Qsym over Sym is free

by
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Astract

We study here the ring $\mathcal{QS}_n$ of Quasi-Symmetric Functions in the variables $x_1, x_2, \ldots, x_n$. F. Bergeron and C. Reutenauer [4] formulated a number of conjectures about this ring, in particular they conjectured that it is free over the ring Λ_n of symmetric functions in $x_1, x_2, \ldots, x_n$. We present here an algorithm that recursively constructs a Λ_n-module basis for $\mathcal{QS}_n$ thereby proving one of the Bergeron-Reutenauer conjectures. This result also implies that the quotient of $\mathcal{QS}_n$ by the ideal generated by the elementary symmetric functions has dimension $n!$. Surprisingly, to show the validity of our algorithm we were led to a truly remarkable connection between $\mathcal{QS}_n$ and the harmonics of S_n.

I. Introduction

Quasi-symmetric functions arose in the theory of $\mathcal{P}$-partitions and were first introduced by Gessel (see [9] p. 401). The space $\mathcal{QS}_n$ of quasi-symmetric functions in the alphabet $X_n = \{x_1, x_2, \ldots, x_n\}$ may be defined as the linear span of the polynomials

$$m_{[p_1,p_2,\ldots,p_k]}[X_n] = \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_k}^{p_k} \qquad (\text{for } k \le n \text{ and } p_1, p_2, \ldots, p_k \ge 1) \qquad \text{I.1}$$

It is customary to call a vector $p = (p_1, p_2, \ldots, p_k)$ with positive integer components a "*composition*", the integer k is called the "*length*" of p and is denoted "$l(p)$". We also set

$$|p| \;=\; p_1 + p_2 + \cdots + p_k$$

and call $|p|$ the "*size* " of p. The collection of all compositions is denoted "$\mathcal{C}$". It will also be convenient to denote by "$\mathcal{C}_{=k}$" and "$\mathcal{C}_{\le k}$" the subcollections of $\mathcal{C}$ consisting of compositions of lengths $= k$ and $\le k$ respectively.

For $p \in \mathcal{C}$, we denote by $\lambda(p)$ the partition obtained by rearranging in decreasing order the parts of p. This given it is easy to see that the polynomial

$$M_\lambda[X_n] \;=\; \sum_{\lambda(p)=\lambda} m_p[X_n] \qquad \text{I.2}$$

is none other that the ordinary "*monomial*" symmetric function. For this reason the polynomials defined by I.1 are called "*quasi-monomials*". It may be shown that the product of any two quasi-monomials is a linear combinations of quasi-monomials and thus $\mathcal{QS}_n$ is also a ring. It follows from I.2 that the ring Λ_n of symmetric functions in $x_1, x_2, \ldots, x_n$ is contained in $\mathcal{QS}_n$. F. Bergeron and C. Reutenauer in [4] conjectured that $\mathcal{QS}_n$ is a free module over Λ_n. They also conjectured that the quotient

$$\mathcal{QS}_n/\mathcal{I}_n \qquad \text{I.3}$$

where $\mathcal{I}_n$ is the ideal generated in $\mathcal{QS}_n$ by the homogeneous symmetric functions of positive degree has dimension $n!$. We give here an algorithmic proof of both conjectures. To give a precise statement of our results we need to make some preliminary remarks.

Since every symmetric function is a polynomial in the elementary symmetric functions

$$e_1[X_n], e_2[X_n], \ldots, e_n[X_n]$$

we may write

$$\mathcal{I}_n \;=\; \big(e_1[X_n], e_2[X_n], \ldots, e_n[X_n]\big)_{\mathcal{QS}_n}. \tag{I.4}$$

It follows from this that if $\{\eta_1[X_n], \eta_2[X_n], \ldots, \eta_N[X_n]\} \subseteq \mathcal{QS}_n$ is any basis of $\mathcal{QS}_n/\mathcal{I}_n$ then any quasi-symmetric function $P \in \mathcal{QS}_n$ has an expansion of the form

$$P \;=\; \sum_{i=1}^{N} \eta_i[X_n] A_i(e_1[X_n], e_2[X_n], \ldots, e_n[X_n]) \tag{I.5}$$

with the $A_i's$ polynomials in their arguments. The Bergeron-Reutenauer conjecture asserts that these expansions are unique. In fact uniqueness for the expansions in I.5 for a single collection $\{\eta_1[X_n], \eta_2[X_n], \ldots, \eta_N[X_n]\}$ yields the Cohen-Macauliness of $\mathcal{QS}_n$ as well as the asserted $n!$ dimension for the quotient $\mathcal{QS}_n/\mathcal{I}_n$.

Our approach is to derive the existence of such bases for $\mathcal{QS}_n$ from the analogous result for the ordinary polynomial ring $\mathbb{Q}[X_n]$. To make this precise we need to recall some basic facts. To begin let us say that an integral vector $p = (p_1, p_2, \ldots, p_k)$ (with $p_i \geq 0$) is "*n-subtriangular*" if

$$p_1 \leq n-1\ ,\ p_2 \leq n-2\ ,\ \cdots, p_k \leq n-k\ ,$$

Here and after we will denote by $\mathcal{SUB}_n$ the collection of all n-subtriangular vectors of length n. A monomial $x^p = x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}$ whose exponent vector $(p_1, p_2, \ldots, p_k)$ is "*n-subtriangular*" will be called "*n-subtriangular*" as well. Now it is well known and proved by E. Artin in [2] that the collection of n-subtriangular monomials $\mathcal{AB}_n = \{x^\epsilon\}_{\epsilon \in \mathcal{SUB}_n}$ is a basis for the quotient ring

$$\mathbf{R}_n \;=\; \mathbb{Q}[X_n]/\mathcal{I}_n\,.$$

Since we shall make extensive use of this result in the sequel it will be convenient to briefly call the elements of $\mathcal{AB}_n$ "*Artin monomials*".

We will study here alternate bases for the quotient ring $\mathbf{R}_n$ which are of the form

$$\Big\{m_p[X_n]\Big\}_{p \in \mathcal{S}_{n-1}} \bigcup \Big\{x^\epsilon\Big\}_{\epsilon \in Z_n} \tag{I.6}$$

with $\mathcal{S}_{n-1}$ and Z_n appropriately chosen subcollections of $\mathcal{C}_{\leq n-1}$ and $\mathcal{SUB}_n$ respectively. To be precise we can obtain such bases by applying the Gauss elimination process modulo $\mathcal{I}_n$ to the collection

$$\Big\{m_p[X_n]\Big\}_{p\,\in\,\mathcal{C}_{\leq n-1}} \tag{I.7}$$

followed by the Artin monomials

$$\Big\{x^{\epsilon}\Big\}_{\epsilon\,\in\,\mathcal{SUB}_n}$$

This done we let $\mathcal{S}_{n-1}$ be simply the set of compositions indexing the elements of I.7 that survive the process and likewise we let Z_n be the set of exponents of the Artin monomials that remain at the end of the process. It immediately follows from the fact that I.7 is a basis for $\mathbf{R}_n$ that the collection

$$\Big\{m_p[X_n]\,e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{p\in\mathcal{S}_{n-1}\\ q_i\geq 0}}\bigcup\Big\{x^{\epsilon}\,e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{\epsilon\in Z_n\\ q_i\geq 0}} \tag{I.8}$$

is a basis for $\mathbb{Q}[X_n]$.

With this notation in place our main result can be stated as follows.

Theorem A

For $n\geq 2$, whenever $\mathcal{S}_{n-1}$ and Z_n are obtained by means of the algorithm described above the resulting subcollection of I.8 given by

$$\Big\{m_p[X_n]\,e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{p\in\mathcal{S}_{n-1}\\ q_i\geq 0}}\bigcup\Big\{x^{\epsilon}\,e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n+1}[X_n]\Big\}_{\substack{\epsilon\in Z_n\\ q_i\geq 0}} \tag{I.9}$$

is basis for $\mathcal{QS}_n$. In particular this implies that $\mathcal{QS}_n$ is a free module over the ring Λ_n of symmetric functions.

Note that since I.9 is contained in I.8, its independence is immediate. Thus to show that I.9 is a basis for $\mathcal{QS}_n$ we need only show that it spans $\mathcal{QS}_n$. This will obtained by a counting argument based on some truly remarkable identities. To appreciate their significance we need some preliminary remarks. To begin it is easily shown that the Hilbert series of $\mathcal{QS}_n$ is given by the rational function

$$F_{\mathcal{QS}_n}(q) \;=\; 1 \;+\; \frac{q}{1-q} \;+\; \frac{q^2}{(1-q)^2} \;+\; \cdots \;+\; \frac{q^n}{(1-q)^n}\,.$$

Now we can show that it may also be written in the form

$$F_{\mathcal{QS}_n}(q) \;=\; \frac{P_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{I.10}$$

with $P_n(q)$ a polynomial with positive integer coefficients. This discovery, due to Bergeron and Reutenauer, already by itself should suggest to the wise the Cohen-Macauliness of $\mathcal{QS}_n$. But we can proceed a bit further and derive from I.10 that we must have

$$\frac{P_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)} \;=\; \frac{P_{n-1}(q)}{(1-q)(1-q^2)\cdots(1-q^{n-1})} \;+\; \frac{q^n}{(1-q)^n}\,. \tag{I.11}$$

Let "$[n]_q!$" denote the customary q-analogue of $n!$, that is

$$[n]_q! \;=\; (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})\,.$$

Putting everything to a common denominator we can rewrite I.11 in the form

$$F_{\mathcal{QS}_n}(q) \;=\; \frac{P_{n-1}(q) \;+\; q^n\left([n]_q! - P_{n-1}(q)\right)}{(1-q)(1-q^2)\cdots(1-q^n)}\,. \tag{I.12}$$

In particular from I.10 we derive that the polynomial $P_n(q)$ must satisfy the recursion

$$P_n(q) = P_{n-1}(q) \;+\; q^n\left([n]_q! - P_{n-1}(q)\right). \tag{I.13}$$

This given, the following two beautiful identities are the culminating point of our efforts.

Theorem I.1

If $\{m_p[X_n]\}_{p\in\mathcal{S}_{n-1}}$ are the quasimonomials that survive the Gauss elimination process applied to the collection $\{m_p[X_n]\}_{p\in\mathcal{C}_{\leq n-1}}$, then

$$\sum_{p\in\mathcal{S}_{n-1}} q^{|p|} \;=\; P_{n-1}(q) \tag{I.14}$$

Equivalently, if $\{x^\epsilon\}_{\epsilon\in Z_n}$ are the Artin monomials that survive the Gauss elimination process described above then

$$\sum_{\epsilon\in Z_n} q^{|p|} \;=\; [n]_q! - P_{n-1}(q) \tag{I.15}$$

In view of I.12 the assertion that the collection in I.9 is a basis for $\mathcal{QS}_n$ is an immediate consequence of I.14 and I.15. Now it develops that these identities follow from a remarkable property of the Harmonics of S_n. To state it we need some definitions. To begin let $\sigma_i(n)$ denote the cycle $(i, i+1, \ldots, n)$ and set

$$\tau_n = \sum_{i=1}^{n} \sigma_i(n)\,, \qquad\qquad \tau'_n = \sum_{i=1}^{n} (-1)^{n-i}\,\sigma_i(n) \tag{I.16}$$

It is well known that the space $\mathbf{H}_n$ of Harmonics of S_n is the linear span of the derivatives of the Vandermonde determinant $\Delta[X_n] = det\|x_i^{n-j}\|_{i,j=1}^n$. In symbols

$$\mathbf{H}_n = \mathcal{L}\big[\partial_x^p \Delta[X_n]\big]\,.$$

Now it is easily seen that τ'_n preserves harmonicity. Taking all this into account the identities in I.14 and I.15 are consequences of the following basic fact

Theorem I.2

The kernel of τ'_n as a map of $\mathbf{H}_{n-1}$ into $\mathbf{H}_n$ has dimension $(n-2)!$.

The connection between Harmonics and Quasy-Symmetric functions stems from the following surprising result.

Theorem I.3

A polynomial $h[X_{n-1}] \in \mathbf{H}_{n-1}$ is in the kernel of τ'_n if and only if it is of the form

$$h[X_{n-1}] \;=\; Q(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}})\Delta[X_{n-1}]$$

with

$$Q(x_1, x_2, \ldots, x_{n-1}) \in \mathcal{QS}[X_{n-1}] \,.$$

We shall see that Theorem I.3 implies Theorem I.2. Indeed it follows from Theorem I.3 that the Hilbert series of the kernel of τ'_n on $\mathbf{H}_{n-1}$ is given by the polynomial

$$P_{n-2}(1/q)\, q^{\binom{n-1}{2}} \,.$$

We have essentially two different proofs of our freeness result, a leasurely one which derives the identities in I.14 and I.15 from Theorem I.3 and a more economical one which derives that the collection in I.9 is a basis directly by proving the dimension result of Theorem I.2.

This paper is divided into six sections. In the first section we prove some useful properties of the ring of Quasi-symmetric functions, and for sake of completeness review some basic facts about Hilbert series and expansions in the polynomial ring $\mathbb{Q}[X_n]$ we shall also give there a combinatorial proof of the positivity of $P_n(q)$. In the second section we give a precise description of our algorithm and establish some criteria that imply its validity. In the third section we develop the connection between Quasi-symmetric functions and Harmonics and show that Theorem I.3 implies Theorem I.1. In the fourth section we give a proof of Theorem I.3 and explore some further consequences of our arguments. In the fifth section we show from general principles that our algorithmic approach is not only sufficient but also necessary and obtain our second proof of the freeness of $\mathcal{QS}_n$. The sixth and last section contains a number of remarks and identities which are by-products of our efforts and we regard to be interesting in themselves and conducive to further investigations.

Acknowledgement

The first author wishes to express his gratitude to Francois Bergeron and Christophe Reutenauer for sharing the contents of their unpublished work, for suggesting the problem and encouragement throughout our efforts. In particular their conjectured basis turned out to be an invaluable tool in many of our computer explorations.

1. Hilbert series and properties of $\mathcal{QS}_n$.

If $\mathbf{V}$ is a graded vector space, we have the direct sum decomposition

$$\mathbf{V} \;=\; \mathcal{H}_o(\mathbf{V}) \oplus \mathcal{H}_1(\mathbf{V}) \oplus \cdots \oplus \mathcal{H}_k(\mathbf{V}) \oplus \cdots$$

where $\mathcal{H}_d(\mathbf{V})$ denotes the subspace of $\mathbf{V}$ spanned by the homogeneous elements of degree d. We recall that the "*Hilbert series*" of $\mathbf{V}$, denoted here by "$F_{\mathbf{V}}(q)$", is the generating functions of dimensions

$$F_{\mathbf{V}}(q) \;=\; \sum_{d\geq 0} dim(\mathcal{H}_d(\mathbf{V}))\, q^d\,. \tag{1.1}$$

Now we have the following basic criterion.

Proposition 1.1

Let $\{\eta\}_{\eta\in\mathcal{B}}$ be a collection of homogeneous elements of $\mathbf{V}$, then

a) If $\{\eta\}_{\eta\in\mathcal{B}}$ is a basis we necessarily have

$$\sum_{\eta\in\mathcal{B}} q^{degree(\eta)} \;=\; F_{\mathbf{V}}(q)\,. \tag{1.2}$$

b) Conversely if 1.2 holds true then the following three properties are equivalent

(i) $\{\eta\}_{\eta\in\mathcal{B}}$ is a basis of $\mathbf{V}$,

(ii) $\{\eta\}_{\eta\in\mathcal{B}}$ spans $\mathbf{V}$,

(iii) $\{\eta\}_{\eta\in\mathcal{B}}$ is an independent set.

Proof

Equating coefficients of q^d in 1.2 yields

$$\#\big\{\eta\in\mathcal{B}\,:\,degree(\eta)=d\big\} = dim(\mathcal{H}_d(V)) \tag{1.3}$$

and this is clearly necessary for $\{\eta\}_{\eta\in\mathcal{B}}$ to be a basis. This proves *a)* . On the other hand if 1.2 his true then 1.3 holds for all d and to show that the subcollection $\{\eta\}_{\eta\in\mathcal{B}\,,degree(\eta)=d}$ is a basis of $\mathcal{H}_d(\mathbf{V})$ we need only show that its spans $\mathcal{H}_d(\mathbf{V})$ or that it is an independent set. This proves *b)*.

The space of quasi-symmetric functions $\mathcal{QS}_n$ is graded by the customary degree of a polynomial and its Hilbert series is given by the rational function

$$F_{\mathcal{QS}_n}(q) \;=\; 1 \;+\; \frac{q}{1-q} \;+\; \frac{q^2}{(1-q)^2} \;+\; \cdots \;+\; \frac{q^n}{(1-q)^n} \tag{1.4}$$

In fact, if we denote by $\mathcal{QS}_{=k}[X_n]$ the linear span of the quasimonomials $\{m_p[X_n]\}_{p\in\mathcal{C}_{=k}}$, we easily see that we have

$$F_{\mathcal{QS}_{=k}[X_n]}(q) \;=\; \sum_{\substack{p\in\mathcal{C}\\ l(p)=k}} q^{|p|} \;=\; \sum_{p_1\geq 1}\sum_{p_2\geq 1}\cdots\sum_{p_k\geq 1} q^{p_1+p_2+\cdots+p_k} \;=\; \frac{q^k}{(1-q)^k}\,, \tag{1.5}$$

and then 1.4 follows from the direct sum decomposition

$$\mathcal{QS}_n \;=\; \mathcal{QS}_{=0}[X_n] \oplus \mathcal{QS}_{=1}[X_n] \oplus \mathcal{QS}_{=2}[X_n] \oplus \cdots \oplus \mathcal{QS}_{=n}[X_n]\,. \tag{1.6}$$

Setting as we did in the introduction for each $n \geq 1$

$$P_n(q) \;=\; (1-q)(1-q^2)\cdots(1-q^n)\, F_{\mathcal{QS}_n}(q) \tag{1.7}$$

we have seen (in I.13) that these polynomials satisfy the recursion

$$P_n(q) \;=\; P_{n-1}(q) \;+\; q^n\left([n]_q! - P_{n-1}(q)\right). \tag{1.8}$$

Starting from the initial conditions

$$P_0(q) = P_1(q) = 1 \tag{1.9}$$

we obtain the following three terms of the sequence $\{P_n(q)\}_{n\geq 1}$:

$$P_2(q) \;=\; 1 + q^3\,,$$

$$P_3(q) \;=\; 1 + q^3 + 2q^4 + 2q^5\,,$$

$$P_4(q) \;=\; 1 + q^3 + 2q^4 + 5q^5 + 5q^6 + 5q^7 + 3q^8 + q^9 + q^{10}\,.$$

Note that setting $q = 1$ in 1.8 we derive that

$$P_n(1) \;=\; n! \tag{1.10}$$

thus $P_n(q)$ may be viewed an another q-analogue of $n!$. This given, it should be of interest to see a combinatorial interpretation of its positivity. There is in fact, a combinatorial interpretation of the positivity of the difference $[n]_q! - P_{n-1}(q)$. It will be good to give it here since it leads in a natural way to the recursive construction of our bases. To begin, let $Art[X_n]$ denote the sum of all Artin monomials in $x_1, x_2, \ldots, x_n$. That is

$$Art[X_n] \;=\; \prod_{i=1}^{n-1}\left(1 + x_i + x_i^2 + \cdots + x_i^{n-i}\right) \;=\; \sum_{\epsilon\,\in\mathcal{SUB}_n} x^{\epsilon}\,. \tag{1.12}$$

With this notation, we let $\{\Pi[X_n]\}_{n\geq 1}$ be the sequence of polynomials defined by setting

$$\begin{aligned} &1)\quad \Pi[X_1] = 1\,,\\ &2)\quad \Pi[X_n] = \Pi[X_{n-1}] \;+\; x_1x_2\cdots x_n\big(Art[X_n] \;-\; \Pi[X_{n-1}]\big) \end{aligned} \tag{1.13}$$

We can now easily see that we must have

Proposition 1.2

a) Each $\Pi[X_n]$ is a sum of $n!$ distinct $(n+1)$-subtriangular monomials,

b) The compositions appearing as exponents of monomials in $\Pi[X_n]$ have length $\leq n$.

c) In particular the polynomial $P_n(q)$ must have positive integer coefficients.

Proof

Since multiplication by $x_1x_2\cdots x_n$ sends an n-subtriangular monomial into an $(n+1)$-subtriangular monomial, then if we recursively assume *a)* and *b)* to be true for $n-1$, it immediately follows from 1.13 2) that a) and b) will also be true for n. To prove *c)* note that if we set all the variables equal to q in $\Pi[X_n]$ we obtain the sequence of polynomials $\{P_n(q)\}_{n\geq 1}$ satisfying the recursion in 1.8 with the initial condition in 1.9 This shows that $P_n(q)$ q-counts the monomials in $\Pi[X_n]$ by degree.

To simplify our language, a set $\mathcal{A}_n$ of homogeneous polynomials with the property that the collection

$$\mathcal{B}_n = \left\{ b[X_n]e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n] \right\}_{b\in\mathcal{A}_n} \tag{1.14}$$

is a basis for the polynomial ring $\mathbb{Q}[X_n]$, will be simply called a "Λ_n-*basis for* $\mathbb{Q}[X_n]$". Likewise, a set of quasi-monomials $\{m_p[X_n]\}_{p\in\mathcal{S}_n}$ (with $\mathcal{S}_n \in \mathcal{C}_{\leq n}$) such that

$$\mathcal{B}_n = \left\{ m_p[X_n]e_1^{q_1}[X_n]e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n] \right\}_{\substack{p\in\mathcal{S}_n \\ q_i\geq 0}} \tag{1.15}$$

is a basis for $\mathcal{QS}_n$ will be briefly called called "Λ_n-*basis of* $\mathcal{QS}_n$" .

It will be good to keep in mind the following basic criteria

Proposition 1.3

(a) A set $\mathcal{A}_n$ of homogeneous polynomials is an Λ_n-basis for $\mathbb{Q}[X_n]$ if and only if

(i) The collection in 1.14 spans $\mathbb{Q}[X_n]$ and

(ii) $\sum_{b\in\mathcal{A}_n} q^{degree(b)} = [n]_q!$

likewise

(b) A set of quasi-monomials $\{m_p[X_n]\}_{p\in\mathcal{S}_n}$ is an Λ_n-basis for $\mathcal{QS}_n$ if and only if

(i) The collection in 1.15 spans $\mathcal{QS}_n$ and

(ii) $\sum_{p\in\mathcal{S}_n} q^{|p|} = P_n(q)$

Proof

In the first case we have

$$\sum_{P\in\mathcal{B}_n} q^{degree(P)} \;=\; \frac{[n]_q!}{(1-q)(1-q^2)\cdots(1-q^n)} \;=\; F_{\mathbb{Q}[X_n]}(q)$$

and in the second case from 1.7 we get

$$\sum_{P\in\mathcal{B}_n} q^{degree(P)} \;=\; \frac{P_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)} \;=\; F_{\mathcal{QS}_n}(q)$$

Thus in both cases the result follows from Proposition 1.2.

As we mentioned in the introduction the following result is well known.

Theorem 1.1

The set of monomials $\{x^\epsilon\}_{\epsilon\in\mathcal{SUB}_n}$ *is a basis for the quotient*

$$\mathbf{R}_n \;=\; \mathbb{Q}[X_n]/\big(e_1[X_n], e_2[X_n], \ldots, e_n[X_n]\big)\,.$$

in particular it follows that $\{x^\epsilon\}_{\epsilon\in\mathcal{SUB}_n}$ *is a* Λ_n*-basis for* $\mathbb{Q}[X_n]$.

A proof of the first assertion may be found in [2]. For our purposes we need only recall how to expand, modulo $(e_1, e_2, \ldots e_n)$, every polynomial in $x_1, x_2, \ldots, x_n$ as a linear combination of n-subtriangular monomials. To this end note that by equating the coefficients of the t^{n-i+1} in the identity

$$(1-x_{i+1}t)\cdots(1-x_nt) \equiv \frac{1}{(1-x_1t)\cdots(1-x_i\,t)} \qquad \big(mod(e_1, e_2, \ldots, e_n)\big) \tag{1.17}$$

we derive that the homogeneous symmetric functions

$$\big\{h_{n-i+1}[x_1, x_2, \ldots, x_i]\big\}_{i=1}^{n}\,,$$

lie all in $(e_1, e_2, \ldots, e_n)$. In fact, they are the Gröbner basis of this ideal. In particular it follows that for $1 \le i \le n$ we have

$$x_i^{n-i+1} \equiv -\sum_{r=1}^{n-i+1} x_i^{n-i+1-r} h_r[x_1 + x_2 + \cdots + x_{i-1}] \quad (mod\ (e_1, e_2, \ldots e_n))\,. \tag{1.18}$$

We can clearly see how this identity may be used to recursively reduce the exponent of x_i to a value $\le n-i$ at the expense of increasing the exponents of $x_1, x_2, \ldots, x_{i-1}$. This algorithm will eventually transfer all the extra powers on x_1 where it will necessarily terminate because for $i = 1$, 1.18 reduces to

$$x_1^n \equiv 0 \quad (mod\ (e_1, e_2, \ldots e_n))\,.$$

Of course this proves the second assertion since it implies that the collection

$$\mathcal{B}_n = \Big\{x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_{n-1}^{\epsilon_{n-1}} e_1^{p_1}[X_n]e_2^{p_2}[X_n]\cdots e_n^{p_n}[X_n] \;:\; 0 \le \epsilon_i \le n-i \,\&\, p_i \ge 0\Big\}$$

spans the polynomial ring $\mathbb{Q}[X_n]$ and then an application of Proposition 1.3 yields uniqueness.

To formalize an argument that will be used several times in the sequel we need to introduce some notation. To begin, it will be convenient to denote by $\mathcal{QS}_{\le k}[X_n]$ the subspace of $\mathcal{QS}[X_n]$ spanned by quasimonomials indexed by compositions of length $\le k$. In symbols

$$\mathcal{QS}_{\le k}[X_n] \;=\; \mathcal{L}\big[m_p[X_n] \,:\, l(p) \le k\big]\,.$$

Likewise we set

$$\mathcal{QS}_{=k}[X_n] \;=\; \mathcal{L}\big[m_p[X_n] \,:\, l(p) = k\big]$$

We shall make frequent use of the following basic property of quasi-symmetric functions:

Proposition 1.4

For $P(x_1, x_2, \ldots, x_n) \in \mathcal{QS}_n$ we have

$$P(x_1, x_2, \ldots, x_{n-1}, 0) = 0 \tag{1.19}$$

if and only if

$$P \in \mathcal{QS}_{=n}[X_n] \;\; = \;\; e_n[X_n]\,\mathbb{Q}[X_n]\,. \tag{1.20}$$

In particular P must be of the form

$$P \;\; = \;\; A[X_n]\,e_n[X_n] \tag{1.21}$$

with $A[X_n] \in \mathbb{Q}[X_n]$.

Proof

We have already observed that when $l(p) = n$ we have

$$m_p[X_n] = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$$

Thus 1.20 implies 1.19. To show the converse note that if P is of degree d the expansion of P in terms of quasimonomials may be written in the form

$$P(x_1, x_2, \ldots, x_n) \;\; = \;\; P(0) + \sum_{k=1}^{n} \sum_{\substack{l(p)=k \\ |p| \le d}} c_p\, m_p(x_1, x_2, \ldots, x_n)$$

This gives

$$P(x_1, x_2, \ldots, x_{n-1}, 0) \;\; = \;\; P(0) + \sum_{k=1}^{n-1} \sum_{\substack{l(p)=k \\ |p| \le d}} c_p\, m_p(x_1, x_2, \ldots, x_{n-1}, 0)$$

Since, the quasimonomials $\{m_p[X_{n-1}]\}_{l(p)\le n-1}$ are independent, the condition in 1.19 forces the vanishing of all the coefficients in this expansion. In other words 1.19 forces P to be of the form

$$P(x_1, x_2, \ldots, x_n) \;\; = \;\; \sum_{\substack{l(p)=n \\ |p| \le d}} c_p\, m_p(x_1, x_2, \ldots, x_n)\,,$$

but this is 1.20.

The step of replacing X_{n-1} by X_n is best viewed as the action of a linear operator

$$\mathcal{E}_n \,:\, \mathcal{QS}[X_{n-1}] \to \mathcal{QS}[X_n]$$

which we call "*extension*" and is defined by setting for every composition p of length $\le n-1$

$$\mathcal{E}_n\, m_p[X_{n-1}] \;\; = \;\; m_p[X_n]\,.$$

The operator $\mathcal{E}_n$ has a number of properties that are worth recording.

Proposition 1.5

(i) *$\mathcal{E}_n$ has a left inverse which simply consists of setting $x_n = 0$. In symbols*

$$\mathcal{E}_n\, P[X_{n-1}]\Big|_{x_n\to 0} \;=\; P[X_{n-1}] \qquad (\,\forall\; P \in \mathcal{QS}[X_{n-1}]\,).$$

(ii) For any two compositions p, q we have

$$\mathcal{E}_n\,(m_p[X_{n-1}]\, m_q[X_{n-1}]) \;-\; (\mathcal{E}_n\, m_p[X_{n-1}])(\mathcal{E}_n\, m_q[X_{n-1}]) \in \mathcal{QS}_{=n}[X_n] \tag{1.22}$$

(iii) The operator "$\mathcal{E}_n e_{n-1}$" which simply consists of multiplication by $e_{n-1}[X_{n-1}]$ followed by $\mathcal{E}_n$ is well defined as a map of

$$\mathbb{Q}[X_{n-1}]/(e_1, e_2, \ldots, e_{n-1})_{Q[X_{n-1}]} \quad \textit{into} \quad \mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)_{Q[X_n]}$$

(iv) For every composition $p \in \mathcal{C}_{\leq n-1}$ we have

$$\mathcal{E}_n e_{n-1} m_p[X_{n-1}] \equiv 0 \qquad (\ \textit{mod}\ (e_1, e_2, \ldots, e_n))$$

Proof

Note that for $p = (p_1, p_2, \ldots, p_k)$ we have the addition formula

$$m_p[X_n] \;=\; m_p[X_{n-1}] + m_{(p_1,\ldots,p_{k-1})}[X_{n-1}]\, x_n^{p_k}\,. \tag{1.23}$$

Thus

$$m_p[X_n]\Big|_{x_n\to 0} = \; m_p[X_{n-1}] + m_{(p_1,\ldots,p_{k-1})}[X_{n-1}]\, x_n^{p_k}\Big|_{x_n\to 0} = \; m_p[X_{n-1}]$$

This proves *(i)*. Note next that if

$$m_p[X_{n-1}]\, m_q[X_{n-1}] \;=\; \sum_r c_{p,q}^r\, m_r[X_{n-1}]$$

then by definition

$$\mathcal{E}_n\,(m_p[X_{n-1}]\, m_q[X_{n-1}]) \;=\; \sum_r c_{p,q}^r\, m_r[X_n]$$

and consequently (by (i)):

$$\mathcal{E}_n\,(m_p[X_{n-1}]\, m_q[X_{n-1}])\Big|_{x_n\to 0} = \sum_r c_{p,q}^r\, m_r[X_{n-1}] = m_p[X_{n-1}]\, m_q[X_{n-1}]\,. \tag{1.24}$$

Likewise

$$\mathcal{E}_n\, m_p[X_{n-1}])(\mathcal{E}_n\, m_q[X_{n-1}])\Big|_{x_n\to 0} = m_p[X_n]\, m_q[X_n]\Big|_{x_n\to 0} = m_p[X_{n-1}]\, m_q[X_{n-1}]\,.$$

Subtracting this from 1.24 and using Proposition 1.4 gives 1.22.

To show *(iii)* let $P[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ and suppose that for $A_i[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ we have

$$P[X_{n-1}] \;=\; \sum_{i=1}^{n-1} A_i[X_{n-1}]e_i[X_{n-1}]\,.$$

Then

$$e_{n-1}[X_{n-1}]P[X_{n-1}] \;=\; \sum_{i=1}^{n-1} e_{n-1}[X_{n-1}]A_i[X_{n-1}]e_i[X_{n-1}]\,.$$

Since $e_{n-1}[X_{n-1}]A_i[X_{n-1}] \in \mathcal{QS}_{n-1}$, we derive from 1.22 that for some $A[X_n] \in \mathbb{Q}[X_n]$ we have

$$\mathcal{E}_n\big(e_{n-1}[X_{n-1}]P[X_{n-1}]\big) \;=\; \sum_{i=1}^{n-1} \mathcal{E}_n\big(e_{n-1}[X_{n-1}]A_i[X_{n-1}]\big)e_i[X_n] \;+\; A[X_n]e_n[X_n]\,.$$

This proves

$$\mathcal{E}_n\big(e_{n-1}[X_{n-1}]P[X_{n-1}]\big) \in (e_1[X_n], e_2[X_n], \ldots, e_n[X_n])\,.$$

Finally note that from *(ii)* it follows that

$$\mathcal{E}_n e_{n-1} m_p[X_{n-1}] \;=\; e_{n-1}[X_n]\, m_p[X_n] \;+\; A[X_n]\, e_n[X_n]$$

for some $A[X_n] \in \mathbb{Q}[X_n]$. This proves *(iv)* and completes the proof.

2. The Algorithm

The validity of our algorithm stems from a sequence of propositions that are of independent interest. The following two basic properties play a crucial role in our developments.

They may be defined as follows

Property $\mathbf{G_n}$

In the quotient ring $\mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)_{Q[X_n]}$, the subspace spanned by the quasi-monomials indexed by compositions of length $\leq n-1$ has Hilbert series $P_{n-1}(q)$.

Property $\mathbf{K_n}$

For $n > 2$ the kernel of $\mathcal{E}_n e_{n-1}$ as a map of

$$\mathbb{Q}[X_{n-1}]/(e_1, e_2, \ldots, e_{n-1})_{Q[X_{n-1}]} \quad \textit{into} \quad \mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)_{Q[X_n]}$$

is the projection of $\mathcal{QS}_{n-1}$ into $\mathbb{Q}[X_{n-1}]/(e_1, e_2, \ldots, e_{n-1})_{Q[X_{n-1}]}$.

Our ultimate goal may be stated as

Property $\mathbf{B_n}$

The space $\mathcal{QS}_n$ has a Λ_n-basis $\{m_p[X_n]\}_{p\in\mathcal{S}_n}$ for a suitable collection $\mathcal{S}_n \in \mathcal{C}_{\leq n}$ with

$$\sum_{p\in\mathcal{S}_n} q^{|p|} \;=\; P_n(q)\,. \qquad 2.2$$

These properties interlace in a remarkable way to yield Theorem A.

Theorem 2.1

$$\mathbf{G}_n \implies \mathbf{B}_n \tag{2.2}$$

Proof

Let $\mathcal{S}_{n-1} \subseteq \mathcal{C}_{\leq n-1}$ be such that $\{m_p[X_n]\}_{p\in\mathcal{S}n-1}$ is a basis for the linear span

$$\mathcal{L}\big[m_p[X_n] : p \in \mathcal{C}_{\leq n-1}\big]$$

in the quotient ring $\mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}$. Note that under $\mathbf{G}_n$. we must have

$$\sum_{p\in\mathcal{S}_{n-1}} q^{|p|} \;=\; P_{n-1}(q)\,. \tag{2.3}$$

Now select $Z_n \subseteq \mathcal{SUB}_n$ so that the colletion

$$\{m_p[X_n]\}_{p\in\mathcal{S}_{n-1}} \;\cup\; \{x^\epsilon\}_{\epsilon\in Z_n} \tag{2.4}$$

is a basis for the quotient $Q[X_n]/(e_1, e_2, \ldots e_n)_{Q[X_n]}$. We may find such a collection by the Gauss elimination process, modulo $(e_1, e_2, \ldots e_n)_{Q[X_n]}$, applied to

$$\{m_p[X_n]\}_{p\in\mathcal{S}_{n-1}} \quad \text{followed by} \quad \{x^\epsilon\}_{\epsilon\in\mathcal{SUB}_n}\,.$$

Then $\{x^\epsilon\}_{\epsilon\in Z_n}$ simply consists of the monomials x^ϵ that survive the process. Since the Hilbert series of $Q[X_n]/(e_1, e_2, \ldots e_n)_{Q[X_n]}$ is $[n]_q!$, from 2.3 we derive that we must have

$$\sum_{\epsilon\in Z_n} q^{|\epsilon|} \;=\; [n]_q! - P_{n-1}(q)\,. \tag{2.5}$$

It also follows that the collection

$$\mathcal{B}_n \;=\; \Big\{m_p[X_n]\, e_1^{q_1}[X_n] e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{p\in\mathcal{S}_{n-1}\\ q_i\geq 0}} \bigcup \Big\{x^\epsilon\, e_1^{q_1}[X_n] e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{\epsilon\in Z_n\\ q_i\geq 0}} \tag{2.6}$$

is a basis for $Q[X_n]$.

We claim that the collection

$$\mathcal{QB}_n \;=\; \Big\{m_p[X_n]\, e_1^{q_1}[X_n] e_2^{q_2}[X_n]\cdots e_n^{q_n}[X_n]\Big\}_{\substack{p\in\mathcal{S}_{n-1}\\ q_i\geq 0}} \bigcup \Big\{x^\epsilon\, e_1^{q_1}[X_n] e_2^{q_2}[X_n]\cdots e_n^{q_n+1}[X_n]\Big\}_{\substack{\epsilon\in Z_n\\ q_i\geq 0}} \tag{2.7}$$

is a basis for $\mathcal{QS}_n$. We first observe that the containment $\mathcal{QB}_n \subseteq \mathcal{B}_n$ implies that $\mathcal{QB}_n$ is independent. Thus to show that it is a basis, by case b) *(iii)* of Proposition 1.1, we need only verify that it has the correct distribution of degrees. To this end note that 2.7 gives

$$\begin{aligned}
\sum_{b\in\mathcal{QB}_n} q^{|p|} \;&=\; \frac{\sum_{p\in\mathcal{S}_{n-1}} q^{|p|} \;+\; q^n\,\big(\,\sum_{\epsilon\in Z_n} q^{|\epsilon|}\,\big)}{(1-q)(1-q^2)\cdots(1-q^n)}\\
\text{(by 2.3 and 2.5)} \;&=\; \frac{P_{n-1}(q) \;+\; q^n\,\big([n]_q! - P_{n-1}(q)\big)}{(1-q)(1-q^2)\cdots(1-q^n)}\\
\text{(by I.13)} \;&=\; F_{\mathcal{QS}_n}(q)\,.
\end{aligned}$$

We have thus proved Property $\mathbf{B_n}$ with

$$\mathcal{S}_n \;=\; \mathcal{S}_{n-1} \cup \left\{\epsilon + 1^n\right\}_{\epsilon \in Z_n} \tag{2.8}$$

where for $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ for convenience we set

$$\epsilon + 1^n \;=\; (\epsilon_1 + 1, \epsilon_2 + 1, \ldots, \epsilon_n + 1)\,. \tag{2.9}$$

Theorem 2.2

$$\mathbf{G}_{n-1} \;+\; \mathbf{K}_n \quad \Longrightarrow \quad \mathbf{G}_n \tag{2.10}$$

Proof

Let $\left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}}$ and $\left\{x^\epsilon\right\}_{\epsilon\in Z_{n-1}}$ be obtained as in the proof of Theorem 2.1 with n replaced by $n-1$. By construction we have the following properties

$$\text{(a)} \quad \sum_{p\in\mathcal{S}_{n-2}} q^{|p|} = P_{n-2}(q)$$

$$\text{(b)} \quad \sum_{\epsilon\in Z_{n-1}} q^{|\epsilon|} = [n-1]_q! - P_{n-2}(q)\,.$$

$$\text{(c)} \quad \left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}} \cup \left\{x^\epsilon\right\}_{\epsilon\in Z_{n-1}} \quad \text{is a } \Lambda_{n-1}\text{-basis for } \mathbb{Q}[X_{n-1}]$$

$$\text{(d)} \quad \left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}} \cup \left\{e_{n-1}[X_{n-1}]x^\epsilon\right\}_{\epsilon\in Z_{n-1}} \quad \text{is a } \Lambda_{n-1}\text{-basis for } \mathcal{QS}_{n-1}$$

$$\text{(e)} \quad \left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}} \cup \left\{x^\epsilon\right\}_{\epsilon\in Z_{n-1}} \quad \text{is a basis for } \mathbb{Q}[X_{n-1}]/(e_1.e_2, \ldots, e_{n-1})_{\mathbb{Q}[X_{n-1}]}$$

This given, we claim that

(i) $\left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}}$ *is a basis for the kernel of* $\mathcal{E}_n e_{n-1}$ *on* $\mathbb{Q}[X_{n-1}]/(e_1, \ldots, e_{n-1})_{\mathbb{Q}[X_{n-1}]}$.

(ii) $\left\{m_p[X_n]\right\}_{p\in\mathcal{S}_{n-2}} \cup \left\{m_{\epsilon+1^{n-1}}[X_n]\right\}_{\epsilon\in Z_{n-1}}$ *is a basis for the the subspace of* $\mathbb{Q}[X_n]/(e_1, \ldots, e_n)_{\mathbb{Q}[X_n]}$. *spanned by the quasi-monomials indexed by compositions of length* $\leq n-1$

Using (d), we obtain that for any quasi-symmetric polynomial $Q[X_{n-1}] \in \mathcal{QS}_{n-1}$ we have the expansion

$$Q[X_{n-1}] = \sum_{p\in\mathcal{S}_{n-2}} c_p(e_1, e_2, \ldots, e_{n-1})\, m_p[X_{n-1}] + \sum_{\epsilon\in Z_{n-1}} d_\epsilon(e_1, e_2, \ldots, e_{n-1})\, e_{n-1}[X_{n-1}]x^\epsilon$$

with $c_p(y_1, y_2, \ldots, y_{n-1})$ and $d_\epsilon(y_1, y_2, \ldots, y_{n-1})$ polynomials in their arguments. This gives

$$Q[X_{n-1}] \equiv \sum_{p\in\mathcal{S}_{n-2}} c_p(0, 0, \ldots, 0)\, m_p[X_{n-1}] \qquad (\ \textit{mod}\ (e_1, e_2, \ldots, e_{n-1})\ )$$

This proves that $\left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}}$ spans $\mathcal{QS}_{n-1}$ modulo $(e_1, e_2, \ldots, e_{n-1})_{\mathbb{Q}[X_{n-1}]}$. Since (c) gives that $\left\{m_p[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}}$ is independent, claim *(i)* then follows from property $\mathbf{K}_n$.

To show claim (ii) we need to prove that the collection

$$\mathcal{A}_n \;=\; \{m_p[X_n]\}_{p\in\mathcal{S}_{n-2}} \cup \{m_{\epsilon+1^{n-1}}[X_n]\}_{\epsilon\in Z_{n-1}}$$

is independent and spans the desired space. To prove independence, suppose, if possible, that for some constants c_p and d_ϵ we have

$$\sum_{p\in\mathcal{S}_{n-2}} c_p\, m_p[X_n] \;+\; \sum_{\epsilon\in Z_{n-1}} d_\epsilon\, m_{\epsilon+1^{n-1}}[X_n] \;=\; \sum_{i=1}^{n} A_i[X_n]\, e_i[X_n]\,. \tag{2.11}$$

Setting $x_n = 0$ we then get that

$$\sum_{p\in\mathcal{S}_{n-2}} c_p\, m_p[X_{n-1}] \;+\; e_{n-1}[X_{n-1}] \sum_{\epsilon\in Z_{n-1}} d_\epsilon\, x^\epsilon \;=\; \sum_{i=1}^{n-1} A_i[X_{n-1}]\, e_i[X_{n-1}]\,.$$

Now this implies that

$$\sum_{p\in\mathcal{S}_{n-2}} c_p\, m_p[X_{n-1}] \equiv 0 \qquad \big(\ mod(e_1, e_2, \ldots, e_{n-1})\ \big)\,. \tag{2.12}$$

But, as we have seen, property (e) in particular asserts that $\{m_p[X_{n-1}]\}_{p\in\mathcal{S}_{n-2}}$ is an independent set. Thus 2.12 forces the vanishing of all the constants c_p. This given, 2.11 now reduces to

$$\sum_{\epsilon\in Z_{n-1}} d_\epsilon\, m_{\epsilon+1^{n-1}}[X_n] \;\equiv\; 0 \qquad \big(\ mod(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}\ \big)\,. \tag{2.13}$$

However, using the relation

$$\mathcal{E}_n e_{n-1}\, x^\epsilon \;=\; m_{\epsilon+1^{n-1}}[X_n]$$

we may rewrite 2.13 as

$$\mathcal{E}_n e_{n-1}\Big(\sum_{\epsilon\in Z_{n-1}} d_\epsilon\, x^\epsilon\Big) \;\equiv\; 0 \qquad \big(\ mod(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}\ \big)$$

Thus from *(i)* we deduce that for some constants a_p we must have

$$\sum_{\epsilon\in Z_{n-1}} d_\epsilon\, x^\epsilon \equiv \sum_{p\in\mathcal{S}_{n-2}} a_p m_p[X_{n-1}] \qquad \big(mod(e_1, e_2, \ldots, e_{n-1})_{\mathbb{Q}[X_{n-1}]}\ \big)$$

But this is inconsistent with our construction of the collection Z_{n-1} unless all the coefficients d_ϵ do vanish. To complete our proof of claim (ii) chose any composition $q \in \mathcal{C}_{\leq n-1}$ and note (from property (d)) that we must have the expansion

$$m_q[X_{n-1}] = \sum_{p\in\mathcal{S}_{n-2}} c_p(e_1, \ldots, e_{n-1})[X_{n-1}]\, m_p[X_{n-1}] + \sum_{\epsilon\in Z_{n-1}} d_\epsilon(e_1, \ldots, e_{n-1})[X_{n-1}] e_{n-1}[X_{n-1}]\, x^\epsilon\,,$$

where $c_p(y_1,\ldots,y_{n-1})$ and $d_\epsilon(y_1,\ldots,d_{n-1})$ are suitable polynomials in their arguments. Applying $\mathcal{E}_n$ to both sides of this identity (and using (ii) of Proposition 1.5) we get

$$m_q[X_n] = \sum_{p\in\mathcal{S}_{n-2}} c_p(e_1,\ldots,e_n)[X_n]\, m_b[X_n] + \sum_{\epsilon\in Z_{n-1}} d_\epsilon(e_1,\ldots,e_{n-1})[X_n] m_{\epsilon+1^{n-1}}[X_n] + A[X_n]\, e_n[X_n].$$

Thus

$$m_q[X_n] \equiv \sum_{p\in\mathcal{S}_{n-2}} c_p(0,\ldots,0)\, m_b[X_n] + \sum_{\epsilon\in Z_{n-1}} d_\epsilon(0,\ldots,0) m_{\epsilon+1^{n-1}}[X_n] \qquad (\mathrm{mod}\ (e_1,e_2,\ldots,e_n)_{\mathbb{Q}[X_n]})\,.$$

This yields the desired spanning property of the basis $\mathcal{A}_n$.

Finally note that from (a) and (b) we derive that

$$\sum_{b\in\mathcal{A}_n} q^{|b|} \;=\; P_{n-2} \;+\; q^{n-1}\big([n-1]_q! - P_{n-2}\big) \;=\; P_{n-1}(q)$$

This proves $\mathbf{G}_n$ and completes the proof of the theorem.

Remark 2.1

Note that the above argument shows that (under $\mathbf{K}_n$) if we start with a subset $\mathcal{S}_{n-2}\subseteq \mathcal{C}_{\le n-2}$ such that the collection

$$\mathcal{A}_{n-2} \;=\; \big\{m_p[X_{n-1}]\big\}_{p\in\mathcal{S}_{n-2}}$$

is a basis for the linear span

$$\mathcal{L}\big[\, m_p[X_{n-1}] \,:\, p\in\mathcal{C}_{\le n-2}\,\big\}$$

and $Z_{n-1}\in\mathcal{SUB}_{n-1}$ is obtained by the Gauss elimination process as indicated in the proof of Theorem 2.1, then a basis for the linear span

$$\mathcal{L}\big[\, m_p[X_n] \,:\, p\in\mathcal{C}_{\le n-1}\,\big\}$$

is given by the collection

$$\mathcal{A}_n \;=\; \big\{m_p[X_n]\big\}_{p\in\mathcal{S}_{n-1}}$$

with

$$\mathcal{S}_{n-1} \;=\; \mathcal{S}_{n-2}\cup\{\epsilon+1^{n-1}\}_{\epsilon\in Z_{n-1}}\,.$$

3. Quasi-symmetric functions and Harmonics of $\mathbf{S_n}$.

Note that by combining Theorems 2.1 and 2.2 we derive the following diagram of

implications

$$\begin{array}{ccccc} & & \mathbf{G}_2 & \rightarrow \mathbf{B}_2 \\ & & \downarrow & \\ \mathbf{K}_3 & \rightarrow & \mathbf{G}_3 & \rightarrow \mathbf{B}_3 \\ & & \downarrow & \\ \mathbf{K}_4 & \rightarrow & \mathbf{G}_4 & \rightarrow \mathbf{B}_4 \\ & & \downarrow & \\ \mathbf{K}_5 & \rightarrow & \mathbf{G}_5 & \rightarrow \mathbf{B}_5 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ & & \downarrow & \\ \mathbf{K}_n & \rightarrow & \mathbf{G}_n & \rightarrow \mathbf{B}_n \end{array} \tag{3.1}$$

It is apparent from this diagram that, after having verified $\mathbf{G}_2$, and proved $\mathbf{K}_n$ for all $n \geq 3$ we will have established the validity of our algorithm for proving the Cohen-Macauliness of all the rings of Quasi-Symmetric functions. In this section we prove that property $\mathbf{K}_n$ is equivalent to the statement of Theorem I.3.

But before we can proceed with our arguments we need a few additional facts about the Harmonics of S_n. To begin, recall that we let $\Delta[X_n]$ denote the Vandermonde determinant in $x_1, x_2, \ldots, x_n$. That is

$$\Delta[x_n] \quad = \quad \|x_i^{n-j}\|_{i,j=1}^{n} \, .$$

Now it is well known that we have

$$P \in (e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]} \quad \Longleftrightarrow \quad P[\partial_n] \, \Delta[X_n] \quad = \quad 0 \, , \tag{3.2}$$

where for a polynomial $P[X_n] \in \mathbb{Q}[X_n]$ we set

$$P[\partial_n] = P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}) \, .$$

This shows the well known fact that the orthogonal complement of the ideal $(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}$ is given by the linear span of derivatives of $\Delta[X_n]$. This space is denoted here by "$\mathbf{H}_n$" and its elements are usually called "*Harmonics of* S_n". In symbols

$$\mathbf{H}_n \quad = \quad \mathcal{L}\big[P(\partial)\Delta[X_n]\big] \, . \tag{3.3}$$

Another consequence of 3.2 is that the Artin monomials yield us a basis for the Harmonics. To be precise a basis for $\mathbf{H}_n$ is given by the collection

$$\big\{\partial_x^{\epsilon} \, \Delta[X_n]\big\}_{\epsilon \in \mathcal{SUB}_n} \, . \tag{3.4}$$

The bridge between $\mathbf{K}_n$ and Theorem I.3 is provided by the following purely combinatorial fact.

Proposition 3.1

For $P[X_{n-1}] \in \mathcal{QS}_{=n-1}$ *we have*

$$\mathcal{E}_n \, P[X_{n-1}] \quad = \quad \tau_n \; P[X_{n-1}] \, . \tag{3.5}$$

where τ_n is as defined in I.16.

Proof

It is sufficient to prove 3.5 for monomials. So let

$$P[X_{n-1}] \;=\; x_1^{p_1} x_2^{p_2} \cdots x_{n-1}^{p_{n-1}}$$

with $p_i \geq 1$ for all $i = 1, 2, \ldots, n-1$. Then

$$\begin{aligned} \mathcal{E}_n P[X_{n-1}] \;&=\; \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n} x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_{n-1}}^{p_{n-1}} \\ &=\; \sum_{i=1}^{n} \sigma_i(n)\; x_1^{p_1} x_2^{p_2} \cdots x_{n-1}^{p_{n-1}} \;=\; \tau_n\; P[X_{n-1}] \end{aligned}$$

Q.E.D.

The connection between Quasi-Symmetric Functions and Harmonics stems from the following basic result:

Proposition 3.2

For any $P[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ we have

$$\mathcal{E}_n e_{n-1} P[X_{n-1}] \equiv 0 \qquad \big(mod(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}\big) \tag{3.6}$$

if and only if

$$\tau_n'\, P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}}) \Delta[X_{n-1}] \;=\; 0 \tag{3.7}$$

Proof

From 3.2 and 3.5 we derive that 3.6 is equivalent to

$$\big(\tau_n e_{n-1} P\big)\big(\partial_n\big) \Delta[X_n] \;=\; 0\,. \tag{3.8}$$

Since

$$\sigma_i(n) \Delta[X_n] = (-1)^{n-i} \Delta[X_n]$$

we may rewrite 3.8 in the form

$$\tau_n' \Big(P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}}) \partial_{x_1} \partial_{x_2} \cdots \partial_{x_{n-1}} \Delta[X_n] \Big) \;=\; 0\,. \tag{3.9}$$

Now it is easily verified that we have

$$\partial_{x_1} \partial_{x_2} \cdots \partial_{x_{n-1}} \Delta[X_n] \;=\; n!\, \Delta[X_{n-1}]\,.$$

Using this identity in 3.9 proves the equivalence of 3.6 and 3.7.

The crucial step in our developments is provided by the following immediate corollary of Proposition 3.2.

Theorem 3.1

Theorem I.3 and $\mathbf{K}_n$ *are equivalent. Moreover On the validity of Theorem I.3 and* G_{n-1} *the Kernel of* τ_n' *as map of*

$$\mathbf{H}_{n-1} \quad \textit{into} \quad \mathbf{H}_n$$

has Hilbert series

$$q^{\binom{n-1}{2}} P_{n-2}(1/q) \tag{3.10}$$

Equivalently, the the Kernel of $\mathcal{E}_n e_{n-1}$ *as map of*

$$Q[X_{n-1}]/(e_1, e_2, \ldots, e_{n-1})_{Q[X_{n-1}]} \quad \textit{into} \quad Q[X_n]/(e_1, e_2, \ldots, e_n)_{Q[X_n]}$$

has Hilbert series

$$P_{n-2}(q) \tag{3.11}$$

Proof

Proposition 3.2 asserts that a polynomial $P[X_{n-1}]$ is in the kernel of $\mathcal{E}_n e_{n-1}$ if and only if the harmonic polynomial

$$P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}})\Delta[X_{n-1}] \tag{3.12}$$

is in the kernel of τ_n'. Now Theorem I.3 implies that this holds true if and only if we have

$$P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}})\Delta[X_{n-1}] \;=\; Q(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}})\Delta[X_{n-1}] \tag{3.13}$$

for some quasi-symmetric polynomial $Q[X_{n-1}] \in \mathcal{QS}_{n-1}$. But from 3.2 we derive that this holds true if and only if

$$P[X_{n-1}] \equiv Q\,[X_{n-1}] \qquad \textit{mod}\ \ (e_1, e_2, \ldots, e_{n-1})_{Q[X_{n-1}]}\,. \tag{3.14}$$

Conversely, the equivalence of 3.7 and 3.6 yields that if the harmonic polynomial in 3.12 is in the kernel of τ_n' then $P[X_{n-1}]$ itself must be in the kernel of $\mathcal{E}_n e_{n-1}$. But then from $\mathbf{K}_n$ we derive that 3.14 must hold true for some $Q[X_{n-1}] \in \mathcal{QS}_{n-1}$. The equivalence of 3.14 and 3.13 then yields that Theorem I.3 holds true for τ_n'. Note further that we have seen in the proof of Theorem 2.2 that under G_{n-1} and $\mathbf{K}_n$ a basis for the kernel of $\mathcal{E}_n e_{n-1}$ is given by the collection $\{m_p[X_{n-1}]\}_{p\in\mathcal{S}_{n-2}}$ it follows then from the equivalence of 3.6 and 3.7 that under $\mathbf{G}_{n-1}$ a basis for the kernel of τ_n' is given by the collection

$$\left\{m_p(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{n-1}})\Delta[X_{n-1}]\right\}_{p\in\mathcal{S}_{n-2}}. \tag{3.15}$$

Thus we see that the assertions in 3.10 and 3.11 regarding the Hilbert series of τ_n' and $\mathcal{E}_n e_{n-1}$ are immediate consequences of property (a) in the proof of Theorem 2.2.

To complete the first proof of the validity of our algorithm we must verify that we can start the inductive process and establish Theorem I.3. The latter will be carried out in the next section. To illustrate our algorithm we end this section by showing the validity of

$\mathbf{G_2}, \mathbf{G_3}$ and $\mathbf{G_4}$. To this end recall that $\mathbf{G}_n$ asserts that, modulo the ideal $(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}$, the linear span

$$\mathcal{L}\big[m_p[X_{n-1}] \,:\, p \in \mathcal{C}_{\leq n-1}\big] \tag{3.16}$$

has Hilbert series $P_{n-1}(q)$. Now recall that we have

$$P_1(q) = 1\ , \quad P_2(q) = 1 + q^3\ , \quad P_3(q) = 1 + q^3 + 2q^4 + 2q^5\ \ ,$$

This given we have the following findings

$\mathbf{G_2}$: For $n = 2$ the linear span in 3.16, modulo the ideal $(e_1, e_2)_{\mathbb{Q}[X_2]}$, reduces to $\mathbb{Q}$. Thus it is of dimension $1 = (2-1)!$. Since $P_1(q) = 1$ we see that $\mathbf{G}_2$ is trivially true. Now for $n = 2$ the Artin basis reduces to $\{1, x_1\}$ and the basic relations are

$$\begin{aligned} x_2 &= e_1 - x_1 \\ x_1^2 &= x_1 e_1 - e_2\,. \end{aligned}$$

Thus from the construction given in the proof of Theorem 2.1, we obtain that our Λ_2-bases for $\mathbb{Q}[X_2]$ and $\mathcal{QS}_2$ are respectively

$$\{1, x_1\} \quad \text{and} \quad \{1, x_1 x_2 \times x_1\}\,.$$

Note that here, the Gauss elimination step eliminates "1" out of $\{1, x_1\}$. Thus our Λ_2-basis for $\mathcal{QS}_2$ may be rewritten as

$$\big\{1, m_{2,1}[X_2]\big\}\,. \tag{3.17}$$

$\mathbf{G_3}$: For $n = 3$ the Artin basis is

$$\big\{1\ ,\ x_1\ ,\ x_2\ ,\ x_1x_2\ ,\ x_1^2\ ,\ x_1^2x_2\big\} \tag{3.18}$$

and the basic relations are

$$\begin{aligned} x_3 &= e_1 - x_1 - x_2 \\ x_2^2 &= -x_2x_1 - x_1^2 - e_2 + e_1x_1 + e_1x_2 \\ x_3^3 &= e_1x_1^2 - e_2x_1 + e_3 \end{aligned} \tag{3.19}$$

Applying the operator $\mathcal{E}_3$ to the collection in 3.17 gives

$$\big\{1\ ,\ m_{2,1}[X_3]\big\} \tag{3.20}$$

Now it is easily verified that this collection spans the linear span in 3.16 for $n = 3$. Indeed, $m_1[X_3]$, $m_{11}[X_3], m_2[X_3]$ and $m_{2,1}[X_3] + m_{2,1}[X_3]$ are symmetric, and all quasi-monomials indexed by 2-part compositions of degree greater than 3 vanish modulo

$(e_1, e_2, e_3)_{\mathbb{Q}[X_3]}$. Clearly, the collection in 3.20 is independent modulo $(e_1, e_2, e_3)_{\mathbb{Q}[X_3]}$ and since $P_2(q) = 1 + q^3$ we see that $\mathbf{G}_3$ is satisfied. Next, from the expansion

$$m_{2,1}[X_3] = x_1^2x_2 + x_1^2x_3 + x_2^2x_3$$

and the relations in 3.1 we derive that

$$m_{2,1}[X_3] \equiv x_1^2x_2 \qquad (\text{mod } (e_1, e_2, e_3)_{\mathbb{Q}[X_3]})$$

This given, if we apply Gauss elimination modulo $(e_1, e_2, e_3)_{\mathbb{Q}[X_3]}$ to 3.20 followed by 3.18, we find that the elements that survive are

$$x_1 \ , \ x_2 \ , \ x_1x_2 \ , \ x_2^2$$

Thus in this case we have

$$Z_3 = \big\{[1,0,0], [0,1,0], [1,1,0], [0,2,0]\big\},$$

and we derive that a Λ_3-basis for $\mathbb{Q}[X_3]$ is given by the collection

$$\big\{1 \ , \ m_{2,1}[X_3]\big\} \cup \big\{x_2 \ , \ x_1 \ , \ x_1x_2 \ , \ x_2^2 \big\}.$$

This implies that the collection

$$\big\{1 \ , \ m_{2,1}[X_3]\big\} \cup \big\{m_{2,1,1}[X_3], m_{1,2,1}[X_3], m_{2,2,1}[X_3], m_{1,3,1}[X_3]\big\}. \tag{3.21}$$

is a Λ_3 basis for $\mathcal{QS}_3$. We may now set

$$\mathcal{S}_3 \ = \ \big\{[\,], [2,1], [2,1,1], [1,2,1], [2,2,1], [1,3,1]\big\} \tag{3.22}$$

$\mathbf{G_4}$: To diminish our work for $n = 4$ and to illustrate another aspect of our algorithm we shall take the shortcut of proving $\mathbf{B}_4$ directly in this case, $\mathbf{G}_4$ then will follow automatically. To this end note that for the Gauss elimination process to deliver Λ_n-bases for $\mathbb{Q}[X_n]$ and $\mathcal{QS}_n$ it is sufficient to show that we can construct a collection $\{m_p[X_n]\}_{p\in\mathcal{S}_{n-1}}$, which is independent modulo $(e_1, e_2, \ldots, e_n)_{\mathbb{Q}[X_n]}$, with $\mathcal{S}_{n-1} \in \mathcal{C}_{\le n-1}$ satisfying the requirement

$$\sum_{p\in\mathcal{S}_{n-1}} q^{|p|} \ = \ P_{n-1}(q) \ .$$

Indeed then the collection of Artin monomials $\{x^\epsilon\}_{\epsilon\in Z_n}$ that survive will necessarily have degree distribution given by $[n]_q! - P_{n-1}(q)$, and that is all that is needed for $\mathbf{B}_n$. Now if we follow the process used in the proof of Theorem 2.2 our choice for $\mathcal{S}_{n-1}$ when $n = 4$ should be the collection in 3.22. We are thus reduced to showing that the collection

$$\mathcal{A}_4 \ = \ \big\{1 \ , \ m_{2,1}[X_4]\big\} \cup \big\{m_{2,1,1}[X_4], m_{1,2,1}[X_4], m_{2,2,1}[X_4], m_{1,3,1}[X_4]\big\}. \tag{3.23}$$

is independent modulo $(e_1, e_2, e_3, e_4)_{\mathbb{Q}[X_4]}$. Clearly, we need only check the independence of the subcolllection

$$\left\{m_{2,1,1}[X_4], m_{1,2,1}[X_4], m_{2,2,1}[X_4], m_{1,3,1}[X_4]\right\}. \tag{3.24}$$

Now, denoting by "$\equiv$" equivalence modulo $(e_1, e_2, e_3, e_4)_{\mathbb{Q}[X_4]}$, from 1.18 we obtain

$$\begin{aligned} x_1^4 &\equiv 0\,, \\ x_2^3 &\equiv -x_1x_2^2 - x_1^2x_2 - x_1^3\,, \\ x_3^2 &\equiv -x_1x_2 - x_1x_3 - x_2x_3 - x_1^2 - x_2^2\,, \\ x_4 &\equiv -x_1 - x_2 - x_3\,. \end{aligned}$$

This gives

$$\begin{aligned} m_{2,1,1}[X_3] &\equiv -x_1^3x_2\,, \\ m_{1,2,1}[X_3] &\equiv x_1^3x_2 + x_1^2x_2x_3 + x_1x_2^2x_3\,, \\ m_{2,2,1}[X_3] &\equiv -x_1^2x_2^2x_3\,, \\ m_{1,3,1}[X_3] &\equiv -x_1^3x_2^2 - x_1^2x_2^2x_3\,, \end{aligned}$$

and the independence of 3.24 is assured. Since

$$\sum_{p\in\mathcal{S}_3} q^{|p|} \;=\; 1 + q^3 + 2q^4 + 2q^5 \;=\; P_3(q)$$

our observations yield that $\mathbf{B}_4$ holds true and in particular the collection $\mathcal{A}_4$ in 3.23 must necessarily a basis for the linear span in 3.16 for $n = 4$.

4. The action of τ_n' on the Harmonics of S_{n-1}

The goal of this section is to determine the kernel of τ_n' on $\mathbf{H}_{n-1}$. Before we can state and prove our results we need to establish a few properties of τ_n' as well as some further facts about harmonics.

We begin with a simple but important observation.

Proposition 4.1

For any polynomial $P[X_{n-2}] \in \mathbb{Q}[X_{n-2}]$ and any exponent $a \geq 0$ we have

$$\tau_n'\, x_{n-1}^a P[X_{n-2}] \;=\; -x_n^a\, \tau_{n-1}' P[X_{n-2}] \;+\; x_{n-1}^a P[X_{n-2}] \tag{4.1}$$

in particular on $\mathbb{Q}[X_{n-2}]$ we have

$$\tau_n' \;=\; 1 - \tau_{n-1}'$$

Proof

The result follows from the simple fact that for $1 \leq i \leq n-1$ the cycles $\sigma_i(n)$ and $\sigma_i(n-1)$ have the same action on polynomials in $x_1, x_2, \ldots, x_{n-2}$. This given, from the definition in I.16 we derive that

$$\begin{aligned} \tau_n'\, x_{n-1}^a P[X_{n-2}] \;&=\; \sum_{i=1}^{n-1} (-1)^{n-i}\, x_n^a\, \sigma_i(n) P[X_{n-2}] \;+\; x_{n-1}^a P[X_{n-2}] \\ &=\; -x_n^a \sum_{i=1}^{n-1} (-1)^{n-1-i}\, \sigma_i(n-1) P[X_{n-2}] \;+\; x_{n-1}^a P[X_{n-2}] \end{aligned}$$

This proves 4.1.

The following identities will play a crucial role in many of our arguments.

Proposition 4.2

For any polynomial $P[X_{n-2}] \in \mathbb{Q}[X_{n-2}]$

$$\begin{aligned} a) \quad & \tau'_n \, \tau'_{n-1} \, P[X_{n-2}] \;=\; 0\,, \\ b) \quad & {\tau'_n}^2 \, P[X_{n-2}] \;=\; \tau'_n \, P[X_{n-2}]\,, \end{aligned} \tag{4.2}$$

Proof

It easily verified that $\tau'_2\tau'_1\,1 = 0$ thus to prove 4.2 a) we can proceed by induction and assume that for $P[X_{n-3}] \in \mathbb{Q}[X_{n-3}]$ we have

$$\tau'_{n-1} \, \tau'_{n-2} \, P[X_{n-3}] \;=\; 0\,. \tag{4.3}$$

Clearly we need only verify 4.2 a) when $P[X_{n-2}]$ is a monomial. Now let $m[X_{n-2}] = x^a_{n-2} m_o[X_{n-3}]$ where $m_o[X_{n-3}]$ is a monomial in $x_1, x_2, \ldots x_{n-3}$, Then from 4.1 (for n-1) we derive that

$$\tau'_{n-1} \, x^a_{n-2} m_o[X_{n-3}] \;=\; -x^a_{n-1} \tau'_{n-2} \, m_o[X_{n-3}] \;+\; x^a_{n-2} \, m_o[X_{n-3}]\,.$$

Thus

$$\begin{aligned} \tau'_n \, \tau'_{n-1} \, x^a_{n-2} m_o[X_{n-3}] \;&=\; -\tau'_n \, x^a_{n-1} \tau'_{n-2} m_o[X_{n-3}] \;+\; \tau'_n \, x^a_{n-2} \, m_o[X_{n-3}] \\ (\text{by 4.1}) \;&=\; -\big(- x^a_n \, \tau'_{n-1} \, \tau'_{n-2} \, m_o[X_{n-3}] + x^a_{n-1} \tau'_{n-2} \, m_o[X_{n-3}]\big) \;+\; \tau'_n \, x^a_{n-2} \, m_o[X_{n-3}] \\ (\text{by 4.3}) \;&=\; -x^a_{n-1} \tau'_{n-2} \, m_o[X_{n-3}] - \tau'_{n-1} \, x^a_{n-2} \, m_o[X_{n-3}] + x^a_{n-2} \, m_o[X_{n-3}] \\ (\text{by 4.1}) \;&=\; -x^a_{n-1} \tau'_{n-2} \, m_o[X_{n-3}] + x^a_{n-1} \, \tau'_{n-2} \, m_o[X_{n-3}] - x^a_{n-2} \, m_o[X_{n-3}] + x^a_{n-2} \, m_o[X_{n-3}] \end{aligned}$$

This completes the induction and the proof of 4.2 a). To prove 4.2 b) we note that, again by 4.1(with $a = 0$) we get for all monomials $m_o[X_{n-2}] \in \mathbb{Q}[X_{n-2}]$

$$\begin{aligned} \tau'_n \tau'_n \; m_o[X_{n-2}] \;&=\; \tau'_n \big(- \tau'_{n-1} m_o[X_{n-2}] \;+\; m_o[X_{n-2}]\big) \\ &=\; -\tau'_n \, \tau'_{n-1} m_o[X_{n-2}] \;+\; \tau'_n \; m_o[X_{n-2}] \\ (\text{ by 4.2 a) }) \;&=\; \tau'_n \; m_o[X_{n-2}]\,. \end{aligned}$$

Remark 4.1

We should note that 4.2 a) implies in particular that for all $n \geq 3$ the Vandermonde $\Delta[X_{n-1}]$ is in the kernel of τ'_n. To see this note that since the cycles $\sigma_i(n)$ may be taken as representatives of the left cosets of S_{n-1} in S_n we may write $\Delta[X_{n-1}]$ in the form

$$\Delta[X_{n-1}] \;=\; \tau'_{n-1} \tau'_{n-2} \cdots \tau'_1 \, x_1^{n-2} x_2^{n-3} \cdots x_{n-2}\,.$$

To see what other harmonics are in the kernel of τ'_n we need further identities.

Proposition 4.3

Denoting by $\Delta^{(r)}[X_{n-1}]$ *the cofactor of* x_n^r *in the matrix* $\|x_i^{n-j}\|_{i,j=1}^n$ *we have*

$$\begin{aligned} &(i) \qquad \Delta[X_n] \;=\; \sum_{r=0}^{n-1}(-x_n)^r\,\Delta^{(r)}[X_{n-1}] \\ &(ii) \qquad \Delta^{(r)}[X_{n-1}] = e_{n-r-1}[X_{n-1}]\Delta[X_{n-1}] \\ &(iii) \qquad \sum_{i=1}^{n-1}\partial_{x_i}\Delta^{(r)}[X_{n-1}] = (r+1)\,\Delta^{(r+1)}[X_{n-1}] \end{aligned} \tag{4.4}$$

Proof

Note that we may write

$$\begin{aligned} \Delta[X_n] \;&=\; (x_1-x_n)(x_2-x_n)\cdots(x_{n-1}-x_n)\,\Delta[X_{n-1}] \\ &=\; \sum_{r=0}^{n-1}(-x_n)^r e_{n-r-1}[X_{n-1}]\,\Delta[X_{n-1}] \end{aligned}$$

comparing with (i) yields (ii). To prove (iii) we note that (ii) gives

$$\sum_{i=1}^{n-1}\partial_{x_i}\Delta^{(r)}[X_{n-1}] \;=\; \Big(\sum_{i=1}^{n-1}\partial_{x_i}e_{n-r-1}[X_{n-1}]\Big)\Delta[X_{n-1}] \;+\; e_{n-r-1}[X_{n-1}]\Big(\sum_{i=1}^{n-1}\partial_{x_i}\Delta[X_{n-1}]\Big)$$

and (iii) is derived from the following two identities that are easily proved

$$\sum_{i=1}^{n-1}\partial_{x_i}\Delta[X_{n-1}] = 0 \qquad \& \qquad \sum_{i=1}^{n-1}\partial_{x_i}e_{n-r-1}[X_{n-1}] = (r+1)e_{n-r-2}[X_{n-1}]\,.$$

These identities yield us an important corollary.

Theorem 4.1

For a polynomial $P[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ *we have*

$$P[\partial_{n-1}]\Delta[X_n] = 0 \quad \Longleftrightarrow \quad P[\partial_{n-1}]\Delta^{(0)}[X_{n-1}] = 0\,, \tag{4.5}$$

In particular it follows that the ideal of polynomials that kill $\Delta^{(0)}[X_{n-1}]$ *is generated by the modified power sums*

$$\sum_{i=1}^{n-1}x_i^k \;+\; \Big(-\sum_{i=1}^{n-1}x_i\Big)^k \qquad (\textit{ for } k=2,3,\ldots,n) \tag{4.6}$$

Thus

$$\sum_{i=1}^{n-1}\partial_{x_i}^k\Delta^{(r)}[X_{n-1}] = (-1)^{k-1}(r+1)\uparrow^{k-1}\Delta^{(r+1)}[X_{n-1}] \tag{4.7}$$

with $(r+1)\uparrow^{k-1} = (r+1)(r+2)\cdots(r+k)$.

Proof

Hitting the expansion in 4.4 (i) with $P[\partial_{n-1}]$ gives

$$P[\partial_{n-1}]\Delta[X_n] \;=\; \sum_{r=0}^{n-1}(-x_n)^r\, P[\partial_{n-1}]\Delta^{(r)}[X_{n-1}] \tag{4.8}$$

setting $x_n = 0$ proves "$\Longrightarrow$". To prove the converse note that from 4.4 (*iii*) we get

$$\Big(\sum_{i=1}^{n-1}\partial_{x_i}\Big)^k \Delta^{(r)}[X_{n-1}] \;=\; (r+1)\uparrow^{k-1}\Delta^{(r+k)}[X_{n-1}]\,, \tag{4.9}$$

thus

$$\Big(\sum_{i=1}^{n-1}\partial_{x_i}\Big)^k P[\partial_{n-1}]\Delta^{(0)}[X_{n-1}] = k!\, P[\partial_{n-1}]\Delta^{(k)}[X_{n-1}]\,.$$

Now we see that

$$P[\partial_{n-1}]\Delta^{(0)}[X_{n-1}] = 0 \qquad\Longrightarrow\qquad P[\partial_{n-1}]\Delta^{(k)}[X_{n-1}] = 0 \qquad (\text{for all } k)\,. \tag{4.10}$$

and $P[\partial_{n-1}]\Delta[X_n] = 0$ then follows from 4.8.

It is well known that the ideal of polynomials that kill $\Delta[X_n]$ is also generated by the power sums

$$\sum_{i=1}^{n} x_i^k \qquad (\text{for } k = 1, 2, \ldots, n)$$

Thus it follows that the modified power sums given in 4.7 must also kill $\Delta[X_n]$. Conversely, assume that $P[x_{n-1}]$ kills $\Delta[X_n]$. We must then have

$$P[x_{n-1}] \;=\; \sum_{k=1}^{n} A_k \Big(\sum_{i=1}^{n} x_i^k\Big)\,. \tag{4.12}$$

Since the left hand side is independent of x_n we may make the replacement $x_n \to -\sum_{i=1}^{n-1} x_i$ in the right hand side and obtain

$$P[x_{n-1}] \;=\; \sum_{k=2}^{n} A_k \left(\sum_{i=1}^{n-1} x_i^k \;+\; \Big(-\sum_{i=1}^{n-1} x_i\Big)^k\right)\,.$$

This proves that the modified power sums generate the ideal of polynomials that kill $\Delta^{(0)}[X_{n-1}]$. Now we also see from 4.10 that the modified power sums kill all of the polynomials $\Delta^{(r)}[X_{n-1}]$. This given, 4.7 immediately follows from 4.9.

Remark 4.2

Note that since for $k+r > n-1$ the alternant $\sum_{i=1}^{n-1} \partial_{x_i}^k \Delta^{(r)}[X_{n-1}]$ has degree less than $\binom{n-1}{2}$, it must necessarily vanish. We derive from this that the alternant $\Delta^{(r)}[X_{n-1}]$ is killed by the collection of polynomials

$$\Big\{ \sum_{i=1}^{n-1} x_i^k \;+\; \Big(-\sum_{i=1}^{n-1} x_i \Big)^k \Big\}_{k=2}^{n-1-r} \quad \bigcup \quad \Big\{ \sum_{i=1}^{n-1} x_i^k \Big\}_{k=n-r}^{n} \quad \bigcup \quad \Big\{ \Big(\sum_{i=1}^{n-1} x_i \Big)^{n-k} \Big\}.$$

In fact it is shown in [1] that this collection generates the ideal of polynomials that kill $\Delta^{(r)}[X_{n-1}]$.

Remark 4.3

Note that since every symmetric function in $\mathbb{Q}[X_{n-1}]$ is a polynomial in the power sums $\sum_{i=1}^{n-1} x_i^k$ it follows from 4.7 that if $Q[X_{n-1}]$ is symmetric and homogeneous of degree k then, for a suitable constant $C_{Q,r}$ we must have

$$Q[\partial_{n-1}]\Delta^{(r)}[X_{n-1}] \;=\; C_{Q,r}\, \Delta^{(r+k)}[X_{n-1}]\,, \tag{4.13}$$

where, for $Q(x_1, x_2, \ldots, x_m) \in \mathbb{Q}[X_m]$ here and after we shall use the symbol "$Q[\partial_m]$" to denote the operator "$Q(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_m})$".

Before we proceed with the next result we should note that every polynonial $h[X_n] \in \mathbf{H}_n$ may be written in the form

$$h[X_n] \;=\; P(\partial_{n-1})\Delta[X_n] \tag{4.14}$$

with $P \in \mathbb{Q}[X_{n-1}]$. This is an immediate consequence of the identity

$$\partial_{x_n}\Delta[X_n] \;=\; -\sum_{i=1}^{n-1} \partial_{x_i}\Delta[X_n] \tag{4.15}$$

which shows that derivations with respect to x_n are not needed in the production of harmonic polynomials.

Proposition 4.4

Every harmonic $h[X_n] \in \mathbf{H}_n$ may be written in the form

$$h[X_n] \;=\; \sum_{r=0}^{n-1} (-x_n)^r\, h_r[X_{n-1}] \tag{4.16}$$

where for a suitable polynomial $P \in \mathbb{Q}[X_{n-1}]$ we have

$$h_r[X_{n-1}] \;=\; P(\partial_{n-1})\, \Delta^{(r)}[X_{n-1}]\,. \qquad (\text{ for } r = 0, 1, \ldots n-1\,) \tag{4.17}$$

In particular, $h[X_n]$ is of maximum degree $n-1$ in x_n if and only if

$$h_{n-1}[X_{n-1}] = P(\partial_{n-1})\Delta[X_{n-1}] \neq 0 \tag{4.18}$$

Proof

These identities follow by combining 4.14 with the expansion in 4.4 (i).

Proposition 4.5

For

$$h[X_{n-1}] \;=\; \sum_{r=0}^{m}(-1)^r x_{n-1}^r h_r[X_{n-2}] \tag{4.19}$$

with $h_r[X_{n-2}] \in \mathbb{Q}[X_{n-2}]$ *for* $r = 0, 1, \ldots, m$ *we have*

$$\tau_n' \, h[X_{n-1}] \;=\; -\tau_{n-1}' h_o[X_{n-2}] \;+\; h[X_{n-1}] \;-\; \sum_{r=1}^{m}(-1)^r x_n^r \tau_{n-1}' h_r[X_{n-2}] \tag{4.20}$$

Proof

Using 4.1on 4.19 gives

$$\begin{aligned}\tau_n' \, h[X_{n-1}] \;&=\; \sum_{r=0}^{m}(-1)^r \big(- x_n^r \, \tau_{n-1}' h_r[X_{n-2}] \;+\; x_{n-1}^r \, h_r[X_{n-2}]\big) \\ &=\; -\tau_{n-1}' h_o[X_{n-2}] \;-\; \sum_{r=1}^{m}(-1)^r x_n^r \, \tau_{n-1}' h_r[X_{n-2}] \;+\; \sum_{r=0}^{m}(-1)^r x_{n-1}^r \, h_r[X_{n-2}]\big)\end{aligned}$$

and this is 4.20.

The next result may be viewed as a first step in the identification of the kernel of τ_n'.

Proposition 4.6

A harmonic polynomial $h[X_{n-1}]$ *is in the kernel of* τ_n' *if and only if*

$$h[X_{n-1}] \;=\; \tau_{n-1}' h_o[X_{n-2}] \tag{4.21}$$

This given, it follows that

$$\tau_{n-1}' h_r[X_{n-2}] \;=\; 0 \qquad \big(\text{for } \; r = 1, \ldots n-2 \big) \tag{4.22}$$

Proof

From Proposition 4.4 we derive that $h[X_{n-1}]$ may be written precisely in the form given by 4.19 with $m = n-2$. We can thus use Proposition 4.5 and conclude that when $\tau_n' h[X_{n-1}] = 0$ we must necessarily have

$$0 \;=\; -\tau_{n-1} h_o[X_{n-2}] \;+\; h[X_{n-1}] \;-\; \sum_{r=1}^{n-2}(-1)^r x_n^r \tau_{n-1} h_r[X_{n-2}] \tag{4.23}$$

and 4.22 follows by setting $x_n = 0$. Conversely, suppose that 4.21 holds true. This given, note that the harmonicity of $h[x_{n-1}]$ yields

$$\sum_{i=1}^{n-1} \partial_{x_i} \, h[x_{n-1}] \;=\; 0 \, .$$

Moreover, since the operator $\sum_{i=1}^{n-1}\partial_{x_i}$ commutes with τ'_{n-1}, hitting 4.21 with $\sum_{i=1}^{n-1}\partial_{x_i}$ gives

$$\tau'_{n-1}h_1[X_{n-2}] \;=\; 0\,.$$

But then all the relations in 4.22 follow from 4.4 (iii) by successive applications of the operator $\sum_{i=1}^{n-1}\partial_{x_i}$. However the validity of 4.21 together with 4.22 yields 4.23, and this, via 4.20, forces $\tau'_n h[X_{n-1}]=0$, completing the proof.

We are thus led to study the space of harmonics in $H[X_{n-1}]$ that can be written in the form given in 4.21. To this end we have the following important auxiliary result.

Theorem 4.2

For given polynomial Δ denote by $\mathcal{L}_\partial[\Delta]$ the linear span of derivatives of Δ. Then a basis for $\mathcal{L}_\partial[\Delta^{(o)}[X_{n-1}]]$ is given by the collection of polynomials

$$\mathcal{B}_0[X_{n-1}] \;=\; \Big\{\partial_x^\epsilon\Delta^{(o)}[X_{n-1}]\Big\}_{\substack{\epsilon\in\mathcal{SUB}_{n-1}\\ 0\le r\le n-1}}\,. \tag{4.24}$$

In particular we derive that

$$\dim\,\mathcal{L}_\partial\big[\Delta^{(o)}[X_{n-1}]\big] \;=\; n!\,. \tag{4.25}$$

Proof

Note that Theorem 4.1 implies in particular that $\mathcal{L}_\partial[\Delta^{(o)}[X_{n-1}]]$ and $\mathcal{L}_\partial[\Delta[X_n]]$ have the same dimension. Since

$$\dim\,\mathcal{L}_\partial\big[\Delta[X_n]\big] \;=\; n!$$

4.25 necessarily follows. Thus to obtain that $\mathcal{B}_o[X_{n-1}]$ is a basis we need only show that it is an independent set. To do this we proceed by contraddiction. Suppose if possible that we have a set of polynomials

$$P_r[X_{n-1}]=\sum_{\epsilon\in\mathcal{SUB}_{n-1}} a_{\epsilon,r}\,x^\epsilon \qquad\qquad \text{for}\quad 0\le r\le n-1 \tag{4.26}$$

such that

$$\sum_{r=0}^{n-1} P_r[\partial_{n-1}]\Delta^{(r)}[X_{n-1}] \;=\; 0\,. \tag{4.27}$$

Now let r_1 be the first r such that $P_r[X_{n-1}]\neq 0$. This given, if we hit 4.27 by $\sum_{i=1}^{n-1}\partial_{x_i}^k$ and use 4.7 we obtain

$$\sum_{r=r_1}^{n-1-k}(-1)^{k-1}(r+k)\uparrow^{k-1} P_r[\partial_{n-1}]\Delta^{(r+k)}[X_{n-1}] \;=\; 0\,,$$

since $\sum_{i=1}^{n-1}\partial_{x_i}^k\Delta^{(r)}[X_{n-1}]=0$ for $r+k>n-1$. Now for $k=n-1-r_1$ this reduces to

$$P_{r_1}[\partial_{n-1}]\Delta^{(n-1)}[X_{n-1}] \;=\; 0\,. \tag{4.28}$$

Now our definition of $\Delta^{(r)}[X_{n-1}]$ gives $\Delta^{(n-1)}[X_{n-1}] = \Delta[X_{n-1}]$ and since the collection in 3.4 is a basis for $\mathcal{L}_\partial\big[\Delta[X_{n-1}]\big]$, we cannot have 4.28 with $P_{r_1}[X_{n-1}] \neq 0$. This yields our contraddiction and completes the proof of the theorem.

Remark 4.4

Although the following additional fact is not needed here we should point out that the collection

$$\mathcal{B}_{r_0}[X_{n-1}] \;=\; \Big\{\partial_x^\epsilon \Delta^{(r)}[X_{n-1}]\Big\}_{\substack{\epsilon\in\mathcal{SUB}_{n-1}\\ r_0\le r\le n-1}}.$$

gives a basis for the subspace $\mathcal{L}_\partial\big[\Delta^{(r_o)}[X_{n-1}]$. In fact, the identical argument that gives the independence of $\mathcal{B}_o[X_{n-1}]$ gives also the independence of $\mathcal{B}_{r_0}[X_{n-1}]$. The result then follows since it was shown in [3] that

$$\dim\, \mathcal{L}_\partial\big[\Delta^{(r_o)}[X_{n-1}] \le (n-r_o)(n-1)!\,.$$

We are finally in a position to give our proof of Theorem I.3. Recall that it states:

Theorem I.3

A polynomial $h[X_{n-1}] \in H[X_{n-1}]$ is in the kernel of τ'_n if and only if

$$h[X_{n-1}] \;=\; Q[\partial_{n-1}]\Delta[X_{n-1}] \qquad 4.30$$

with $Q \in \mathcal{QS}_{n-1}$.

Proof

Combining case *(iv)* of Proposition 1.5 with Proposition 3.2 we derive that for every composition $p \in \mathcal{C}_{\le n-1}$ we have

$$\tau'_n\, m_p[\partial_{n-1}]\Delta[X_{n-1}] \;=\; 0\,.$$

This proves the sufficiency. Thus we need only show the necessity. The special cases we worked out in section 3. yield the validity of the theorem for $n = 2, 3, 4$. We can thus proceed by induction on n and assume it is valid up to $n-1$.

Now, given that

$$h[X_{n-1}] \;=\; P[\partial_{n-2}]\Delta[X_{n-1}]$$

from Proposition 4.6 we derive that

$$h[X_{n-1}] = \tau'_{n-1}P[\partial_{n-2}]\Delta[X_{n-1}]\,\big|_{x_{n-1}=0} = \tau'_{n-1}P[\partial_{n-2}]\Delta^{(0)}[X_{n-2}]\,.$$

Theorem 4.2 then implies that there is a unique set of polynomials

$$P_r[X_{n-2}] = \sum_{\epsilon\in\mathcal{SUB}_{n-2}} a_{\epsilon,r}\, x^\epsilon \qquad\qquad (\text{for } \; r = 1, 2, \ldots, n-2)$$

giving

$$P[\partial_{n-2}]\Delta^{(0)}[X_{n-2}] \;=\; \sum_{r=0}^{n-2} P_r[\partial_{n-2}]\Delta^{(r)}[X_{n-2}]\,.$$

Thus we may write $h[X_{n-1}]$ in the form

$$h[X_{n-1}] \;=\; \sum_{r=0}^{n-2} \tau'_{n-1} P_r[\partial_{n-2}] \Delta^{(r)}[X_{n-2}]\,. \tag{4.31}$$

Let us suppose that for some $0 \le r_o < n-2$ we have

$$P_{r_o}[X_{n-2}] \neq 0 \qquad \text{and} \qquad P_r[X_{n-2}] = 0 \qquad (\text{ for } \quad 0 \le r < r_o)$$

so

$$h[X_{n-1}] \;=\; \sum_{r=r_0}^{n-2} \tau'_{n-1} P_r[\partial_{n-2}] \Delta^{(r)}[X_{n-2}]\,. \tag{4.32}$$

It will be convenient here and after to set

$$D_m^{(k)} \;=\; \sum_{i=1}^{m} x_i^k\,,$$

this given, hitting 4.23 with $D_{n-1}^{(k)}$ the harmonicity of $h[X_{n-1}]$ gives

$$\begin{aligned}
0 \;&=\; \sum_{r=r_0}^{n-2} D_{n-1}^{(k)} \tau'_{n-1} P_r[\partial_{n-2}] \Delta^{(r)}[X_{n-2}] \\
&=\; \sum_{r=r_0}^{n-2} \tau'_{n-1} P_r[\partial_{n-2}] D_{n-2}^{(k)} \Delta^{(r)}[X_{n-2}] \qquad\qquad . \\
(\text{ by 4.7}) \;&=\; \sum_{r=r_0}^{n-2} (-1)^{k-1} (r+1) \uparrow^{k-1} \tau'_{n-1} P_r[\partial_{n-2}] \Delta^{(r+k)}[X_{n-2}]
\end{aligned}$$

For $k = n-2-r_o$ this forces

$$\tau'_{n-1} P_{r_0}[\partial_{n-2}] \Delta^{(n-2)}[X_{n-2}] \;=\; 0$$

Since $\Delta^{(n-2)}[X_{n-2}] = \Delta[X_{n-2}]$, the induction hypothesis gives that

$$P_{r_0}[\partial_{n-2}] \Delta[X_{n-2}] \;=\; Q_{r_0}[\partial_{n-2}] \Delta[X_{n-2}]$$

with $Q_{r_0}[X_{n-2}] \in \mathcal{QS}_{n-2}$. This implies that

$$P_{r_0}[\partial_{n-2}] \;=\; Q_{r_0}[\partial_{n-2}] \;+\; \sum_{i=1}^{n-2} A_i[X_{n-2}] e_i[X_{n-2}]$$

Thus

$$\begin{aligned}
P_{r_0}[\partial_{n-2}] \Delta^{(r_0)}[X_{n-2}] \;&=\; Q_{r_0}[\partial_{n-2}]\, \Delta^{(r_0)}[X_{n-2}] \;+\; \sum_{i=1}^{n-2} A_i[\partial_{n-2}] e_i[\partial_{n-2}] \Delta^{(r_0)}[X_{n-2}] \\
(\text{by Remark 4.3}) \;&=\; Q_{r_0}[\partial_{n-2}]\, \Delta^{(r_0)}[X_{n-2}] \;+\; \sum_{i=1}^{n-2} C_{i,r_0} A_i[\partial_{n-2}] \Delta^{(r_0+i)}[X_{n-2}]\,.
\end{aligned}$$

This gives

$$\tau'_{n-1}P_{r_0}[\partial_{n-2}]\Delta^{(r_0)}[X_{n-2}] = \tau'_{n-1}Q_{r_0}[\partial_{n-2}]\,\Delta^{(r_0)}[X_{n-2}] + \sum_{i=1}^{n-2} C_{i,r_0}\;\tau'_{n-1}A_i[\partial_{n-2}]\Delta^{(r_0+i)}[X_{n-2}]\,. \qquad 4.33$$

Now note that we have

$$\Delta^{(r_0)}[X_{n-2}] = C_{r_0,n}\,\partial_{x_1}\partial_{x_2}\cdots\partial_{x_{n-2}}\Delta^{(r_0+1)}[X_{n-1}]\,.$$

with $C_{r_0,n} = (n-1)!/(r_0+1)$. Thus

$$\begin{aligned}\tau'_{n-1}Q_{r_0}[\partial_{n-2}]\Delta^{(r_0)}[X_{n-2}] &= \alpha_{r_0,n}\;\tau'_{n-1}Q_{r_0}[\partial_{n-2}]\partial_{x_1}\partial_{x_2}\cdots\partial_{x_{n-2}}\Delta^{(r_0+1)}[X_{n-1}]\\ &= \alpha_{r_0,n}\;\tau'_{n-1}Q_{r_0}[\partial_{n-2}]e_{n-2}[\partial_{n-2}]\Delta^{(r_0+1)}[X_{n-1}]\end{aligned}\,. \qquad 4.34$$

But

$$\tau'_{n-1}Q_{r_0}[\partial_{n-2}]e_{n-2}[\partial_{n-2}]\Delta^{(r_0+1)}[X_{n-1}] \;=\; \big(\tau_{n-1}Q_{r_0}[X_{n-2}]e_{n-2}[X_{n-2}]\big)[\partial_{n-1}]\Delta^{(r_0+1)}[X_{n-1}]\,. \qquad 4.35$$

Now 3.20 and (ii) of Proposition 1.6 give that for a suitable $A[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ we have

$$\tau_{n-1}Q_{r_0}[X_{n-2}]e_{n-2}[X_{n-2}] \;=\; Q_{r_0}[X_{n-1}]e_{n-2}[X_{n-1}] \;+\; A[X_{n-1}]\,e_{n-1}[X_{n-1}]\,.$$

Substituting this in 4.35 gives

$$\tau'_{n-1}Q_{r_0}[\partial_{n-2}]e_{n-2}[\partial_{n-2}]\Delta^{(r_0+1)}[X_{n-1}] \;=\; \big(Q_{r_0}[\partial_{n-1}]e_{n-2}[\partial_{n-1}] \;+\; A[\partial_{n-1}]\,e_{n-1}[\partial_{n-1}]\big)\Delta^{(r_0+1)}[X_{n-1}]\,.$$

Now it is easily verified that

$$e_{n-1}[\partial_{n-1}]\big)\Delta^{(r_0+1)}[X_{n-1}] \;=\; 0\,,$$

while for a suitable constant C we have

$$e_{n-2}[\partial_{n-1}]\,\Delta^{(r_0+1)}[X_{n-1}] \;=\; \begin{cases} 0 & \text{if } r_o > 0\ ,\\ C\,\Delta[X_{n-1}] & \text{if } r_0 = 0\ .\end{cases}$$

Combining this result with 4.34 we finally obtain

$$\tau'_{n-1}Q_{r_0}[\partial_{n-2}]\Delta^{(r_0)}[X_{n-1}] \;=\; \begin{cases} 0 & \text{if } r_o > 0\ ,\\ C'\,Q_0[\partial_{n-1}]\Delta[X_{n-1}] & \text{if } r_0 = 0\ .\end{cases} \qquad 4.36$$

for possibly another constant C'.

Now suppose first that $r_o = 0$. In this case, combining 4.36 with 4.33 and 4.32 we derive that $h[X_{n-1}]$ may be rewritten in the form

$$h[X_{n-1}] \;=\; C'Q_0[\partial_{n-1}]\Delta[X_{n-1}] \;+\; \sum_{r=1}^{n-2}\tau'_{n-1}P'_r[\partial_{n-2}]\Delta^{(r)}[X_{n-2}]$$

with a new set of polynomials

$$P'_r[X_{n-2}] = \sum_{\epsilon \in \mathcal{SUB}_{n-2}} a'_{\epsilon,r}\, x^{\epsilon}\,.$$

We can now work with the difference

$$k[X_{n-1}] \;=\; h[X_{n-1}] - C'Q_0[\partial_{n-1}]\Delta[X_{n-1}] \;=\; \sum_{r=1}^{n-2} \tau'_{n-1} P'_r[\partial_{n-2}]\Delta^{(r)}[X_{n-2}]\,. \tag{4.37}$$

Applying the same reasoning to $k[X_{n-1}]$, we see that the first alternative in 4.36 will apply in this case and we will be able to rewrite $k[X_{n-1}]$ in the form

$$k[X_{n-1}] \;=\; \sum_{r=2}^{n-2} \tau'_{n-1} P''_r[\partial_{n-2}]\Delta^{(r)}[X_{n-2}]\,.$$

with a second set of polynomials

$$P''_r[X_{n-2}] = \sum_{\epsilon \in \mathcal{SUB}_{n-2}} a''_{\epsilon,r}\, x^{\epsilon}\,.$$

Clearly we are in a position to repeat this process and reduce the expansion of $k[X_{n-1}]$ to a single summand. To be precise we ultimately obtain that for some polynomial $R[X_{n-1}]$ we have

$$k[X_{n-1}] \;=\; \tau'_{n-1} R[\partial_{n-2}]\Delta[X_{n-2}]\,.$$

But now we can use the further determinantal identity

$$\Delta[X_{n-2}] \;=\; \tfrac{1}{(n-2)!}\, \partial_{x_1}\partial_{x_1}\cdots\partial_{x_{n-2}}\Delta[X_{n-1}]$$

and obtain that

$$\begin{aligned} k[X_{n-1}] &= \tfrac{1}{(n-2)!}\, \tau'_{n-1} R[\partial_{n-2}]\partial_{x_1}\partial_{x_2}\cdots\partial_{x_{n-2}}\Delta[X_{n-1}] \\ &= \tfrac{1}{(n-2)!}\, \Big(\tau_{n-1} R[X_{n-2}]e_{n-2}[X_{n-2}]\Big)[\partial_{n-1}]\Delta[X_{n-1}] \\ \text{(by Proposition 3.1)} \quad &= \tfrac{1}{(n-2)!}\, \Big(\mathcal{E}_{n-1} R[X_{n-2}]e_{n-2}[X_{n-2}]\Big)[\partial_{n-1}]\Delta[X_{n-1}] \\ &= Q_1[\partial_{n-1}]\Delta[X_{n-1}] \end{aligned} \tag{4.38}$$

where for convenience we have set

$$Q_1[X_{n-1}] \;=\; \tfrac{1}{(n-2)!}\, \Big(\mathcal{E}_{n-1} R[X_{n-2}]e_{n-2}[X_{n-2}]\Big)\,.$$

Combining 4.38 with 4.37 we finally derive that

$$h[X_{n-1}] \;=\; C'Q_0[\partial_{n-1}]\Delta[X_{n-1}] \;+\; Q_1[\partial_{n-1}]\Delta[X_{n-1}]\,.$$

Since $Q_1[X_{n-1}]$ is Quasi-Symmetric this final identity completes the induction and the proof of the theorem.

5. The alternate proof of Theorem A.

In this section we will use an alternate approach for establishing that

$$\mathcal{QS}_n \text{ is a free module over } \Lambda_n.$$

We will prove this by showing that Theorem A is equivalent to Theorem I.2 and then proving Theorem I.2. In the course of the derivation several results will be proved that show that the algorithm in the introduction is forced by the structure of the algebra of quasi-symmetric polynomials.

A few caveats concerning the contents of this section are necessary at this point. To begin with, some of the constructs introduced in previous sections will be dealt with here with a slightly different notation. The style and the tools used may be more germane to contemporary commutative algebra litterature than the algebraic combinatorial litterature of recent years. We hope that this lack of uniformity in the paper will make its contents accessible to a wider audience.

Recall that if X_n is the alphabet $x_1, ..., x_n$ then we have the elementary symmetric functions $e_1[X_n], ..., e_n[X_n]$. If the alphabet is understood we will drop here the $[X_n]$. We will also look upon $\mathbb{Q}[X_n]$ as a module for $e_1, ..., e_{n+1}$ with e_{n+1} acting by 0. We denote by φ_n the map of $\mathbb{Q}[X_n]$ onto $\mathbb{Q}[X_{n-1}]$ given by $\varphi_n(f)(x_1, ..., x_{n-1}) = f(x_1, ..., x_{n-1}, 0)$. We also recall the map $\mathcal{E}_n : \mathcal{QS}_{n-1} \to \mathcal{QS}_n$ given by $\mathcal{E}_n(m_{[p_1,...,p_k]}[X_{n-1}]) = m_{[p_1,...,p_k]}[X_n]$. Then $\varphi_n\mathcal{E}_n(f) = f$ for $f \in \mathcal{QS}_{n-1}$ (see Proposition 1.5). We will also denote by π the natural projection of $\mathbb{Q}[X_n]$ onto $\mathbb{Q}[X_n]/\sum e_i\mathbb{Q}[X_n]$.

We observe that we have an exact sequence of $e_1, ..., e_{n+1}$ modules

$$0 \quad \to \quad \mathbb{Q}[X_{n+1}] \quad \overset{e_{n+1}}{\to} \quad \mathcal{QS}_{n+1} \quad \overset{\varphi_{n+1}}{\to} \quad \mathcal{QS}_n \quad \to \quad 0.$$

Here the first map is given by multiplication by e_{n+1} and e_{n+1} acts by 0 on the last space.

If M is a vector space over $\mathbb{Q}$ that is a module for commuting operators $e_1, ..., e_n$ then we can form the Koszul complex of M as follows. We consider the free vector space V over $\mathbb{Q}$ with basis $e_1, ..., e_n$. We set $C_r([e_1, ..., e_n]M) = M \bigotimes \bigwedge^r V$ if $e_1, ..., e_n$ are understood we will just write $C_r(M)$. We define the coboundary operator $\partial : C_r(M) \to C_{r-1}(M)$ by

$$\partial(m \bigotimes e_{j_1} \bigwedge e_{j_2} \bigwedge \cdots \bigwedge e_{j_r}) = \sum_{i=1}^{r} (-1)^{i+1} e_{j_i} m \bigotimes e_{j_1} \bigwedge e_{j_2} \bigwedge \cdots \bigwedge \hat{e}_{j_i} \cdots \bigwedge e_{j_r}$$

here (as is usual) the " hat" means remove. The r-th homology of this complex will be denoted $H_r([e_1, ..., e_n], M)$ or $H_r(M)$ if the e_i are understood. In particular we have

$$H_0(M) = M/\sum e_i M;$$

$$H_1(M) = \{\sum m_i \bigotimes e_i | \sum e_i m_i = 0\}/\partial(C_2(M)).$$

As usual, $\mathcal{QS}_n$ and Λ_n are graded by degree and we look upon $\mathcal{QS}_n$ as a graded Λ_n-module. The standard theory of the Koszul complex (*cf.*H. Matsumura, Commutative Ring Theory, Cambridge,1986, Theorem 16.5) implies

Lemma 5.1. *The Λ_n-module $\mathcal{QS}_n$ is free if and only if*

$$H_1([e_1[X_n], ..., e_n[X_n]], \mathcal{QS}_n) = 0.$$

We note that $\mathcal{QS}_1 = \Lambda_1$. We will now assume (until further notice) that we know that $\mathcal{QS}_m$ is free as a Λ_m module for all $m \leq n$. We will now embark on an inductive proof of the freeness for $\mathcal{QS}_{n+1}$. The exact sequence above leads to the long exact sequence

$$\begin{aligned} H_1(\mathbb{Q}[X_{n+1}]) \quad \rightarrow \quad H_1(\mathcal{QS}_{n+1}) \quad \rightarrow \quad H_1([e_1[X_n], ..., e_{n+1}[X_n]], \mathcal{QS}_n) \quad \rightarrow \\ \rightarrow \quad H_0(\mathbb{Q}[X_{n+1}]) \quad \rightarrow \quad H_0(\mathcal{QS}_{n+1}) \quad \rightarrow \quad H_0(\mathcal{QS}_n) \quad \rightarrow \quad 0\,. \end{aligned}$$

We first observe that the previous lemma and the fact that $\mathbb{Q}[X_{n+1}]$ is a free Λ_n-module implies that $H_1(\mathbb{Q}[X_{n+1}]) = 0$. We next calculate

$$H_1([e_1[X_n], ..., e_{n+1}[X_{n+1}]], \mathcal{QS}_n)$$

and the connecting homomorphism. We first note that

$$H_0([e_1[X_n], ..., e_{n+1}[X_{n+1}]], \mathcal{QS}_n) = H_0([e_1[X_n], ..., e_n[X_n]], \mathcal{QS}_n).$$

We are assuming that $\mathcal{QS}_n$ is a free Λ_n module and the lemma above imply that if $m_i \in \mathcal{QS}_n$ and $\sum_{i \leq n+1} e_i m_i = 0$ then (since $e_{n+1} m_{n+1} = 0$) there exists

$$v \in \bigoplus_{1 \leq i < j \leq n} M \bigotimes e_i \bigwedge e_j$$

with $\partial v = \sum_{i \leq n} m_i \bigotimes e_i$. Thus modulo $\partial C_1(\mathcal{QS}_n)$ every class in

$$H_1([e_1[X_n], ..., e_{n+1}[X_{n+1}]], \mathcal{QS}_n)$$

is represented by an element of the form $m \bigotimes e_{n+1}$. We note that if $u_i \in \mathcal{QS}_n$ then we have $\partial \sum_{i<n+1} u_i \bigotimes e_i \bigwedge e_n = \sum_{i<n} e_i u_i \bigotimes e_n$. We therefore see that

$$H_1([e_1[X_n], ..., e_{n+1}[X_{n+1}]], \mathcal{QS}_n) = H_0(\mathcal{QS}_n) \bigotimes e_{n+1}.$$

We next calculate the connecting homomorphism. Let $v = m \bigotimes e_{n+1}$ be a representative of a class in $H_1([e_1[X_n], ..., e_{n+1}[X_{n+1}]], \mathcal{QS}_n)$. Then $m = \varphi_{n+1}(u)$ with $u \in \mathcal{QS}_{n+1}$. We can take $u = \mathcal{E}_{n+1}(m)$. Then the element $e_{n+1}\mathcal{E}_{n+1}(m)$ is in the image of the map of $\mathbb{Q}[X_{n+1}]$ to $\mathcal{QS}_{n+1}$.

Hence the connecting homomorphism is just $m \bigotimes e_{n+1} \longmapsto \pi\mathcal{E}_{n+1}(m)$. We therefore have the exact sequence

$$\begin{array}{ccccccc} 0 & \to & H_1(\mathcal{QS}_{n+1}) & \to & H_0(\mathcal{QS}_n)\bigotimes e_{n+1} & \to & \\ & & & & \to\ H_0(\mathbb{Q}[X_{n+1}]) & \to & H_0(\mathcal{QS}_{n+1}) \quad \to \quad H_0(\mathcal{QS}_n) \quad \to \quad 0 \end{array}$$

with the connecting homomorphism (which will be called δ) as described. At this point we have the following

Proposition 5.1.

The map $u \longmapsto \mathcal{E}_{n+1}(u) + \sum_{i\leq n+1} e_i\,\mathbb{Q}[X_{n+1}]$ from $H_0(\mathcal{QS}_n)$to $H_0(\mathbb{Q}[X_{n+1}])$is well defined. Furthermore, $\mathcal{QS}_{n+1}$is a free Λ_{n+1}-module if and only if this map is injective. Furthermore, if it is free it is free on $(n+1)!$ generators.

Proof.

If the map is injective then, since its kernel is $H_1(\mathcal{QS}_{n+1})$, $H_1(\mathcal{QS}_{n+1}) = 0$. If $H_1(\mathcal{QS}_{n+1}) = 0$ then the map is injective. The last assertion is also clear since $\dim H_0(\mathbb{Q}[X_{n+1}]) = (n+1)!$ and if $H_1(\mathcal{QS}_{n+1}) = 0$ then $\dim H_0(\mathcal{QS}_{n+1}) = \dim H_0(\mathbb{Q}[X_{n+1}])$.

Remark 5.1.

The above exact sequence shows that our construction of the basis in I.9 is essentially the " only way" of getting such a basis. This observation will become clearer in the proof of the next proposition.

We will now use Proposition 5.1 to establish a more direct relationship with the chain of ideas in the previous sections. Recall that $\mathbf{H}_n$ denotes the space of S_n-harmonic polynomials in $\mathbb{Q}[X_n]$.

Proposition 5.2.

A necessary and sufficient condition for $\mathcal{QS}_{n+1}$to be a free module over the ring of invariants for the symmetric group is that

$$\dim \ker \pi \circ \mathcal{E}_{n+1|e_n\mathbf{H}_n} \leq (n-1)! \tag{5.1}$$

Proof.

If we apply the above exact sequence to the case of n then we see that since we are assuming that $H_1(\mathcal{QS}_n) = 0$ we have the exact sequence

$$0 \quad \to \quad H_0(\mathcal{QS}_{n-1})\bigotimes e_n \quad \to \quad H_0(\mathbb{Q}[X_n]) \quad \to \quad H_0(\mathcal{QS}_n) \quad \to \quad H_0(\mathcal{QS}_{n-1}) \quad \to \quad 0\,.$$

We note that this implies that the dimension of the kernel of the map $H_0(\mathbb{Q}[X_n]) \to H_0(\mathcal{QS}_n)$ induced by multiplication by e_n is $(n-1)!$ and that there is a subspace, W_n, of the S_n harmonic polynomials in $\mathbb{Q}[X_n]$ such that e_nW_n projects bijectively onto the image V_n of $H_0(\mathbb{Q}[X_n])$. It also implies that if apply $\pi \circ \mathcal{E}_n$ to $\mathcal{QS}_{n-1}$ we get a subspace of $H_0(\mathcal{QS}_n)$ complementary to the image of $H_0(\mathbb{Q}[X_n])$. Let G_n denote this subspace. Then $H_0(\mathcal{QS}_n) = G_n \oplus V_n$. Then we

note that $\pi \circ \mathcal{E}_{n+1}(G_n) \cap \pi \circ \mathcal{E}_{n+1}(e_n\mathbb{Q}[X_n]) = (0)$. Indeed, let f be in that intersection. Then since it is in $\pi \circ \mathcal{E}_{n+1}(V_n)$ we see that if g is a representative then $\varphi_n \circ \varphi_{n+1}(g) = 0$. But since it is in $\pi \circ \mathcal{E}_{n+1}(G_n)$ it has a representative of the form $\mathcal{E}_{n+1} \circ \mathcal{E}_n(h)$ with h in $\mathcal{QS}_{n-1}$. But $\varphi_n \circ \varphi_{n+1}(\mathcal{E}_{n+1} \circ \mathcal{E}_n(h)) = h$. This proves the assertion. If $H_1(\mathcal{QS}_{n+1}) = 0$ the dimension of the image of $e_n\mathbb{Q}[X_n]$ under $\pi \circ \mathcal{E}_{n+1}$ is at least $(n)! - (n-1)!$. Suppose it is greater. This implies that $\pi \circ \mathcal{E}_{n+1}(\mathcal{QS}_n)$ has dimension $d > (n)!$. Applying φ_{n+1} we see that $\dim H_0(\mathcal{QS}_n) \geq d > n!$. But we are assuming that $\mathcal{QS}_n$ is free as a Λ_n module. This implies that it must have $n!$ generators as a free Λ_n-module. Thus if $\mathcal{QS}_{n+1}$ is free as a Λ_n module then the map $\pi \circ \mathcal{E}_{n+1}$ restricted to $e_n\mathbb{Q}[X_n]$ has an $(n-1)!$ dimensional kernel. This certainly proves the necessity.

We now prove the sufficiency. If $H_1(\mathcal{QS}_{n+1}) \neq 0$ then $\dim\ Im\pi \circ \mathcal{E}_{n+1}(\mathcal{QS}_n) < n!$. Since $G_n \cap e_n\mathbf{Q}[X_n] = (0)$ we see that

$$(G_n \bigoplus e_n\mathbf{H}_n) + \sum_i e_{i,}\mathcal{QS}_n = \mathcal{QS}_n.$$

This implies that $\dim \pi \circ \gamma_{n-1}(\mathcal{QS}_{n-1}) = (n-2)! + \dim \pi \circ \gamma_{n-1}(e_{n-1,n-1}H_{n-1})$. Now assuming the upper bound in 5.1 for the dimension of the kernel we see that

$$\dim \pi \circ \mathcal{E}_{n+1}(\mathcal{QS}_n) \geq (n-1)! + n! - \dim \ker \pi \circ \mathcal{E}_{n+1|e_n\mathbf{H}_n} \geq n!.$$

This contradiction implies the sufficiency.

As in the introduction we define

$$\tau_n \;=\; \sum_{i=1}^{n}(i, i+1, \ldots, n) \qquad \text{and} \qquad \tau'_n \;=\; \sum_{i=1}^{n}(-1)^{n-i}(i, i+1, \ldots, n)\,.$$

The basic role (for our purposes) of these elements of the group algebra of S_n is that if $f \in \mathbb{Q}[X_n]$ then (Proposition 3.1)

$$\mathcal{E}_n e_n f = \tau_n e_n f.$$

This implies that

$$\dim \ker \pi \circ \mathcal{E}_{n+1|\mathbf{H}_n} = \dim \ker(f \mapsto \pi\tau_{n+1}e_n f)$$

the latter being considered as a map from $\mathbf{H}_n$ to $\mathbb{Q}[X_{n+1}]/\sum e_i\mathbb{Q}[X_{n+1}]$.

The Vandermonde determinant (or discriminant) in X_n, that is $\prod_{1i<i<j\leq n}(x_i - x_j)$, will be denoted here Δ_n. If $f \in \mathbb{Q}[X_n]$ then we shall set here

$$\partial(f) \;=\; f(\partial_{x_1}, ..., \partial_{x_n})\,.$$

We recall that we have $f \in \sum_{i\leq n} e_i\mathbb{Q}[X_n]$ if and only if $\partial(f)\Delta_n = 0$. With this notation in mind we see that the condition of Proposition 5.2 is

$$\dim \ker(f \to \partial(\tau_{n+1}(e_n f))\,\Delta_{n+1}) \leq (n-1)!$$

where the above map is considered to be a map of H_{n-1} to H_n. We recall that we have

$$\partial(e_n)\Delta_{n+1} = (n+1)!\ \Delta_n\ .$$

This yields the identity (see the proof of Proposition 3.2)

$$\partial(\tau_{n+1}(e_n f))\,\Delta_{n+1}) = (n+1)!\ \tau'_{n+1}(\partial(f)\Delta_n)$$

for $f \in \mathbb{Q}[X_n]$. Finally we note that $\partial(\mathbf{H}_n)\Delta_n = \mathbf{H}_n$. Putting this together we see that Proposition 5.2 can be refomulated as

Proposition 5.3.

A necessary and sufficient condition that $\mathcal{QS}_{n+1}$ be a free module over Λ_{n+1} is that

$$\dim\ker\tau'_{n+1|\mathbf{H}_n} \leq (n-1)!.$$

As in section 4 we introduce the differential operators $D_n^{(j)} = \sum_{1\leq k\leq n} \partial_{x_k^j}$, we will write $D^{(j)}$ if we are differentiating all of the variables. We have

Lemma 5.2.

We have $f \in \ker\tau'_{n+1|H_n}$ if and only if $f = e^{-x_n D^{(1)}_{n-1}}\phi$ with $\phi \in \mathbb{Q}[X_{n-1}]$ satisfying the conditions

$$(1)\quad (-\tau'_n + e^{-x_n D^{(1)}_{n-1}})\,\phi = 0$$

$$(2)\quad (D^{(j)} + (-D^{(1)})^j)\,\phi = 0 \text{ for } j \geq 2\,.$$

Proof.

Let $f \in \mathbb{Q}[X_n]$ be such that $D^{(1)}f = 0$. We write $f = \sum f_j x_n^j$ with $f_j \in \mathbb{Q}[X_{n-1}]$. Then the condition that $D^{(1)}f = 0$ means that

$$D^{(1)}f_j = -(j+1)f_{j+1}.$$

Thus if we set $\phi = f_0$ then $f = e^{-x_n D^{(1)}_{n-1}}\phi$. The converse is also easily checked, that is if f is given as $e^{-x_n D^{(1)}_{n-1}}\phi$ then $D^{(1)}f = 0$. We are now ready to prove the Lemma. We note that

$$\tau'_{n+1}e^{-x_n D^{(1)}_{n-1}}\phi = e^{-x_n D^{(1)}_{n-1}}\phi - e^{-x_{n+1} D^{(1)}_{n}}\tau'_n\phi\,.$$

Thus if $f = e^{-x_n D^{(1)}_{n-1}}\phi$ is such that $\tau'_{n+1}f = 0$, comparing coefficients the powers of x_{n+1}, we get

$$0 = e^{-x_n D^{(1)}_{n-1}}\phi - \tau'_n\phi\,.$$

Thus $f = e^{-x_n D^{(1)}_{n-1}}\phi$ satisfies $\tau'_{n+1}f = 0$ if and only if condition (1) is satisfied. We next look at the condition that f is harmonic. We note that

$$D^{(j)}e^{-x_n D^{(1)}_{n-1}}\phi = e^{-x_n D^{(1)}_{n-1}}((D^{(j)} + (-D^{(1)})^j)\phi).$$

Thus $f = e^{-x_n D^{(1)}_{n-1}}\phi$ is harmonic if and only if the condition (2) is satisfied.

We are finally ready to complete the induction that has been weighing on us throughout this section. That is to say we will now prove $\dim\ker\tau'_{n+1|\mathbf{H}_n} \leq (n-1)!$. We first observe that one can check that this is true directly and without much difficulty for $n \leq 2$. We also recall that we are assuming that $\mathcal{QS}_m$ is free as a Λ_m module for all $m \leq n$.

Let V denote the space of a ϕ satisfying the two conditions of Lemma 5.2. Let V_j denote the subspace of V consisting of those elements $\phi \in V$ such that $(D^{(1)})^j\phi = 0$. We write $c_{n-1} = \dim\ker(\tau'_n - I)_{|\mathbf{H}_{n-1}}$. We prove by induction on j that $\dim V_j \leq (j-1)(n-2)! + c_{n-1}$. We first look at V_1. Then if $\phi \in V_1$ we have $\phi \in \mathbf{H}_{n-1}$ and

$$\tau'_n\phi = \phi.$$

Then $\dim V_1 = c_{n-1}$. Now consider $\phi \in V_2$ then

$$-\tau'_n\phi + \phi - x_n\, D^{(1)}\phi = 0.$$

If $\phi = \sum_{j\geq 0} \phi_j x_{n-1}^j$ with $\phi_j \in \mathbb{Q}[X_{n-2}]$ then we have

$$-\phi + \sum_{j\geq 0} \tau'_{n-1}(\phi_j)\, x_n^j + \phi - x_n\, D^{(1)}\phi = 0\,.$$

Thus

$$\tau'_{n-1}\phi_0 = 0 \qquad \text{and} \qquad D^{(1)}\phi = \tau'_{n-1}\phi_1\,.$$

Thus we have $D^{(1)}\phi \in \mathbf{H}_{n-1}$ and $\tau'_n D^{(1)}\phi = 0$. This implies by the inductive hypothesis that $\dim D^{(1)}V_2 \leq (n-2)!$. Since $\ker D^{(1)}{}_{|V_2} = V_1$ we see that $\dim V_2 \leq (n-2)! + c_{n-1}$. Suppose that we have shown that

$$\dim V_j \leq (j-1)(n-2)! + c_{n-1}.$$

Let us consider $\alpha = (D^{(1)})^j\phi$ with $\phi \in V_{j+1}$. If we write $\phi = \sum_j \phi_j x_{n-1}^j$ (as usual) and write out condition (1) above we have

$$\tau'_{n-1}(\phi_j) = \pm(D^{(1)})^j\phi = \pm\alpha.$$

Thus $\alpha \in \ker\tau'_{n|H_{n-1}}$. Thus $\dim(D^{(1)})^j V_{j+1} \leq (n-2)!$. Now $\ker(D^{(1)})^j{}_{|V_{j+1}} = V_j$. Hence

$$\dim V_{j+1} \leq \dim V_j + (n-2)!\,.$$

Thus to complete the proof we must show that $c_{n-1} \leq (n-2)!$. This will be proved by an argument analogous to the one above. If $\xi \in \mathbb{Q}[X_{n-2}]$ and if $\phi = e^{-x_{n-1}D^{(1)}_{n-2}}\xi$ then $\phi \in \ker(\tau'_n - I)_{|\mathbf{H}_{n-1}}$ if and only if ξ satisfies the two conditions

$$\begin{aligned} &a) \quad \tau'_{n-1}\,\xi = 0 \\ &b) \quad \left(D^{(j)} \;+\; (-D^{(1)})^j\right)\,\xi = 0\,. \end{aligned}$$

Let W be the space of those elements ξ of $\mathbb{Q}[X_{n-2}]$ that satisfy (a) and (b). Let $W_j = \{\phi \in W | (D^{(1)})^j \phi = 0\}$. We now show that $\dim W_j \leq j(n-3)!$ by induction on j. This will complete the proof since $W_{n-2} = W$. We have $W_1 = \ker \tau'_{n-1|\mathbf{H}_{n-2}}$, so $\dim W_1 = (n-3)!$ by the inductive hypothesis. Now $(D^{(1)})^j W_{j+1} \subset W_1$ and $\ker (D^{(1)})^j{}_{|W_{j+1}} = W_j$. Thus $\dim W_{j+1} \leq \dim W_j + (n-3)!$.

The proof is complete.

6. Final Remarks and identities

It should be apparent at this point that our algorithm for constructing Λ_n-module bases for $\mathcal{QS}_n$ follows closely the recursion satisfied by the the numereator of its Hilbert series. Namely

$$P_n(q) \;=\; P_{n-1}(q) \;+\; q^n \left([n]_q! - P_{n-1}(q) \right). \tag{6.1}$$

Now it happens that there is a very intriguing further recursion satisfied by the polynomial $P_n(q)$ which we have been unable to translate into an alternate algorithm for proving the freeness of $\mathcal{QS}_n$. We shall present here since it may be conducive to further findings concerning this remarkable algebra. More precisely we have the following result

Theorem 6.1

The expression

$$Q_n(q) \;=\; \frac{[n+1]_q! - P_n(q)}{[n]_q} \tag{6.2}$$

yields a polynomial in $\mathbb{N}[q]$. *In fact, the pair* $P_n(q)$, $Q_n(q)$ *satisfies the following recursions*

$$\begin{aligned} a) \quad P_n(q) &= P_{n-1}(q) + q^n [n-1]_q Q_{n-1}(q) \\ b) \quad Q_n(q) &= q\, P_{n-1}(q) + [n-1]_q Q_{n-1}(q) \end{aligned} \tag{6.3}$$

with initial conditions $P_1(q) = 1$ *and* $Q_1(q) = q$. *In particular we derive that*

$$P_n(q) \;=\; P_{n-1}(q) \;+\; q^{n+1} \sum_{r=2}^{n} [n-1]_q \cdots [n-r+1]_q \, P_{n-r}(q). \tag{6.4}$$

Proof

Substituting 6.1 into 6.2 gives

$$\begin{aligned} Q_n(q) &= \frac{1}{[n]_q} \Big([n+1]_q - P_{n-1}(q) - q^n [n]_q! + q^n P_{n-1}(q) \Big) \\ &= \frac{1}{[n]_q} \Big([n]_q [n]_q - (1-q^n) P_{n-1}(q) \Big) \\ &= [n]_q - (1-q) P_{n-1}(q) \\ &= [n]_q - P_{n-1}(q) + q\, P_{n-1}(q) \\ \Big(\text{by 6.2 for } n \to n-1 \Big) &= [n-1]_q Q_{n-1}(q) + q\, P_{n-1}(q). \end{aligned}$$

This proves 6.3 b). Clearly, 6.3 a) is obtained by combining 6.1 with 6.2 for $n \to n-1$. We have seen in 1.9 that $P_1(q) = 1$ and then 6.2 for $n = 1$ gives $Q_1(q) = q$. The identity in 6.4 is obtained by combining 6.3 a) with recursive applications of 6.3 b).

Remark 6.1

The recursion in 6.4 suggests the existence of a very special Λ_n-basis for $\mathcal{QS}_n$ consisting of a Λ_{n-1}-basis for $\mathcal{QS}_{n-1}$ followed by certain Artin monomials multiplying e_n times Λ_{n-r}-bases for $\mathcal{QS}_{n-r}$. Such a basis might be found if we could only find a setting that explains why the ratio in 6.2 turns out to be polynomial with non-negative integer coefficients.

In the context of special bases we should mention that a substantial part of our computer data was obtained using the conjectured Λ_n-basis of Bergeron-Reutenauer [4]. Assuming the validity of their conjecture we were able to carry out computer explorations of a size that appeared forbidding by any other means. The construction of the Bergeron-Reutenauer basis may be obtained by a process which closely follows the constrution of the polynomials $\Pi[X_n]$ defined in 1.12 and 1.13 with Artin monomials replaced by "*descent monomials*".

Let us recall that for a given permutation $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ we define the descent monomial $d_\sigma[X_n]$ corresponding to σ by setting

$$d_\sigma[X_n] \;=\; \prod_{\substack{1\le i\le n-1\\ \sigma_i>\sigma_{i+1}}} x_{\sigma_1}x_{\sigma_2}\cdots x_{\sigma_i}\,. \tag{6.9}$$

This given let

$$\mathcal{DM}[X_n] \;=\; \sum_{\sigma\in S_n} d_\sigma[X_n] \tag{6.9}$$

and recursively define the sequence of polynomials $\Xi[X_n]$ by setting

$$\begin{aligned} &1)\quad \Xi[X_1] = 1\,,\\ &2)\quad \Xi[X_n] = \Xi[X_{n-1}] \;+\; x_1x_2\cdots x_n\big(\mathcal{DM}[X_n] \;-\; \Xi[X_{n-1}]\big) \end{aligned} \tag{6.10}$$

It develops that there is a complete analogue of Proposition 1.2. More precisely, setting

$$\mathcal{S}_n \;=\; \{p : x^p \quad \text{is a summand in } \Xi[X_n]\,\} \tag{6.11}$$

we have

Proposition 6.1

a) Each $\Xi[X_n]$ is a sum of $n!$ distinct monomials each of which is a summand in $\mathcal{DM}[X_{n+1}]$

b) The compositions in $\mathcal{S}_n$ have length $\le n$.

c) $\sum_{p\in\mathcal{S}_n} q^{|p|} = P_n(q)$.

Proof

It is easily seen that multiplication by $x_1, x_2, \ldots, x_n$ of a descent monomial in $\mathcal{DM}[X_n]$ yields a descent monomial in $\mathcal{DM}[X_{n+1}]$. Thus if we inductively assume a) and b) to be true for $n-1$ the recursions in 6.10 will assure them to be true for n. It is well known that we have

$$\sum_{\sigma\in S_n} q^{maj(\sigma)} \;=\; [n]_q!$$

where "$maj(\sigma)$" denotes the major index of σ. Since degree$(d_\sigma[X_n]) = maj(\sigma)$, setting all the variables x_i equal to q in 6.10 2) yields the recursion

$$\sum_{p\in\mathcal{S}_n} q^{|p|} \;=\; \sum_{p\in\mathcal{S}_{n-1}} q^{|p|} \;+\; q^n\Big([n]_q! - \sum_{p\in\mathcal{S}_{n-1}} q^{|p|}\Big)\,.$$

This proves that both sides of c) satify the same recursion. Since both sides satisfy the same initial conditions the identity in must hold true for all n. This completes our proof.

F. Bergeron and C. Reutenauer conjectured that the collection

$$\mathcal{B}_n \;=\; \big\{m_p[X_n]\big\}_{p\in\mathcal{S}_n} \tag{6.12}$$

is a Λ_n-basis for $\mathcal{QS}_n$. Note that if we set

$$DZ_n \;=\; \Big\{p : x^p \text{ is a monomial in } \mathcal{DM}[X_n] - \Xi[X_{n-1}]\Big\}$$

then the collection in 6.12 may be written in the form

$$\mathcal{B}_n \;=\; \Big\{m_p[X_n]\Big\}_{p\in\mathcal{S}_{n-1}} \cup \Big\{\,e_n[X_n]\,x^p\Big\}_{p\in DZ_n} \tag{6.13}$$

which is completely analogous to the bases constructed in the proof of Theorem 2.1. Now we conjecture that the collection

$$\mathcal{A}_n \;=\; \Big\{m_p[X_n]\Big\}_{p\in\mathcal{S}_{n-1}} \cup \Big\{\,x^p\Big\}_{p\in DZ_n} \tag{6.14}$$

is itself a Λ_n-basis for $\mathbb{Q}[X_n]$. We have extensive data in support of this conjecture. In particular, computer explorations based on its validity, predicted a variety of facts which eventually led us to the proof of Theorem I.3.

Remark 6.2

It was shown in [6] that the collection of descent monomials $\{d_\sigma[X_n]\}_{\sigma\in S_n}$ is a basis for the quotient $\mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)$. In going from $\{d_\sigma[X_n]\}_{\sigma\in S_n}$ to $\mathcal{A}_n$ we see from 6.10 2) that we have replaced the monomials in $\mathcal{DM}[X_n]$ that are in $\{x^p\}_{p\in\mathcal{S}_{n-1}}$ by the quasi-monomials in $\{m_p[X_n]\}_{p\in\mathcal{S}_{n-1}}$. If this replacement caused no loss of independence modulo $(e_1, e_2, \ldots, e_n)$, then $\mathcal{A}_n$ would necessarily be a Λ_n-basis for $\mathbb{Q}[X_n]$ and it would then follow that $\mathcal{B}_n$ is a Λ_n-basis for $\mathcal{QS}_n$. The same conclusions can be drawn if $\mathcal{A}_n$ is shown to span the quotient $\mathbb{Q}[X_n]/(e_1, e_2, \ldots, e_n)$. Thus a direct proof of the independence or the spanning property of $\mathcal{A}_n$ not only would provide a further proof of the freeness of $\mathcal{QS}_n$ but it would as well establish the Bergeron-Reutenauer conjecture about $\mathcal{B}_n$.

We should also mention that our quest for a proof of Theorem I.2 led us to a most surprising fact concerning the action of the group algebra element τ_n'. In fact, some of the identities proved in Section 4. yield the following result

Theorem 6.2

The Hilbert series $F_{\mathbf{K}_n}(q)$ of the kernel of τ'_n as a map from $\mathbb{Q}[X_{n-1}]$ into $\mathbb{Q}[X_n]$ satisfies the following recursion.

$$F_{\mathbf{K}_n}(q) + F_{\mathbf{K}_{n-1}}(q) \;=\; \frac{1}{(1-q)^{n-2}} \tag{6.15}$$

with initial condition $F_{\mathbf{K}_2}(q) = 1$. In particular it follows that

$$F_{\mathbf{K}_n}(q) \;=\; E(n) + \sum_{r=1}^{n-2} E(n-r)\,\frac{q}{(1-q)^r} \tag{6.16}$$

with $E(m) = 1 - m \bmod 2$.

Proof

From Propositions 4.2 and 4.5 it follows that a polynomial $h[X_{n-1}] \in \mathbb{Q}[X_{n-1}]$ is in the kernel of τ'_n if and only if

$$h[X_{n-1}] \;=\; \tau'_{n-1}\, h_o[X_{n-2}] \tag{6.17}$$

with

$$h_o[X_{n-2}] \;\in\; \mathbb{Q}[X_{n-2}]\,. \tag{6.18}$$

Indeed, any polynomial $h[X_{n-1}]$ of degree m in x_{n-1} may be written in the form given by 4.19 that is

$$h[X_{n-1}] \;=\; \sum_{r=0}^{m} (-1)^r x_{n-1}^r h_r[X_{n-2}]\,.$$

If $\tau'_n\, h[X_{n-1}] = 0$ formula 4.20 (for $x_n = 0$) gives 6.17 with 6.18. Conversely, if 6.17 and 6.18 hold true then 4.2 a) of Proposition 4.2 gives $\tau'_n\, h[X_{n-1}] = 0$. In other words the kernel of τ'_n is the range of τ'_{n-1}. Since the Hilbert series of the co-kernel and the range of τ'_{n-1} on $\mathbb{Q}[X_{n-2}]$ are the same we must necessarily have

$$F_{\mathbf{K}_n}(q) \;=\; \frac{1}{(1-q)^{n-2}} \;-\; F_{\mathbf{K}_{n-1}}(q)\,. \tag{6.19}$$

This proves 6.15. Now 6.16 is trivially true for $n = 2$ because the only elements of $\mathbb{Q}[X_1]$ that are killed by τ'_2 are the constants. We can thus proceed by indution and assume 6.16 true for $n-1$, and 6.19 gives

$$F_{\mathbf{K}_n}(q) \;=\; \frac{1}{(1-q)^{n-2}} \;-\; E(n-1) - \sum_{r=1}^{n-3} E(n-1-r)\,\frac{q}{(1-q)^r}\,. \tag{6.20}$$

Since for $n \geq 3$ we have

$$\frac{1}{(1-q)^{n-2}} \;=\; 1 + \frac{q}{1-q} + \frac{q}{(1-q)^2} + \cdots + \frac{q}{(1-q)^{n-2}}$$

substituting this in 6.20 gives

$$F_{\mathbf{K}_n}(q) \;=\; 1 - E(n-1) + \sum_{r=1}^{n-3} \Big(1 - E(n-1-r)\Big)\,\frac{q}{(1-q)^r} \;+\; \frac{q}{(1-q)^{n-2}}.$$

and this is just another way of writing 6.16 completing the proof of the theorem.

What is surprising about the relation in 6.15 is that it yields the simplest mechanism for producing the polynomials $P_n(q)$. More precisely 6.15 implies that

Theorem 6.3

$$F_{\mathbf{K}_{n+2}}(q) \;=\; \frac{q^{\binom{n+1}{2}} P_n(1/q)}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{6.21}$$

Proof

For convenience set

$$F_{\mathbf{K}_{n+2}}(q) \;=\; \frac{\Gamma_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)}\,,$$

and 6.15 becomes

$$\frac{\Gamma_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)} \;+\; \frac{\Gamma_{n-1}(q)}{(1-q)(1-q^2)\cdots(1-q^{n-1})} \;=\; \frac{1}{(1-q)^n}\,.$$

Making the replacement $q \to 1/q$ we may thus rewrite this in the form

$$\frac{(-1)^n \Gamma_n(1/q)\; q^{\binom{n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} \;+\; \frac{\Gamma_{n-1}(1/q)\,(-1)^{n-1} q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^{n-1})} \;=\; \frac{(-1)^n q^n}{(1-q)^n}\,.$$

Multiplying both sides by $(1-q)(1-q^2)\cdots(1-q^n)$ converts this to

$$\begin{aligned} \Gamma_n(1/q)\; q^{\binom{n+1}{2}} \;&=\; (1-q)\Gamma_{n-1}(1/q)\, q^{\binom{n}{2}} \;+\; q^n\;[n]_q!\,. \\ &=\; \Gamma_{n-1}(1/q)\, q^{\binom{n}{2}} \;+\; q^n\Big(\;[n]_q! - \Gamma_{n-1}(1/q)\, q^{\binom{n}{2}}\Big)\,, \end{aligned} \tag{6.22}$$

which we can easily recognize to be the recursion satisfied by $P_n(q)$. Note that for $n = 3$ formula 6.17 gives

$$F_{\mathbf{K}_3}(q) \;=\; \frac{q}{1-q} \;=\; \frac{\Gamma_1(q)}{1-q}\,.$$

Therefore

$$q^{\binom{1+1}{2}}\Gamma_1(1/q) \;=\; 1 \;=\; P_1(q)\,.$$

Combining this with 6.22 yields the equality

$$q^{\binom{n+1}{2}}\Gamma_n(1/q) \;=\; P_n(q)\,, \tag{6.23}$$

completing the proof of the theorem.

The reappearence of $P_n(q)$ in this further context begs for an explanation. Comparing formula 3.10 with the equality

$$F_{\mathbf{K}_n}(q) \;=\; \frac{q^{\binom{n-1}{2}} P_{n-2}(1/q)}{(1-q)(1-q^2)\cdots(1-q^{n-2})}$$

suggests using the freeness of $\mathbb{Q}[X_{n-1}]$ over Λ_{n-1} to relate the kernel of τ_n' on $\mathbf{H}_{n-1}$ to kernel of τ_n' on $\mathbb{Q}[X_{n-1}]$. However we found this path fraught with technical difficulties.

The group algebra elements τ_n and τ_n' have appeared in previous literature. Indeed, denoting by L the left regular representation of S_n, it was shown in [9] that the matrix $L(\tau_n)$ is diagonable with eigenvalues

$$0\ ,\ 1\ ,\ 2\ ,\ \ldots\ ,\ n-2\ ,\ n\,. \tag{6.24}$$

It was later shown by Diaconis et al. in [5] that the multiplicity of i in $L(\tau_n)$ is equal to the number of permutations with i fixed points, beautifully explaining the absence of $n-1$ in 6.24.

In [5] Diaconis et al. imbed a conjugate of τ_n as the first member of a one-parameter family $\mathbf{B}_a$ of group algebra elements naturally arising in a card shuffling context. This yielded them an explicit formula for all the successive powers of τ_n and a number of interesting identities. It develops that these identities and further ones can be established in a very simple, elementary way. We will end this writing with a brief presentation of this further development.

If α and β are two words in an alphabet A, we denote by "$\alpha \cup\cup \beta$" the formal sum of all the words that can be obtained by shuffling α and β as it is done with two decks of cards. This given, let us set for $1 \le a \le n$

$$\mathbf{B}_a \;=\; \sum_{\alpha \in S_a} \alpha \cup\cup \beta_{a,n} \tag{6.25}$$

where a permutation $\alpha \in S_a$ is viewed here as a word in the alphabet $\{1, 2, \ldots, a\}$ and $\beta_{k,n}$ denotes the word $(a+1)(a+2)\cdots(n)$. It is easy to see that the right hand side of 6.25 is none other than the sum of all the permutations of S_n with $a+1, a+2, \ldots n$ occurring in their natural order. It is also seen that B_1 reduces to the sum of cycles

$$\mathbf{B}_1 \;=\; \sum_{i=1}^{n} (1, 2, \ldots, i)\,.$$

This implies that τ_n and $\mathbf{B}_1$ are conjugate elements of the group algebra of S_n. To be precise

$$\tau_n \;=\; \sigma^{(n)}\ \mathbf{B}_1\ \sigma^{(n)} \tag{6.26}$$

with $\sigma^{(n)}$ the top element of S_n.

Interpreting all the $\mathbf{B}_a$'s as elements of the group algebra of S_n we have the following basic identities.

Proposition 6.1

For $1 \leq a \leq n-1$

$$\mathbf{B}_1 \, \mathbf{B}_a \;=\; a\mathbf{B}_a \;+\; \mathbf{B}_{a+1}\,, \tag{6.27}$$

thus

$$\mathbf{B}_a \;=\; \mathbf{B}_1(\mathbf{B}_1 - \mathbf{I})(\mathbf{B}_1 - 2\mathbf{I}) \cdots (\mathbf{B}_1 - (a-1)\mathbf{I})\,, \tag{6.28}$$

where $\mathbf{I}$ *represents the identity permutation. This in turn gives*

$$\mathbf{B}_1^k \;=\; \sum_{r=1}^{k} S_{k,a}\, \mathbf{B}_a \tag{6.29}$$

where $S_{k,a}$ *as customary denotes the Stirling number of the second kind. We also have*

$$\mathbf{B}_a \times \mathbf{B}_b \;=\; \sum_{r=a\vee b}^{a+b} \binom{a}{r-b}\binom{b}{r-a}\,(a+b-r)!\;\mathbf{B}_r \tag{6.30}$$

Proof

The identity in 6.27 can be obtained by grouping the permutations σ occurring in $\mathbf{B}_a$ according to the value of σ_1. Clearly σ_1 can only take the values $1, 2, \ldots, a+1$. Each of the groups where $1 \leq \sigma_1 \leq a$ yield a term $\mathbf{B}_a$ upon multiplication by $\mathbf{B}_1$ and the group where $\sigma_1 = a+1$ is easily seen to give the term $\mathbf{B}_{a+1}$. This given, we can rewrite 6.27 in the form

$$\mathbf{B}_{a+1} \;=\; (\mathbf{B}_1 - a\,\mathbf{I})\mathbf{B}_a$$

and 6.28 then follows by iteration. Formulas 6.29 and 6.30 are immediate consequences of the classical identities (see [7] V. I p. 35)

$$x^k \;=\; \sum_{a=1}^{k} S_{k,a}\, x(x-1)(x-2)\cdots(x-a+1)$$

$$(x)\!\downarrow_a \times\, (x)\!\downarrow_b \;=\; \sum_{r=a\vee b}^{a+b} \binom{a}{r-b}\binom{b}{r-a}\,(a+b-r)!\;(x)\!\downarrow_r$$

where for convenience we have set

$$(x)\!\downarrow_k \;=\; x(x-1)(x-2)\cdots(x-k+1)\,.$$

Remark 6.3

We should note that formula 6.28 already implies that τ_n, as a group algebra element, is diagonable with eigenvalues a subset of the integers in 6.24. Indeed, since both $\mathbf{B}_{n-1}$ and $\mathbf{B}_n$ reduce to the sum of all the permutations of S_n, formula 6.27 for $a = n-1$ may be rewritten as

$$\mathbf{B}_1\mathbf{B}_{n-1} \;=\; n\;\mathbf{B}_{n-1}$$

and from 6.28 with $a = n-1$ we get that

$$\mathbf{B}_1(\mathbf{B}_1 - \mathbf{I})(\mathbf{B}_1 - 2\,\mathbf{I})\cdots(\mathbf{B}_1 - (n-2)\mathbf{I})(\mathbf{B}_1 - n\,\mathbf{I}) \;=\; 0$$

This implies the diagonability of $\mathbf{B}_1$. The diagonability of τ_n then follows from 6.26. To get more precise information about the multiplicities of the eigenvalues we may follow the argument given in [5]. This is based on the following identity of Diaconis et Al. which can be obtained here as Corollary of 6.29.

Proposition 6.3

$$\mathbf{B}_1^k \;=\; \sum_{i=1}^{n} i^k \,\frac{1}{i!}\sum_{a=i}^{n}\frac{(-1)^{a-i}}{(a-i)!}\,\mathbf{B}_a \qquad\qquad (\text{for } 1\le k\le n) \tag{6.31}$$

Proof

It is well know that the Stirling numbers of the second kink may be expressed in the form

$$S_{k,a} \;=\; \frac{k!}{a!}\,(e^t-1)^a\,\big|_{t^k}$$

Substituting this into 6.29 gives

$$\mathbf{B}_1^k \;=\; \sum_{a=1}^{k}\mathbf{B}_a\,\frac{k!}{a!}\,(e^t-1)^a\,\big|_{t^k} \tag{6.32}$$

However note that since for $a > k$ we clearly have

$$\frac{k!}{a!}\,(e^t-1)^a\,\big|_{t^k} \;=\; 0$$

we may rewrite 6.32 in the form

$$\begin{aligned}
\mathbf{B}_1^k &= \sum_{a=1}^{n}\mathbf{B}_a\,\frac{k!}{a!}\,(e^t-1)^a\,\big|_{t^k}\\
&= \sum_{a=1}^{n}\mathbf{B}_a\,\frac{k!}{a!}\sum_{i=0}^{a}\binom{a}{i}e^{it}(-1)^{a-i}\Big|_{t^k}\\
&= \sum_{a=1}^{n}\mathbf{B}_a\,\frac{k!}{a!}\sum_{i=0}^{a}\binom{a}{i}\frac{i^k}{k!}(-1)^{a-i}\\
&= \sum_{a=1}^{n}\mathbf{B}_a\,\sum_{i=1}^{a}\frac{i^k}{i!(k-i)!}(-1)^{a-i}
\end{aligned}$$

and 6.31 is obtained by changing order of summation.

It is easily shown that the operators

$$\mathbf{E}_i \;=\; \frac{1}{i!}\sum_{a=i}^{n}\frac{(-1)^{a-i}}{(a-i)!}\,\mathbf{B}_a$$

do sum to the identity. Moreover it follows from 6.31 that they are the orthogonal projections onto the eigenspaces of $\mathbf{B}_1$. Denoting by L the left regular represedntation of S_n, it follows from this that the multiplicity of i in the matrix $L(\mathbf{B}_1)$ is given by the trace of $L(\mathbf{E}_i)$. Now we have

$$trace\ L(\mathbf{E}_i) \;=\; \frac{1}{i!}\sum_{a=i}^{n}\frac{(-1)^{a-i}}{(a-i)!}\ trace\, L(\mathbf{B}_a)$$

and since the left regular represenation has trace zero except at the identity this formula reduces to

$$\begin{aligned} trace\ L(\mathbf{E}_i) &= \frac{1}{i!}\sum_{a=i}^{n}\frac{(-1)^{a-i}}{(a-i)!}\ n! \\ &= \frac{n!}{i!}\sum_{a=0}^{n-i}\frac{1}{a!}\,(-1)^a \\ &= \binom{n}{i}(n-i)!\sum_{a=0}^{n-i}\frac{1}{a!}\,(-1)^a \;=\; \binom{n}{i}\ D_{n-i} \end{aligned}$$

where D_m denotes the number of fixed-point-free permutations of S_m. The desired conclusion concerning multiplicities follows since the number of permutation of S_n with i fixed points is precisely equal to $\binom{n}{i}\ D_{n-i}$.

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