed in the present development. It may be stated as follows

Theorem 1.2

For all $m \geq 1$ the operators in the family

$$\{[Q_{b,-1}, Q_{a,1}]\}_{\substack{a \geq 0; b \geq 0 \\ a+b=m}} \quad 1.19$$

all commute with D_0 and act identically on Λ .

The proof will be given in the next section. Here we will prove what is necessary to define the operators $Q_{b,-1}$ and derive some of their properties.

To begin, for any co-prime pair (m, n) we will set

$$Q_{m,-n} = Q_{m,n}^{\perp *} \quad 1.20$$

where “ $\perp^*$ ” denotes the operation of the taking the adjoint of a symmetric function operator with respect to the $*$ -scalar product. In other words $Q_{m,-n}$ is the unique operator which satisfies

$$\langle Q_{m,-n} f, g \rangle_* = \langle f, Q_{m,n} g \rangle_* \quad (\text{for all } f, g \in \Lambda). \quad 1.21$$

To obtain an explicit formula for $Q_{b,-1}$ our starting point are the following two auxiliary identities

Lemma 1.1

For all $f, g \in \Lambda$ we have

$$\langle f, g \rangle_* = \langle \phi \omega f, g \rangle = \langle \omega \phi f, g \rangle$$

where ϕ is the operator defined by the plethystic substitution

$$\phi f[X] = f[MX] = f[(1-t)(1-q)X]. \quad 1.22$$

This is an easy consequence of the definition in I.19 (see [6] for a proof).

Lemma 1.2

$$Q_{0,-1} = -M e_1^\perp \quad 1.23$$

where $e_1^\perp$ is the adjoint of e_1 with respect to the Hall scalar product.

Proof

Recall from 1.13 a) that $Q_{0,1} = -e_1$ thus by 1.22

$$\langle Q_{0,1} f, g \rangle_* = -\langle e_1 f, g \rangle_* = -\langle \phi \omega e_1 f, g \rangle = -M \langle e_1 \phi \omega f, g \rangle = -M \langle \phi \omega f, e_1^\perp g \rangle_* = -M \langle f, e_1^\perp g \rangle_*.$$

As a corollary we get

Proposition 1.3

For all $m \geq 1$ we have

$$Q_{m,-1} = -M \nabla^{-m} e_1^\perp \nabla^m. \quad 1.24$$

Proof

From the particular case $n = 1$ of 1.20 we have

$$\langle Q_{m,-1} f, g \rangle_* = \langle f, Q_{m,1} g \rangle_* \quad (\text{for all } f, g \in \Lambda)$$

and 1.15 gives

$$\langle Q_{m,-1} f, g \rangle_* = \langle f, \nabla^m Q_{0,1} \nabla^{-m} g \rangle_* = -M \langle \nabla^{-m} e_1^\perp \nabla^m f, g \rangle_*.$$

The last equality is due to the self-adjointness of ∇ with respect to the $*$ -scalar product.

The following identities will also play a role in the sequel

Proposition 1.4

For all $a, b \geq 1$ we have

$$\begin{aligned} a) \quad Q_{a,1} &= \frac{1}{M} [Q_{a-1,1}, D_0] & b) \quad Q_{b,-1} &= \frac{1}{M} [D_0, Q_{b-1,-1}]. \end{aligned} \quad 1.25$$

Proof

The identity in 1.25 a) is an instance of 1.16. The identity in 1.25 b) is the $*$ -adjoint of 1.25 a) together with the fact that taking “ $\perp^*$ ” reverses order.

For $F[X; q, t] \in \Lambda$ let us set

$$\downarrow F[X; q, t] = \omega F[X; 1/q, 1/t] = F[-\epsilon X; 1/q, 1/t]. \quad 1.26$$

It is easily seen that the operator “ $\downarrow$ ” is an involution. It also has the following useful properties, proved in [6].

Proposition 1.5

$$\begin{aligned} a) \quad \downarrow \nabla \downarrow &= \nabla^{-1} \\ b) \quad \downarrow D_k \downarrow &= (-1)^k D_k^*. \end{aligned} \quad 1.27$$

2. Proofs of basic identities for the algebra $\mathcal{A}$

Our goal in this section is to prove Theorem I.1 and its corollary Theorem I.2. This requires establishing first Theorem 1.2. To carry all this out, we need some preliminary observations and establish some auxiliary properties of the family of operators

$$Q_{m,-1} = \nabla^{-m} Q_{0,-1} \nabla^m = -M \nabla^{-m} e_1^\perp \nabla^m. \quad 2.1$$

For notational convenience we need to set

$$\nabla^{-m} e_1^\perp \nabla^m = V_m. \quad 2.2$$

This given, 1.24 and 1.25 b), namely the two identities

$$Q_{m,-1} = -M \nabla^{-m} e_1^\perp \nabla^m, \quad Q_{b,-1} = \frac{1}{M} [D_0, Q_{b-1,-1}]$$

combine to give us the recursion

$$V_m = \frac{1}{M} [D_0, V_{m-1}]. \quad 2.3$$

Surprisingly, a simple conjugation by the “ $\downarrow$ ” operators reverses this recursion. More precisely we have

Proposition 2.3

$$V_{m-1} = \frac{1}{M} [D_0^*, V_m]. \quad 2.4$$

Proof

The definition in 2.2 and 2.3 for $m = 1$ give

$$\nabla^{-1} e_1^\perp \nabla = \frac{1}{M} [D_0, e_1^\perp].$$

Since we trivially have $\downarrow e_1^\perp \downarrow = e_1^\perp$ (as it is easily verified by applying both sides to any of the standard symmetric function bases), from Proposition 1.4 we derive that

$$\nabla e_1^\perp \nabla^{-1} = \frac{1}{M} [D_0^*, e_1^\perp]. \quad 2.5$$

Since ∇ and D_0^* are both eigen-operators for the modified Macdonald basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$, they commute. Thus conjugating both sides of 2.5 by ∇^{-m} gives

$$\nabla^{-m+1} e_1^\perp \nabla^{m-1} = \frac{1}{M} [D_0^*, \nabla^{-m} e_1^\perp \nabla^m],$$

which is another way of writing 2.4.

Theorem 2.1

The operator

$$U_m = [\nabla^{-m} e_1^\perp \nabla^m, e_1] \tag{2.6}$$

commutes with both D_0 and D_0^ and therefore it is an eigen-operator of the basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$.*

Proof

We will prove by induction that we have for all $m \geq 1$

$$\begin{cases} a) & U_m = [D_1, V_{m-1}] \\ b) & [U_m, D_0^*] = 0 \end{cases}. \tag{2.7}$$

Note that setting $m = 1$ in 2.6 and using 1.11 (iii) and then 1.11 (ii) we get

$$U_1 = [\nabla^{-1} e_1^\perp \nabla, e_1] = \frac{1}{M} [D_{-1}, e_1] = D_0.$$

This proves 2.7 b) for $m = 1$. To prove 2.7 in the base case, we are left to show that

$$D_0 = [D_1, e_1^\perp],$$

but this is precisely 1.11(v). We can thus inductively assume 2.7 true up to $m - 1$.

Now by definition we have

$$U_m = [V_m, e_1]$$

and 2.3 gives

$$\begin{aligned} U_m &= \frac{1}{M} [D_0, V_{m-1}, e_1] \\ (\text{by Jacobi}) &= -\frac{1}{M} [[V_{m-1}, e_1], D_0] - [[e_1, D_0], V_{m-1}] \\ (\text{by 1.11 (ii)}) &= -\frac{1}{M} [U_{m-1}, D_0] + [D_1, V_{m-1}] \\ (\text{by 2.7 b) for } m-1) &= [D_1, V_{m-1}]. \end{aligned}$$

This proves 2.7 a) for m . To show 2.7 b) for m we use this and get

$$\begin{aligned} [U_m, D_0^*] &= [[D_1, V_{m-1}], D_0^*] \\ (\text{by Jacobi}) &= -[[V_{m-1}, D_0^*], D_1] - [[D_0^*, D_1], V_{m-1}]. \end{aligned} \tag{2.8}$$

For the first term Proposition 2.3 gives (using induction)

$$-[[V_{m-1}, D_0^*], D_1] = \widetilde{M}[V_{m-2}, D_1] = -\widetilde{M}U_{m-1}. \tag{2.9}$$

For the second term in 2.8 we have, using 1.11 (iii),

$$\begin{aligned} [D_0^*, D_1] &= [\nabla e_1 \nabla^{-1}, D_0^*] \\ (D_0^* \text{ and } \nabla \text{ commute}) &= \nabla [e_1, D_0^*] \nabla^{-1} \\ (\text{by 1.11 (ii)}^*) &= \widetilde{M} \nabla D_1^* \nabla^{-1} \\ (\text{by 1.11 (iii)}^*) &= \widetilde{M} e_1. \end{aligned}$$

Thus

$$- [D_0^*, D_1], V_{m-1}] = -\widetilde{M}[e_1, V_{m-1}] = \widetilde{M}U_{m-1}$$

and this together with 2.9 reduces 2.8 to

$$[U_m, D_0^*] = -\widetilde{M}U_{m-1} + \widetilde{M}U_{m-1} = 0$$

completing the induction.

We are now finally in a position to prove the following sharpening of Theorem 1.2.

Theorem 2.2

For all $m \geq 1$ the operators in the family

$$\{[Q_{b,-1}, Q_{a,1}]\}_{\substack{a \geq 0; b \geq 0 \\ a+b=m}} \quad 2.10$$

act identically on Λ and we can set

$$Q_{m,0} = \frac{1}{M} [Q_{m,-1}, Q_{0,1}] = [\nabla^{-m} e_1^\perp \nabla^m, \underline{e}_1]. \quad 2.11$$

In particular we see that the collection $\{Q_{m,0}\}_{m \geq 0}$ is a commuting family of eigen-operators for the modified Macdonald basis.

Proof

Notice first that the second equality in 2.11 follows from 2.1 and 1.13 a). Moreover, since Theorem 2.1 assures that the right hand side of 2.11 commutes with D_0 and D_0^* , the last assertion follows from the fact that the eigenvalues of D_0 (or D_0^*) are all distinct. Thus the only thing that remains to prove is the equality of these operators. But this is now easily seen if we write the operators in 2.10 using 1.15 and 1.24, that is

$$[Q_{b,-1}, Q_{a,1}] = M[\nabla^{-b} e_1^\perp \nabla^b, \nabla^a e_1 \nabla^{-a}].$$

In fact, we know from Theorem 2.1 that this operator, for $b = m$ and $a = 0$, commutes with ∇ thus

$$[Q_{m,-1}, Q_{0,1}] = M\nabla[\nabla^{-m} e_1^\perp \nabla^m, e_1] \nabla^{-1} = M[\nabla^{-m+1} e_1^\perp \nabla^{m-1}, \nabla e_1 \nabla^{-1}] = [Q_{m-1,-1}, Q_{1,1}].$$

Proceeding by descent induction on $b \in [1, m]$, assume that for $a = m - b$ we have

$$[Q_{m,-1}, Q_{0,1}] = [Q_{b,-1}, Q_{a,1}].$$

Conjugating by ∇ we similarly obtain

$$[Q_{m,-1}, Q_{0,1}] = \nabla[Q_{b,-1}, Q_{a,1}] \nabla^{-1} = [Q_{b-1,-1}, Q_{a+1,1}].$$

This completes the induction and our proof.

Our next and final task in this section is the identification of the eigenvalues of the operators

$$Q_{m,0} = [\nabla^{-m} e_1^\perp \nabla^m, \underline{e}_1].$$

This task, as well as the developments in the next section are heavily dependent on the Pieri rules for $\underline{e}_1$ and $e_1^\perp$ and their summation formulas. These remarkable identities may be stated as follows.

Theorem 2.3

For $\mu \vdash n$ and $\nu \vdash n-1$ we have

$$a) \quad e_1 \tilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu,\nu} \tilde{H}_\mu, \quad b) \quad e_1^\perp \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu,\nu} \tilde{H}_\nu \quad 2.12$$

where “ $\nu \rightarrow \mu$ ” means that μ is obtained by adding a corner square to ν . Moreover we also have the basic relation

$$d_{\mu,\nu} = M \frac{w_\nu}{w_\mu} c_{\mu,\nu}. \quad 2.13$$

Theorem 2.4

$$a) \quad \sum_{\nu \rightarrow \mu} c_{\mu\nu}(q, t) (T_\mu/T_\nu)^k = \begin{cases} \frac{tq}{M} h_{k+1} [D_\mu(q, t)/tq] & \text{if } k \geq 1 \\ B_\mu(q, t) & \text{if } k = 0 \end{cases} \quad 2.14$$

$$b) \quad \sum_{\mu \leftarrow \nu} d_{\mu\nu}(q, t) (T_\mu/T_\nu)^k = \begin{cases} (-1)^{k-1} e_{k-1} [D_\nu(q, t)] & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \end{cases}$$

Remark 2.1

We must mention that 2.12 b) and 2.13 a) were first proved in [10], directly from the original Stanley-Macdonald Pieri rules. On the other hand for 2.12 a), and 2.13 b), a direct proof was never published, although extensively used in several publications. These proofs will be included in the next section for sake of completeness. An indirect derivation of 2.14 a) and b) was also given in [6].

Our goal, which is Theorem I.1, may be simply restated as follows

Theorem 2.5

For all $k \geq 1$ and partitions μ we have

$$[\nabla^{-k} e_1^\perp \nabla^k, e_1] \tilde{H}_\mu[X; q, t] = \frac{qt}{qt-1} h_k [D_\mu(q, t)(\frac{1}{qt} - 1)] \tilde{H}_\mu[X; q, t]. \quad 2.15$$

Proof

Notice first that if we know already that a certain symmetric function operator $\mathbf{Z}$ is an eigen-operator for the basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$, then given the expansion

$$e_n^*[X] = e_n[\frac{X}{M}] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu}$$

we can simply identify its eigenvalues $\mathbf{z}_\mu$ from the formula

$$\mathbf{Z} e_n[\frac{X}{M}] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \mathbf{z}_\mu.$$

This given, let us start with

$$\begin{aligned} \nabla^k e_1 e_n^* &= \sum_{\mu \vdash n} \frac{1}{w_\mu} \sum_{\gamma \leftarrow \mu} d_{\gamma\mu} \tilde{H}_\gamma T_\gamma^k \\ &= \sum_{\gamma \vdash n+1} \frac{\tilde{H}_\gamma T_\gamma^k}{w_\gamma} \sum_{\mu \rightarrow \gamma} \frac{w_\gamma}{w_\mu} d_{\gamma\mu} \\ (\text{by 2.13}) &= M \sum_{\gamma \vdash n+1} \frac{\tilde{H}_\gamma T_\gamma^k}{w_\gamma} \sum_{\mu \rightarrow \gamma} c_{\gamma\mu} \\ (\text{by 2.14 a)}) &= M \sum_{\gamma \vdash n+1} \frac{\tilde{H}_\gamma T_\gamma^k}{w_\gamma} B_\gamma(q, t). \end{aligned}$$

Thus

$$\begin{aligned}
\nabla^{-k} e_1^\perp \nabla^k e_1 e_n^* &= M \sum_{\gamma \vdash n+1} \frac{\nabla^{-k} e_1^\perp \tilde{H}_\gamma T_\gamma^k}{w_\gamma} B_\gamma(q, t) \\
&= M \sum_{\gamma \vdash n+1} \frac{T_\gamma^k}{w_\gamma} B_\gamma(q, t) \sum_{\delta \rightarrow \gamma} c_{\gamma\delta} \tilde{H}_\delta T_\delta^{-k} \\
&= M \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} \sum_{\gamma \leftarrow \delta} \frac{w_\delta}{w_\gamma} c_{\gamma\delta} \frac{T_\gamma^k}{T_\delta^k} B_\gamma(q, t) = \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} \sum_{\gamma \leftarrow \delta} d_{\gamma\delta} \frac{T_\gamma^k}{T_\delta^k} B_\gamma(q, t).
\end{aligned} \tag{2.16}$$

But the relation $B_\gamma = B_\delta + \frac{T_\gamma}{T_\delta}$ gives

$$\sum_{\gamma \leftarrow \delta} d_{\gamma\delta} \frac{T_\gamma^k}{T_\delta^k} B_\gamma = B_\delta \sum_{\gamma \leftarrow \delta} d_{\gamma\delta} \frac{T_\gamma^k}{T_\delta^k} + \sum_{\gamma \leftarrow \delta} d_{\gamma\delta} \frac{T_\gamma^{k+1}}{T_\delta^{k+1}}$$

and 2.14 b) delivers

$$\sum_{\gamma \leftarrow \delta} d_{\gamma\delta} \frac{T_\gamma^k}{T_\delta^k} B_\gamma = (-1)^{k-1} B_\delta e_{k-1}[D_\delta(q, t)] + (-1)^k e_k[D_\delta(q, t)].$$

Using this in 2.16 we obtain

$$\nabla^{-k} e_1^\perp \nabla^k e_1 e_n^* = (-1)^{k-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} B_\delta e_{k-1}[D_\delta(q, t)] + (-1)^k \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_k[D_\delta(q, t)]. \tag{2.17}$$

Next we start with

$$\begin{aligned}
\nabla^{-k} e_1^\perp \nabla^k e_n^* &= \sum_{\mu \vdash n} \frac{T_\mu^k}{w_\mu} \sum_{\nu \rightarrow \mu} c_{\mu\nu} T_\nu^{-k} \tilde{H}_\nu \\
&= \sum_{\nu \vdash n-1} \frac{\tilde{H}_\nu}{w_\nu} \sum_{\mu \rightarrow \nu} c_{\mu\nu} \frac{w_\nu}{w_\mu} \frac{T_\mu^k}{T_\nu^k} = \frac{1}{M} \sum_{\nu \vdash n-1} \frac{\tilde{H}_\nu}{w_\nu} \sum_{\mu \leftarrow \nu} d_{\mu\nu} \frac{T_\mu^k}{T_\nu^k}
\end{aligned}$$

and 2.14 b) gives

$$\nabla^{-k} e_1^\perp \nabla^k e_n^* = \frac{(-1)^{k-1}}{M} \sum_{\nu \vdash n-1} \frac{\tilde{H}_\nu}{w_\nu} e_{k-1}[D_\nu(q, t)].$$

Multiplying on both sides by e_1 yields

$$\begin{aligned}
e_1 \nabla^{-k} e_1^\perp \nabla^k e_n^* &= \frac{(-1)^{k-1}}{M} \sum_{\nu \vdash n-1} \frac{1}{w_\nu} e_{k-1}[D_\nu(q, t)] \sum_{\delta \leftarrow \nu} d_{\delta\nu} \tilde{H}_\delta \\
&= \frac{(-1)^{k-1}}{M} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} \sum_{\nu \rightarrow \delta} \frac{w_\delta}{w_\nu} d_{\delta\nu} e_{k-1}[D_\nu(q, t)] = (-1)^{k-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} \sum_{\nu \rightarrow \delta} c_{\delta\nu} e_{k-1}[D_\nu(q, t)].
\end{aligned} \tag{2.18}$$

To deal with the sum

$$\sum_{\nu \rightarrow \delta} c_{\delta\nu} e_{k-1}[D_\nu(q, t)]$$

the identity $D_\nu = D_\delta - M \frac{T_\delta}{T_\nu}$ and the addition formula for the e_k 's gives

$$e_{k-1}[D_\nu] = e_{k-1}[D_\delta] + \sum_{r=1}^{k-1} e_{k-1-r}[D_\delta] e_r[-M] \left(\frac{T_\delta}{T_\nu}\right)^r = e_{k-1}[D_\delta] + \sum_{r=1}^{k-1} (-1)^r e_{k-1-r}[D_\delta] h_r[M] \left(\frac{T_\delta}{T_\nu}\right)^r.$$

Thus

$$\sum_{\nu \rightarrow \delta} c_{\delta\nu} e_{k-1}[D_\nu(q, t)] = \sum_{\nu \rightarrow \delta} c_{\delta\nu} e_{k-1}[D_\delta] + \sum_{r=1}^{k-1} (-1)^r e_{k-1-r}[D_\delta] h_r[M] \sum_{\nu \rightarrow \delta} c_{\delta\nu} \left(\frac{T_\delta}{T_\nu}\right)^r$$

and 2.14 b) gives

$$\begin{aligned} \sum_{\nu \rightarrow \delta} c_{\delta\nu} e_{k-1}[D_\nu(q, t)] &= e_{k-1}[D_\delta] B_\delta + \sum_{r=1}^{k-1} (-1)^r e_{k-1-r}[D_\delta] h_r[M] \frac{qt}{M} h_{r+1}[D_\delta/qt] \\ (\text{by 1.10}) &= e_{k-1}[D_\delta] B_\delta + \frac{qt}{(1-qt)} \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[D_\delta] (1 - (qt)^r) h_{r+1}[D_\delta/qt] \\ &= e_{k-1}[D_\delta] B_\delta - \frac{qt}{(1-qt)} \sum_{r=1}^k (-1)^r e_{k-r}[D_\delta] (1 - (qt)^{r-1}) h_r[D_\delta/qt] \\ &= e_{k-1}[D_\delta] B_\delta - \frac{qt}{(1-qt)} \sum_{r=1}^k e_{k-r}[D_\delta] e_r[-D_\delta/qt] + \frac{1}{(1-qt)} \sum_{r=1}^k e_{k-r}[D_\delta] e_r[-D_\delta] \\ &= e_{k-1}[D_\delta] B_\delta - \frac{qt}{(1-qt)} e_k[D_\delta(1 - 1/qt)] + \frac{qt}{(1-qt)} e_k[D_\delta] - \frac{1}{(1-qt)} e_k[D_\delta] \\ &= e_{k-1}[D_\delta] B_\delta - \frac{qt}{(1-qt)} e_k[D_\delta(1 - 1/qt)] - e_k[D_\delta]. \end{aligned}$$

Using this in 2.18 we finally obtain

$$e_1 \nabla^{-k} e_1^\perp \nabla^k e_n^* = (-1)^{k-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_{k-1}[D_\delta] B_\delta + \frac{(-1)^k qt}{(1-qt)} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_k[D_\delta(1 - 1/qt)] + (-1)^k \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_k[D_\delta].$$

Recalling that the identity in 2.17 is

$$\nabla^{-k} e_1^\perp \nabla^k e_1 e_n^* = (-1)^{k-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} B_\delta e_{k-1}[D_\delta] + (-1)^k \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_k[D_\delta],$$

subtracting this from the former yields the final identity

$$[\nabla^{-k} e_1^\perp \nabla^k, e_1] e_n^* = \frac{(-1)^k qt}{qt-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} e_k[D_\delta(1 - 1/qt)] = \frac{qt}{qt-1} \sum_{\delta \vdash n} \frac{\tilde{H}_\delta}{w_\delta} h_k[D_\delta(\frac{1}{qt} - 1)]$$

completing our proof of 2.15.

Remark 2.2

The identity of Proposition 2.3, namely

$$[D_0^*, V_k] = \widetilde{M} V_{k-1}, \tag{2.19}$$

is quite remarkable when viewed geometrically. In fact from 2.2 and 2.1 we derive that

$$V_m = \nabla^{-m} e^\perp \nabla^m = -\frac{1}{M} Q_{m,-1}$$

and the definition in I.11 gives

$$D_0^* = -\frac{1}{qt} Q_{-1,0}$$

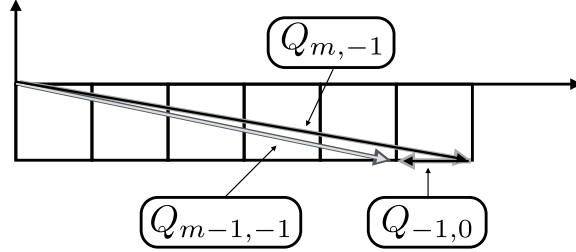
Thus 2.19 is none other than

$$-\frac{1}{qt}[Q_{-1,0}, Q_{m,-1}] = \frac{M}{qt}Q_{m-1,1}$$

or better

$$Q_{m-1,1} = \frac{1}{M}[Q_{m,-1}, Q_{-1,0}], \quad 2.20$$

which may be viewed as the identity resulting from the following splitting of the vector $(m-1, -1)$.



3. Pieri Rules and Standard Tableaux expansions

Explicit formulas for the coefficients $d_{\mu,\nu}$ and $c_{\mu,\nu}$ in 2.12 were first obtained in [10] from Macdonald's formula for the multiplication by e_1 of his original $P_\mu[X; q, t]$ basis. More precisely the latter formula was used in [10] to obtain the identity

$$d_{\mu\nu}(q, t) = \prod_{s \in R_{\mu\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} \prod_{s \in C_{\mu\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}} \quad 3.1$$

where $R_{\mu\nu}$ and $C_{\mu\nu}$ denote the collections of cells of ν that are respectively in the row and the column of the cell μ/ν . This done, an easy use of the orthogonality relations in I.18 gave that

$$c_{\mu\nu}(q, t) = \frac{1}{M} \frac{w_\mu}{w_\nu} d_{\mu\nu}(q, t). \quad 3.2$$

Finally, this relation combined with 3.1, after many cancellations, yielded the identity

$$c_{\mu\nu}(q, t) = \prod_{s \in R_{\mu\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s \in C_{\mu\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}}. \quad 3.3$$

It is not difficult to see that, in both 3.1 and 3.2, there are still many cancellations remaining. This observation led to extremely useful formulas for these Pieri coefficients. To state them we need notation. For a partition μ with l removable corners, let $x_0, x_1, x_2, \dots, x_l$ and $u_0, u_1, u_2, \dots, u_l$ denote the weights ^(†) of the cells of μ as illustrated in the adjacent diagram for $l = 5$. This given, it is shown in [10] that carrying out these cancellations 3.3 reduces to

$$c_{\mu, \nu^{(k)}} = \frac{x_k}{(1 - \frac{1}{t})(1 - \frac{1}{q})} \frac{\prod_{i=0}^l (1 - \frac{u_i}{x_k})}{\prod_{i=1; i \neq k}^l (1 - \frac{x_i}{x_k})}, \quad 3.4$$

where $\nu^{(k)}$ is the partition obtained by removing from μ the corner with weight $x_k = T_\mu / T_{\nu^{(k)}}$.

Moreover, it is shown in [10] that massive cancellations also yield the truly remarkable identity

$$(1 - \frac{1}{t})(1 - \frac{1}{q})B_\mu(q, t) = x_0 + x_1 + \dots + x_l - u_0 - u_1 - \dots - u_l. \quad 3.5$$

^(†) $weight(c) = t^{l'_\mu(c)} q^{a'_\mu(c)}$

Somewhat later the last named author derived the companion formula for the Pieri coefficients $d_{\mu,\nu}$. This formula, which can be obtained by carrying out the appropriate cancellations in 3.1, can be written as

$$d_{\mu^{(k)}, \nu} = \frac{1}{q t u_k} \frac{\prod_{i=1}^l (1 - \frac{x_i}{u_k})}{\prod_{i=0: i \neq k}^l (1 - \frac{u_i}{u_k})} \quad 3.6$$

provided the shape in 3.4 is now interpreted as the Ferrers' diagram of the partition ν and $\mu^{(k)}$ is now the partition that is obtained from ν by adding the (addable) corner square that is the NE shift by one cell unit of the cell labelled by u_k in 3.4. In particular this gives that

$$T_{u^{(k)}}/T_\nu = tq u_k. \quad 3.7$$

Since the proof of 3.6 was never published, it will be good to include it here and at the same time illustrate the process that yields 3.6 from 3.1.

To begin let us start from the figure in 3.4 interpreted as the Ferrers' diagram of ν but shift all the labelled cells NE by one cell unit, placing a bar on each of their labels. That is we are setting $\bar{x}_i = qtx_i$ and $\bar{u}_i = qtu_i$. With these conventions, multiplying both sides of the identity in 3.5 by qt and replacing μ by ν we obtain

$$MB_\nu(q, t) = \bar{x}_0 + \bar{x}_1 + \cdots + \bar{x}_l - \bar{u}_0 - \bar{u}_1 - \cdots - \bar{u}_l \quad 3.8$$

In the following display we have on the right the labelled Ferrers' diagram of ν and on the left we have 3.1 atop the identity we will derive from it.

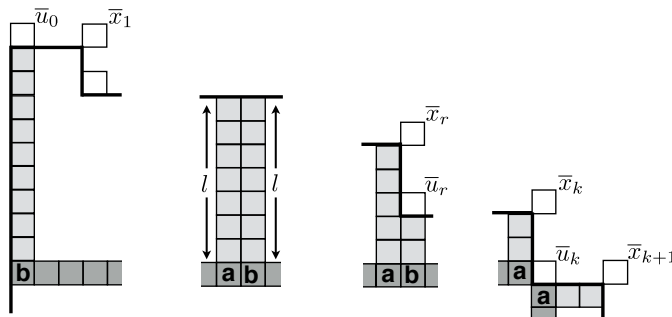
$$d_{\mu\nu}(q, t) = \prod_{s \in R_{\mu\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} \prod_{s \in C_{\mu\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}} \quad (3.9)$$

$$d_{\mu^{(k)}, \nu} = \frac{1}{\bar{u}_k} \frac{\prod_{i=1}^l \left(1 - \frac{\bar{x}_i}{\bar{u}_k}\right)}{\prod_{i=0; i \neq k}^l \left(1 - \frac{\bar{u}_i}{\bar{u}_k}\right)}$$

Here, for convenience, we set

$$\bar{x}_i = t^{\alpha_i} q^{\beta_i} \quad \& \quad \bar{u}_i = t^{\alpha_{i+1}} q^{\beta_i} \quad (\text{for } 0 \leq i \leq l, \text{ with } \beta_0 = \alpha_{l+1} = 0). \quad 3.10$$

To get across the cancellations that occur in S.21 along the row $R_{\mu\nu}$, we need only focus our attention on the following four figures. To help visualize where these figures are located in the diagram of ν we have depicted in 3.9 the row $R_{\mu^{(k)},\nu}$ when the cell added to ν to obtain μ is the one whose weight is $\bar{u}_3$



To begin note that for two cells $a, b \in R_{\mu^{(k)}, \nu}$, both of whose legs are l (as in the second figure above), we have

$$q^{a_\nu(a)} - t^{l_\nu(a)+1} = q^{(\beta_k-1)-a} - t^{l+1} = q^{\beta_k-b} - t^{l+1} = q^{a_\mu(b)} - t^{l_\mu(b)+1}.$$

Thus these two factors cancel each other in $d_{\mu^{(k)}, \nu}$.

On the other hand for two cells $a, b \in R_{\mu^{(k)}, \nu}$, with a preceding b and b below the cell labelled $\bar{u}_r$ (as in the third figure above), we have

$$q^{a_\nu(a)} - t^{l_\nu(a)+1} = q^{\beta_k-\beta_r} - t^{\alpha_r-\alpha_{k+1}} \quad \text{and} \quad q^{a_\mu(b)} - t^{l_\mu(b)+1} = q^{\beta_k-\beta_r} - t^{\alpha_{r+1}-\alpha_{k+1}}.$$

Thus these two cells of $R_{\mu^{(k)}, \nu}$ contribute the following ratio to $d_{\mu^{(k)}, \nu}$

$$\frac{q^{\beta_k-\beta_r} - t^{\alpha_r-\alpha_{k+1}}}{q^{\beta_k-\beta_r} - t^{\alpha_{r+1}-\alpha_{k+1}}} = \frac{q^{\beta_k} t^{\alpha_{k+1}} - t^{\alpha_r} q^{\beta_r}}{q^{\beta_k} t^{\alpha_{k+1}} - t^{\alpha_{r+1}} q^{\beta_r}} = \frac{\bar{u}_k - \bar{x}_r}{\bar{u}_k - \bar{u}_r}. \quad 3.11$$

These cases take care of all the factors contributed by $R_{\mu^{(k)}, \nu}$ except for the first factor in the denominator and the last factor in the numerator, respectively contributed by the cell b in the first figure above and the first cell a in the last figure above. These two factors yield the ratio

$$\frac{q^{a_\nu(a)} - t^{l_\nu(a)+1}}{q^{a_\mu(b)} - t^{l_\mu(b)+1}} = \frac{q^{\beta_k-\beta_k} - t^{\alpha_k-\alpha_{k+1}}}{q^{\beta_k-\beta_0} - t^{\alpha_1-\alpha_{k+1}}} = \frac{q^{\beta_0} q^{\beta_k} t^{\alpha_{k+1}} - t^{\alpha_k} q^{\beta_k}}{q^{\beta_k} q^{\beta_k} t^{\alpha_{k+1}} - t^{\alpha_1} q^{\beta_0}} = \frac{1}{q^{\beta_k}} \frac{\bar{u}_k - \bar{x}_k}{\bar{u}_k - \bar{u}_0}. \quad 3.12$$

We thus obtain that

$$\prod_{s \in R_{\mu\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} = \frac{1}{q^{\beta_k}} \frac{\prod_{i=1}^k (\bar{u}_k - \bar{x}_i)}{\prod_{i=0}^{k-1} (\bar{u}_k - \bar{u}_i)}. \quad 3.13$$

To compute the contribution of the column $C_{\mu^{(k)}, \nu}$ to $d_{\mu^{(k)}, \nu}$ we start as in the previous case to note that for two adjacent cells a, b with the same arm (as indicated in the third figure on the right), the ratio

$$\frac{t^{l_\nu(a)} - q^{a_\nu(a)+1}}{t^{l_\mu(b)} - q^{a_\mu(b)+1}}$$

contributes nothing, since $l_\nu(a) = l_\mu(b)$ in this case.

For the two cells $a, b \in C_{\mu^{(k)}, \nu}$ as indicated in the second figure on the right, reasoning as we did above we see that their contribution to $d_{\mu^{(k)}, \nu}$ is the ratio

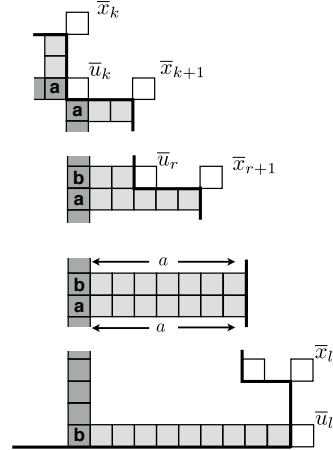
$$\frac{t^{l_\nu(a)} - q^{a_\nu(a)+1}}{t^{l_\mu(b)} - q^{a_\mu(b)+1}} = \frac{t^{\alpha_{k+1}-\alpha_{r+1}} - q^{\beta_{r+1}-\beta_k}}{t^{\alpha_{k+1}-\alpha_{r+1}} - q^{\beta_r-\beta_k}} = \frac{\bar{u}_k - \bar{x}_{r+1}}{\bar{u}_k - \bar{u}_r}.$$

Finally the bottom cell $b \in C_{\mu^{(k)}, \nu}$ and its top cell a , as indicated in the bottom and top figure in the above display, contribute the ratio

$$\frac{t^{l_\nu(a)} - q^{a_\nu(a)+1}}{t^{l_\mu(b)} - q^{a_\mu(b)+1}} = \frac{t^{\alpha_{k+1}-\alpha_{k+1}} - q^{\beta_{k+1}-\beta_k}}{t^{\alpha_{k+1}-\alpha_{l+1}} - q^{\beta_l-\beta_k}} = \frac{1}{t^{\alpha_{k+1}}} \frac{\bar{u}_k - \bar{x}_{k+1}}{\bar{u}_k - \bar{u}_l}.$$

Thus collecting all these ratios gives

$$\prod_{s \in C_{\mu\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}} = \frac{1}{t^{\alpha_{k+1}}} \frac{\prod_{i=k+1}^l (\bar{u}_k - \bar{x}_i)}{\prod_{i=k+1}^l (\bar{u}_k - \bar{u}_i)} \quad 3.14$$



and we finally can see that the combination of 3.13 and 3.14 gives precisely the identity displayed in 3.9. This completes our proof of 3.6.

The most important consequence of the identity in 3.6 is the summation formula in 2.14 b) which we are now in a position to derive with a minimum of efforts. The idea is to consider the rational function

$$R(y) = \frac{\prod_{i=1}^l (1 - y\bar{x}_i)}{\prod_{i=0}^l (1 - y\bar{u}_i)}$$

and note that its partial fraction expansion may be written in the form

$$R(y) = \sum_{j=0}^l \frac{A_j}{1 - y\bar{u}_j}$$

with

$$A_j = (1 - y\bar{u}_j)R(y) \Big|_{y=1/\bar{u}_j} = \frac{\prod_{i=1}^l (1 - \bar{x}_i/\bar{u}_j)}{\prod_{i=0, i \neq j}^l (1 - \bar{u}_i/\bar{u}_j)} = \bar{u}_j d_{\mu^{(j)}, \nu}.$$

Since from 3.7 we derive that $\bar{u}_j = \frac{T_{\mu^{(j)}}}{T_\nu}$ we see that the left hand side of 2.14 b) is none other than

$$\sum_{j=0}^l A_j \bar{u}_j^{k-1} = R(y) \Big|_{y^{k-1}}.$$

But now the identity in 1.9 together with 3.8 gives

$$\begin{aligned} R(y) \Big|_{y^{k-1}} &= \Omega[-y(\bar{x}_1 + \dots + \bar{x}_l - \bar{u}_0 - \dots - \bar{u}_l)] \Big|_{y^{k-1}} \\ &= \Omega[-y(MB_\mu - 1)] \Big|_{y^{k-1}} = h_{k-1}[-D_\mu] = (-1)^{k-1} e_{k-1}[D_\mu]. \end{aligned}$$

This proves the first case of 2.14 b). The second case is immediate.

Here and in the following it will be convenient to set

$$\Pi(a_1, a_2, \dots, a_n) = \nabla^{a_n} e_1 \nabla^{-a_n} \dots \nabla^{a_2} e_1 \nabla^{-a_2} \nabla^{a_1} e_1 \nabla^{-a_1}. \quad 3.15$$

As we have seen Theorem 1.1 assures that every operator $Q_{m,n}$ may be expressed as a non commutative polynomial in the operators $Q_{m,1}$, thus the operators in 3.15 span the algebra $\mathcal{A}$ generated by the operators D_k .

It turns out that the action of these operators have a remarkably beautiful Macdonald polynomial expansion, a particular case of which may be stated as follows

Theorem 3.1

For any weak composition $a_1, a_2, \dots, a_n$ we have

$$\Pi(a_1, a_2, \dots, a_n) \mathbf{1} = \sum_{\mu \vdash n} \tilde{H}_\mu[X; q, t] \sum_{T \in ST(\mu)} \prod_{i=2}^n \frac{x_i^{1-a_i}}{1-x_i} \prod_{1 \leq i < j \leq n} \Omega[-Mx_j/x_i] \prod_{i=2}^n (1 - x_i w_T(i)) \Big|_{S_T} \quad 3.16$$

Proof

We may write

$$\begin{aligned}\Pi(a_1, a_2, \dots, a_n) \mathbf{1} &= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \langle \Pi(a_1, a_2, \dots, a_n) \mathbf{1}, \tilde{H}_\mu \rangle_* \\ &= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \Pi(a_1, a_2, \dots, a_n)^{\perp*} \tilde{H}_\mu.\end{aligned}$$

Since the identity $P_{0,1} = -e_1$ gives $\nabla^a e_1 \nabla^{-a} = -\nabla^a P_{0,1} \nabla^{-a}$, from 1.23 we derive that

$$(\nabla^a e_1 \nabla^{-a})^{\perp*} = -\nabla^{-a} P_{0,-1} \nabla^a = M \nabla^{-a} e_1^{\perp} \nabla^a.$$

Thus

$$\Pi(a_1, a_2, \dots, a_n)^{\perp*} \tilde{H}_\mu = M^n \nabla^{-a_1} e_1^{\perp} \nabla^{a_1} \nabla^{-a_2} e_1^{\perp} \nabla^{a_2} \dots \nabla^{-a_n} e_1^{\perp} \nabla^{a_n} \tilde{H}_\mu,$$

and we can write

$$\Pi(a_1, a_2, \dots, a_n) \mathbf{1} = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} M^n \nabla^{-a_1} e_1^{\perp} \nabla^{a_1} \nabla^{-a_2} e_1^{\perp} \nabla^{a_2} \dots \nabla^{-a_n} e_1^{\perp} \nabla^{a_n} \tilde{H}_\mu. \quad 3.17$$

To apply the operator $\nabla^{-a} e_1^{\perp} \nabla^a$ to $\tilde{H}_\mu$ we use 2.12 b) and obtain, for any integer $a \geq 0$,

$$\nabla^{-a} e_1^{\perp} \nabla^a \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu, \nu} \left(\frac{T_\mu}{T_\nu} \right)^a \tilde{H}_\nu.$$

Starting from this identity, a straightforward induction argument (carried out first in [9]) gives

$$M^n \nabla^{-a_1} e_1^{\perp} \nabla^{a_1} \nabla^{-a_2} e_1^{\perp} \nabla^{a_2} \dots \nabla^{-a_n} e_1^{\perp} \nabla^{a_n} \tilde{H}_\mu = M^n \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} c_{T^{(k)}, T^{(k-1)}} \quad 3.18$$

where $T^{(k)}$ denotes the tableau obtained from T by removing all the entries larger than k and for notational convenience we have set

$$c_{T^{(k)}, T^{(k-1)}} = c_{\text{shape}(T^{(k)}), \text{shape}(T^{(k-1)})}.$$

Next notice that a telescoping effect based on the fact that $w_{T(1)} = M$ yields the identity

$$\frac{M}{w_\mu} = \prod_{k=2}^n \frac{w_{T^{(k-1)}}}{w_{T^{(k)}}} \quad (\text{with } w_{T^{(k)}} = w_{\text{shape}(T^{(k)})}),$$

which together with 2.13 written in the form

$$M c_{\mu, \nu} = \frac{w_\mu}{w_\nu} d_{\mu, \nu}$$

allows us to carry out the following steps

$$\begin{aligned}\frac{M^n}{w_\mu} \nabla^{-a_1} e_1^{\perp} \nabla^{a_1} \nabla^{-a_2} e_1^{\perp} \nabla^{a_2} \dots \nabla^{-a_n} e_1^{\perp} \nabla^{a_n} \tilde{H}_\mu &= \frac{M}{w_\mu} \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} M c_{T^{(k)}, T^{(k-1)}} \\ &= \frac{M}{w_\mu} \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{w_{T^{(k)}}}{w_{T^{(k-1)}}} d_{T^{(k)}, T^{(k-1)}} \\ &= \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} d_{T^{(k)}, T^{(k-1)}}.\end{aligned} \quad 3.19$$

The next move is to express $d_{T^{(k)}, T^{(k-1)}}$ by means of the formula given in 2.12, rewritten as follows

$$\begin{aligned}
d_{\mu^{(k)}, \nu} &= \frac{1}{\bar{u}_k} \frac{\prod_{i=1}^l (1 - \frac{\bar{x}_i}{\bar{u}_k})}{\prod_{i=0; i \neq k}^l (1 - \frac{\bar{u}_i}{\bar{u}_k})} \\
&= \frac{z}{1-z} \frac{\prod_{i=0}^l (1 - \bar{x}_i z)}{\prod_{i=0}^l (1 - \bar{u}_i z)} (1 - \bar{u}_k z) \Big|_{z=\frac{1}{\bar{u}_k}} \\
(\text{by 1.9}) &= \frac{z}{1-z} \Omega[z(\bar{u}_0 + \dots + \bar{u}_l - \bar{x}_0 - \dots - \bar{x}_l)] (1 - \bar{u}_k z) \Big|_{z=\frac{1}{\bar{u}_k}} \\
(\text{by 3.8}) &= \frac{z}{1-z} \Omega[-zMB_\nu] (1 - \bar{u}_k z) \Big|_{z=\frac{1}{\bar{u}_k}}.
\end{aligned} \tag{3.20}$$

Since $\underline{u}_k = w_T^{(k)}$, multiple substitutions of 3.20 gives

$$\begin{aligned}
\frac{M^n}{w_\mu} \nabla^{-a_1} e_1^\perp \nabla^{a_1} \nabla^{-a_2} e_1^\perp \nabla^{a_2} \dots \nabla^{-a_n} e_1^\perp \nabla^{a_n} \tilde{H}_\mu &= \\
&= \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{z_k}{1-z_k} \Omega[-z_k MB_\nu] (1 - w_T(k) z_k) \Big|_{z_k=\frac{1}{w_T(k)}} \\
&= \sum_{T \in ST(\mu)} \prod_{k=2}^n \frac{1}{z_k^{a_k-1} (1-z_k)} \prod_{1 \leq h < k} \Omega[-Mz_k/z_h] \prod_{k=2}^n (1 - w_T(k) z_k) \Big|_{z_k=1/w_T(k)}
\end{aligned}$$

which used in 3.18 gives 3.16 as desired. This completes our proof.

At this point it is worthwhile also seeing what tableaux expansions can be obtained directly from the $c_{\mu, \nu}$ formula given in 3.4. Recall that in 3.4 we have depicted the diagram of the partition μ together with the $m(=5)$ cells that must be removed from μ to obtain the partitions ν immediately preceding μ in the Young order. We denoted there by $x_1, \dots, x_m$ their respective weights (from left to right). In addition to the removable cells, we have depicted the addable cells of μ that are SW shifted by one unit, whose weights we denote by $u_0, u_1, \dots, u_m$. To the former sequence we have added the cell $(-1, -1)$ whose weight $1/qt$ we denoted by x_0 . This given the $c_{\mu, \nu}$ formula given in 3.4 can be more conveniently rewritten as

$$c_{\mu, \nu^{(i)}} = \frac{qt x_i}{M} \frac{\prod_{j=0}^l (1 - u_j/x_i)}{\prod_{j=1; j \neq i}^l (1 - x_j/x_i)} = \frac{qt}{zM} (1 - z/qt) \frac{\prod_{j=0}^l (1 - u_j z)}{\prod_{j=0}^l (1 - x_j z)} (1 - x_i z) \Big|_{z=\frac{1}{x_i}}, \tag{3.21}$$

where $\nu^{(i)}$ is the partition obtained by removing from μ the cell of weight x_i . Now to carry out the same sequence of steps that yielded the identity in 3.16 we start by using 3.5 and subject the right hand side of 3.21 to the following successive transformations

$$\begin{aligned}
c_{\mu, \nu^{(i)}} &= \frac{qt}{zM} (1 - z/qt) \Omega[z(x_0 + x_1 + \dots + x_l - u_0 - u_1 - \dots - u_l)] (1 - x_i z) \Big|_{z=\frac{1}{x_i}} \\
&= \frac{qt}{zM} (1 - z/qt) \Omega[\frac{z}{qt} MB_\mu] (1 - x_i z) \Big|_{z=\frac{1}{x_i}} \\
(\text{Using } B_\mu &= B_{\nu^{(i)}} + x_i) = \frac{qt}{zM} (1 - z/qt) \Omega[\frac{z}{qt} MB_{\nu^{(i)}} + \frac{z}{qt} Mx_i] (1 - x_i z) \Big|_{z=\frac{1}{x_i}} \\
&= \frac{qt}{zM} (1 - z/qt) \Omega[\frac{z}{qt} MB_{\nu^{(i)}}] \Omega[zx_i - zx_i/t - zx_i/q + zx_i/qt - x_i z] \Big|_{z=\frac{1}{x_i}} \\
&= \frac{qt}{zM} (1 - z/qt) \Omega[\frac{z}{qt} MB_{\nu^{(i)}}] \Omega[-1/t - 1/q + 1/qt] \Big|_{z=\frac{1}{x_i}} \\
&= \frac{qt}{zM} (1 - z/qt) \frac{(1-1/t)(1-1/q)}{1-1/qt} \Omega[\frac{z}{qt} MB_{\nu^{(i)}}] = x_i \frac{1-1/x_i qt}{1-1/qt} \Omega[\frac{M}{qt} B_{\nu^{(i)}}/x_i].
\end{aligned}$$

Using this identity for $i \rightarrow k$ and $x_i \rightarrow w_T(k)$, 3.18 becomes

$$\begin{aligned}
& \nabla^{-a_1} e_1^\perp \nabla^{a_1} \nabla^{-a_2} e_1^\perp \nabla^{a_2} \dots \nabla^{-a_n} e_1^\perp \nabla^{a_n} \tilde{H}_\mu \\
&= \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k+1} \frac{1-1/x_k q t}{1-1/q t} \Omega \left[\frac{M}{q t} \sum_{1 \leq h \leq k-1} w_T(h)/w_T(k) \right] \\
&= \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{1-w_T(k) q t}{1-q t} \Omega \left[\frac{M}{q t} \sum_{1 \leq h \leq k-1} w_T(h)/w_T(k) \right] \\
&= \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{1-w_T(k) q t}{1-q t} \prod_{1 \leq h < k \leq n} \Omega \left[\frac{M}{q t} \frac{w_T(h)}{w_T(k)} \right]
\end{aligned}$$

which in turn, substituted in 3.17, gives

$$\frac{1}{M^n} \Pi(a_1, a_2, \dots, a_n) \mathbf{1} = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu} \sum_{T \in ST(\mu)} \prod_{k=2}^n w_T(k)^{a_k} \frac{1-w_T(k) q t}{1-q t} \prod_{1 \leq h < k \leq n} \Omega \left[\frac{M}{q t} \frac{w_T(h)}{w_T(k)} \right].$$

This proves Theorem I.3 stated in the introduction.

The above results give only a glimpse of the computational power of $d_{\mu, \nu}$ and $c_{\mu, \nu}$. Note that in the original paper [16] Macdonald derived Pieri formulas for multiplication of his polynomials by $e_r[X]$ and $h_r[\frac{1-t}{1-q}X]$, for any $r \geq 1$. Nevertheless, for many years we have had Pieri coefficients corresponding only to multiplication (or skewing) of $\tilde{H}_\mu[X; q, t]$ by e_1 . Actually there is a simple reason for this. In passing from $P_\mu[X; q, t]$ to $\tilde{H}_\mu[X; q, t]$ the Macdonald formulas yield Pieri coefficients corresponding to multiplication (or skewing) by $h_r[\frac{X}{1-t}]$ and $e_r[\frac{X}{1-q}]$ and it is only when $r = 1$ that the Macdonald formulas yield Pieri coefficients corresponding to multiplication by an elementary or the homogeneous symmetric function.

Actually it turns out that multiplication (or skewing) by e_1 is not at all as limited as it may appear on the surface. Indeed, in a recent (2012) paper [5], Bergeron-Haiman, guided by Hilbert Scheme considerations, discovered that multiplication by $e_r[\frac{X}{M}]$ and skewing by $h_r[X]$ for any $r \geq 1$ may be recursively expressed in terms of the $d_{\mu, \nu}$ and $c_{\mu, \nu}$. Since this development is very closely related to our operators $Q_{m, -1}$ and is conducive to a wide variety of standard tableaux expansions, we will terminate this section and the paper with a brief presentation of some applications of the Bergeron-Haiman identities.

For a given $k \geq 1$ let us set

$$a) \quad h_k^\perp \tilde{H}_\mu[X; q, t] = \sum_{\nu \subset_k \mu} c_{\mu, \nu}^{(k)} \tilde{H}_\nu[X; q, t] \quad b) \quad e_k[\frac{X}{M}] \tilde{H}_\nu[X; q, t] = \sum_{\mu \supset_k \nu} d_{\mu, \nu}^{(k)} \tilde{H}_\mu[X; q, t] \quad 3.22$$

where “ $\nu \subset_k \mu$ ” means that ν is contained in μ (as Ferrers diagrams) and μ/ν has k lattice cells. The symbol “ $\mu \supset_k \nu$ ” is analogously defined. It follows from the orthogonality of the basis $\{\tilde{H}_\mu[X; q, t]\}_\mu$ that in this case 3.2 becomes

$$c_{\mu, \nu}^{(k)} = \frac{w_\mu}{w_\nu} d_{\mu, \nu}^{(k)} \quad 3.23$$

This given, the Bergeron-Haiman identities may be stated as follows

Theorem 3.2

For any $k \geq 1$ and $\mu \vdash n$ we have

$$c_{\mu, \nu}^{(k+1)} = \frac{1}{B_{\mu/\nu}} \sum_{\nu \subset_1 \alpha \subset_k \mu} c_{\mu, \alpha}^{(k)} c_{\alpha, \nu}^{(1)} T_\alpha / T_\nu \quad (\text{with } B_{\mu/\nu} = B_\mu - B_\nu) \quad 3.24$$

In spite of the cumbersome denominator, this is a remarkably rapid way of obtaining the $c_{\mu,\nu}^{(k)}$ in terms of our $c_{\mu,\nu}'$ s. In particular, 3.24 yields by far the fastest algorithm for computing the $\tilde{H}_\mu[X; q, t]$ to this date. With any efficient Symbolic Manipulation software, in a few seconds we may obtain the monomial expansion of $\tilde{H}_\mu[X; q, t]$ for $\mu \vdash n$ with n as large as 18. In fact, the coefficients $M_{\lambda,\mu}(q, t)$ in the expansion

$$\tilde{H}_\mu[X; q, t] = \sum_{\lambda \vdash n} m_\lambda[X] M_{\lambda,\mu}(q, t) \quad (\text{for } \mu \vdash n) \quad 3.25$$

can be simply calculated by means of the identities

$$M_{\lambda,\mu}(q, t) = \langle h_\lambda, \tilde{H}_\mu \rangle = h_\lambda^\perp \tilde{H}_\mu. \quad 3.26$$

The most surprising development concerning this 2012 discovery is that it is an immediate consequence of a simple identity obtained in the 1990's! The latter identity is best stated, as expressed in [6], in terms of the generating functions

$$\mathcal{T}(U)P[X] = P[X + U] \quad D(z) = \sum_r z^r D_r = \Omega[-zX] \mathcal{T}\left(\frac{M}{z}\right), \quad 3.27$$

since the commutativity of the two translations $\mathcal{T}(u)$ and $\mathcal{T}(\frac{M}{z})$ immediately yields

$$\mathcal{T}(u)D(z) = \Omega[-z(X + u)] \mathcal{T}\left(\frac{M}{z}\right) \mathcal{T}(u) = (1 - uz)D(z) \mathcal{T}(u)$$

or better

$$[D(z), \mathcal{T}(u)] = uzD(z) \mathcal{T}(u). \quad 3.28$$

This brings us in position to include here a proof of Theorem 3.2. In fact equating the coefficients of $u^{k+1}z^0$ in 3.28 gives (using 1.11 (i) and (iii))

$$[(I - M\Delta_{e_1}), h_{k+1}^\perp] = [D_0, h_{k+1}^\perp] = D_{-1}h_k^\perp = M\nabla^{-1}e_1^\perp \nabla h_k^\perp$$

which may be rewritten as

$$[h_{k+1}^\perp, \Delta_{e_1}] = \nabla^{-1}e_1^\perp \nabla h_k^\perp \quad 3.29$$

and 3.24 immediately follows from the definition in 3.22 a) by applying both sides of 3.29 to $\tilde{H}_\mu$.

It will be instructive to see in a more explicit way how the recursion in 3.24 expresses $c_{\mu,\nu}^{(k)}$ as a polynomial in the $c_{\mu,\nu}'$ s. This is best carried out as a standard tableaux expansion. To begin, given a standard tableau T with n cells let $T^{(k)}$ as before denote the standard tableau obtained from T by removing all the entries greater than k , in particular we may set $T = T^{(n)}$. Also set

$$c_{T^{(k)}, T^{(k-1)}} = c_{\text{shape}(T^{(k)}), \text{shape}(T^{(k-1)})} \quad \text{and} \quad B_{T^{(k)}/T^{(h)}} = B_{\text{shape}(T^{(k)})} - B_{\text{shape}(T^{(h)})}$$

and define for $1 \leq m \leq n$

$$\Pi_m(T; q, t) = \prod_{k=0}^{m-1} \left(\frac{c_{T^{(n-k)}, T^{(n-k-1)}}}{B_{T^{(n)}/T^{(n-k-1)}}} wt(T^{(n-k)}) \right) \quad 3.30$$

where " $wt(T^{(s)})$ " denotes the weight of the cell we must remove from $T^{(s)}$ to obtain $T^{(s-1)}$.

Using this notation we can state the following consequence of the recursion in 3.24

Proposition 3.1

$$h_m^\perp \tilde{H}_\mu[X; q, t] = \sum_{T \in \mathcal{ST}(\mu)} \Pi_m(T; q, t) \tilde{H}_{\text{shape}(T^{(n-m)})}[X; q, t] \quad 3.31$$

From this we can immediately derive the following corollary.

Theorem 3.3

If $\mu \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ then setting $p_i = \lambda_1 + \lambda_2 + \dots + \lambda_{i-1}$, the coefficients $M_{\lambda, \mu}(q, t)$ in 3.25 yielding the monomial expansion of $\tilde{H}_\mu[X; q, t]$ are given by the following standard tableaux sums

$$M_{\lambda, \mu}(q, t) = \sum_{T \in \mathcal{ST}(\mu)} \prod_{i=1}^k \Pi_{\lambda_i}(T^{(n-p_i)}; q, t). \quad 3.32$$

This leads to the following efficient way to obtain the expansion of any symmetric function in terms of the modified Macdonald basis.

For any given $F[X] \in \Lambda^n$, start with the expansion

$$F[X] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X, q, t]}{w_\mu} \langle \tilde{H}_\mu, F \rangle_*,$$

then use 3.25 and obtain

$$F[X] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X, q, t]}{w_\mu} \sum_{\lambda \vdash n} \langle m_\lambda, F \rangle_* M_{\lambda, \mu}(q, t).$$

With precomputed $M_{\lambda, \mu}(q, t)$ according to 3.32 this permits experimenting with various combinatorial questions in Macdonald polynomial theory working with symmetric functions of substantially larger degrees than was possible before the Bergeron-Haiman 2012 discovery.

Bibliography

- [1] F. Bergeron and A. M. Garsia *Science Fiction and Macdonald's Polynomials*. CRM Proceedings & Lecture Notes, American Mathematical Society, 22:, (1999), 1–52.
- [2] F. Bergeron, A. M. Garsia, M. Haiman and G. Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, Methods Appl. Anal. 6, (1999), 363–420.
- [3] F. Bergeron, A. M. Garsia, E. Leven and G. Xin, *Some remarkable new Plethystic Operators in the Theory of Macdonald Polynomials*, arXiv:1405.0316 [math.CO]
- [4] F. Bergeron, A. M. Garsia, E. Leven and G. Xin, *A Compositional (km, kn)-Shuffle Conjecture*, arXiv:1404.4616 [math.CO]
- [5] F. Bergeron, M. Haiman, *Tableaux Formulas for Macdonald Polynomials*, Special edition in honour of Christophe Reutenauer 60 birthday, International Journal of Algebra and Computation, Volume 23, Issue 4, (2013), pp 833–852.
- [6] A. M. Garsia, M. Haiman, and G. Tesler. *Explicit plethystic formulas for Macdonald (q, t)-Kostka coefficients*. Séminaire Lotharingien de Combinatoire [electronic only], 42:B42m, 1999.
- [7] A. M. Garsia and M. Haiman, *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A **90** no. 8, (1993), 3607–3610.

- [8] A. M. Garsia and M. Haiman, *Some Natural Bigraded S_n -Modules and q, t -Kostka Coefficients. (1996) (electronic) The Foata Festschrift*, [http://www.combinatorics.org/Volume 3/volume 3 2.html#R24](http://www.combinatorics.org/Volume%203/volume%203.html#R24)
- [9] A. M. Garsia, J. Haglund and G. Xin, *Constant term methods in the theory of Tesler matrices and Macdonald polynomial operators*, Annals of Combinatorics, DOI: 10.1007/s00026-013-0213-6, (2013).
- [10] A. M. Garsia and G. Tesler, *Plethystic formulas for Macdonald q, t -Kostka coefficients*, Adv. Math. **123** no. 2, (1996), 144–222.
- [11] A. M. Garsia, M. Zabrocki and G. Xin, *Hall-Littlewood Operators in the Theory of Parking Functions and Diagonal Harmonics*, Int Math Res Notices (2011) doi: 10.1093/imrn/rnr060
- [12] E. Gorsky and A. Negut. *Refined knot invariants and Hilbert schemes*. arXiv preprint arXiv:1304.3328, 2013.
- [13] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. *A combinatorial formula for the character of the diagonal coinvariants*. Duke J. Math., 126:, (2005), 195–232.
- [14] J. Haglund, J. Morse, and M. Zabrocki. *A compositional refinement of the shuffle conjecture specifying touch points of the dyck path*. Canadian J. Math, 64:, (2012), 822–844.
- [15] T. Hikita. *Affine springer fibers of type a and combinatorics of diagonal coinvariants*. arXiv preprint arXiv:1203.5878, 2012.
- [16] I. G. Macdonald, *A new class of symmetric functions*, Actes du 20e Séminaire Lotharingien, Publ. I.R.M.A. Strasbourg 372/S20, (1988), 131–171.
- [17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, New York, 1995.
- [18] A. Negut, *The shuffle algebra revisited*, arXiv preprint arXiv:1209.3349, 2012.
- [19] O. Schiffmann and E. Vasserot. *The elliptical Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*. Compos. Math., 147.1:, (2011), 188–234.
- [20] O. Schiffmann and E. Vasserot. *The elliptical Hall algebra and the equivariant K -theory of the Hilbert scheme of A^2* . Duke J. Math., 162.2:, (2013), 279–366.
- [21] O. Schiffmann, *On the Hall algebra of an elliptic curve, II*, Preprint (2005), arXiv:math/0508553.
- [22] R. Stanley, *Some Combinatorial Properties of Jack Polynomials*, Advances in Mathematics **77**, (1989), 76–115.