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A q-ANALOGUE OF THE LAGRANGE INVERSION FORMULA

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ABSTRACT. A natural problem in the theory of q-series is to find a q-analogue of the Lagrange inversion formula. A special case was treated by Carlitz [12]. Andrews [3] and Gessel [18] have studied what appear to be different versions of the problem. We study here a third version intimately related to q-analogues of Jabotinski [23] matrices. We prove an inversion theorem for these matrices and derive a number of q-analogues of classical identities, we also show that the problems studied by Andrews and Gessel and the present one are all equivalent. The methods we introduce can be used in a more general context. We illustrate this by deriving some results from the theory of partitions. In particular we obtain new proofs of some identities of Sylvester [29], Rogers-Ramanujan [25], Rogers-Selberg [25], [28] and Bressoud [10].

In the classical Lagrange inversion problem a series

I.1 $F(z) = \Sigma_{n\geqslant 1} F_n z^n, \quad F_1 \neq 0,$

is given and a new series

I.2 $f(z) = \Sigma_{n\geqslant 1} f_n z^n, \quad f_1 \neq 0,$

is to be constructed such that

I.3 $F(f(z)) = f(F(z)) = z.$

It can be shown that if we set

I.4 $\beta_{nk} = F^k(z)|_{z^n}, \alpha_{nk} = f^k(z)|_{z^n}, \quad n,k \geqslant 1,$

where "$|_{z^n}$" denotes the operation of taking a coefficient, then I.3 is equivalent to the statement that the matrices

$$\alpha = \|\alpha_{nk}\|, \beta = \|\beta_{nk}\|$$

are inverses of each other.

The Lagrange inversion formula is essentially a recipe for constructing the matrix

α given F(z). There are two versions of the formula which can be stated as follows

I.5 (a) $\Phi(f(z))|_{z^n} = \frac{F'(z)}{F^{n+1}(z)}\Phi(z)|_{z^{-1}}$,

(b) $\Phi(f(z))|_{z^n} = \frac{1}{n}\frac{\Phi'(z)}{F^n(z)}|_{z^{-1}}$.

These formulas are to hold for an arbitrary formal power series $\Phi(z) = \Phi_0 + \Phi_1 z + \Phi_2 z^2 + \cdots$. In particular setting $\Phi(z) = z^k$ we get

I.6 $\alpha_{nk} = \frac{F'(z)}{F^{n+1}(z)}|_{z^{-k-1}} = \frac{k}{n}\frac{1}{F^n(z)}|_{z^{-k}}$.

It is worthwhile noting that if we extend the definition of α_{nk}, β_{nk} to all pairs of integers n,k then I.6 can also be stated in the form

I.7 $\alpha_{nk} = \frac{k}{n}\beta_{-k,-n}$.

In the present setting we shall work with series $F(z) = F_1 z + F_2 z^2 + \cdots$ whose coefficients are given rational functions of a further variable q. Our goal is to find analogues of formulas I.3, I.5, I.6 when the matrix $\beta = \|\beta_{nk}\|$ has entries of the form

I.8 $\beta_{nk} = F(z)F(zq)\cdots F(zq^{k-1})|_{z^n}$.

The reason for considering such matrices is that in certain combinatorial models [17] expressions such as

$$\Phi(\overline{F}) = \Sigma_{n\geqslant 0}\Phi_n F(z)F(zq)\cdots F(zq^{n-1})$$

appear as natural q-analogues of functional *composition*.

The first basic question which arises then is whether or not the matrix $\alpha = \|\alpha_{nk}\|$ inverse of $\beta = \|\beta_{nk}\|$ can be associated to some formal power series f(z). The main result of Section 1 is that in point of fact there is a unique formal power series f(z) such that

I.9 $\alpha_{nk} = f(z)f(z/q)\cdots f(z/q^{k-1})|_{z^n}$.

In particular f(z) can be constructed by means of any of the two equations below

I.10 (a) $\Sigma_{n\geqslant 1} F_n f(z)f(z/q)\cdots f(z/q^{n-1}) = z$,

(b) $\Sigma_{n\geqslant 1} f_n F(z)F(zq)\cdots F(zq^{n-1}) = z$.

We shall therefore call here and after f(z) the *right* inverse of F(z) and F(z) the *left*

inverse of f(z).

An important consequence of the above mentioned result is that given a sequence $\{\Phi_n\}$ the sequence $\{c_n\}$ defined by the equation

$$\Sigma_{k\geqslant 0}\Phi_k f(z)f(z/q)\cdots f(z/q^{k-1}) = \Sigma_{n\geqslant 0}c_n z^n$$

is the same as that defined by

I.11 $\Sigma_{k\geqslant 0}\Phi_k z^k = \Sigma_{n\geqslant 0}c_n F(z)F(zq)\cdots F(zq^{n-1})$.

It is customary to write the formal power series F(z) in the form

I.12 $F(z) = z/R(z)$, $R(z) = R_0 + R_1 z + R_2 z^2 + \cdots$, $(R_0 \neq 0)$.

We shall do so here and note that I.11 can then be written in the form

$$\Sigma_{k\geqslant 0}\Phi_k z^k = \Sigma_{n\geqslant 0}c_n \frac{q^{\binom{n}{2}} z^n}{R(z)R(zq)\cdots R(zq^{n-1})}.$$

Now in [3] Andrews essentially gives a formula for the c_n's in such an expression in terms of the Φ_n's. Thus we see that Andrews indirectly was concerned with the inversion of the matrix defined by I.8.

In [18] Gessel starts with a given formal power series $G(z) = G_0 + G_1 z + G_2 z^2 + \cdots$ $(G_0 \neq 0)$ and considers the task of finding a formal power series f(z) satisfying the equation

$$f(z) = zq\Sigma_{n\geqslant 0}G_n f(z)f(zq)\cdots f(zq^{n-1}).$$

Let f(z) be the right inverse of F(z). Dividing I.10(b) by F(z) and using I.12 we obtain

$$\Sigma_{n\geqslant 1}f_n F(zq)\cdots F(zq^{n-1}) = R(z);$$

replacing zq by z we get

$$f_1 + \Sigma_{n\geqslant 2}f_n F(z)\cdots F(zq^{n-2}) = R(z/q) = \Sigma_{n\geqslant 0}R_n/q^n \quad z^n.$$

By the inversion result we see that this is equivalent to

$$f(z)/z = \Sigma_{n\geqslant 0}\frac{R_n}{q^n}f(z)f(z/q)\cdots f(z/q^{n-1}).$$

In other words, modulo replacing q by $1/q$, the solution f(z) of I.10(b) solves Gessel's equation for

$$G(z) = \frac{1}{q}R(zq).$$

Our next result is a q-analogue of formula I.5(a). Indeed as we shall see a

surprising number of classical identities can be extended to the present case.

To give some examples we need further notation. For a given formal power series $\theta(z) = \Sigma_{n\geqslant 0}\theta_n z^n$ we set

I.13 (a) $\hat{\theta}(z) = \Sigma_{n\geqslant 0}\theta_n q^{-\binom{n}{2}} z^n$,

(b) $\theta^*(z) = \theta(z)\theta(zq)\theta(zq^2)\cdots$.

(Here as well as in some of our later formulas for convergence purposes it will be necessary to assume that the constant term is equal to one. This involves no loss since we can always reduce ourselves to that situation.) It is not difficult to see that the operators *roofing* and *starring* defined above act "naturally" on the family of formal power series in z with coefficients rational functions of q. It is convenient along with the operators in I.13 to introduce the *reciprocal* operators obtained by replacing q by 1/q. We shall therefore set

I.14 (a) $\overset{\vee}{\theta}(z) = \Sigma_{n\geqslant 0}\theta_n q^{\binom{n}{2}} z^n$,

(b) $^*\theta(z) = \theta(z)\theta(z/q)\theta(z/q^2)\cdots$.

It is also convenient to introduce a shorthand notation for the two types of functional substitution occurring in our formulas. Thus if $\Phi(z)$ is a formal power series we set

$$\Phi(\underline{f}) = \Sigma_{n\geqslant 0}\Phi_n f(z)f(z/q)\cdots f(z/q^{n-1}),$$

$$\Phi(\overline{F}) = \Sigma_{n\geqslant 0}\Phi_n F(z)F(zq)\cdots F(zq^{n-1}).$$

We shall refer to $\Phi(\underline{f})$ and $\Phi(\overline{F})$ respectively as the formal power series obtained by 1/q-*substitution* and q-*substitution* of f and F *into* Φ.

This notion of substitution leads naturally to an operation which has a certain analogy with differentiation. This operation is easiest to express by means of the roofing and starring operations. Namely if F and R are related by I.12 we set

I.15 $F^{\mathrm{O}}(z) = R^*(zq)\left(\frac{1}{\hat{R}^*(z/q)}\right)\hat{}$.

(Roofing to be carried out before starring, in the inner expression.) We shall refer to $F^{\mathrm{O}}(z)$ as the q-*compositional derivative* of F(z). Of course a parallel definition is obtained upon replacing q by 1/q. In particular, if

I.16 $f(z) = z/r(z)$, $r(z) = 1 + r_1 z + r_2 z^2 + \cdots$,

we set

$$\text{I.17}\quad {}^{o}f(z) = {}^{*}r(z/q)\left(\frac{1}{{}^{\vee}_{*}r(zq)}\right)^{\vee}.$$

This given our q-analogue of I.5(a) can be stated as follows

$$\text{I.18}\quad \Phi(\underline{f})|_{z^n} = q^n \left.\frac{F^{o}(q^n z)\Phi(z)}{F(z)F(zq)\cdots F(zq^n)}\right|_{z^{-1}}$$

where this is to hold for all formal power series $\Phi(z)$. Of course an analogous formula holds upon replacing q by $1/q$ and reversing the roles of f and F.

Perhaps we should point out that the operation $F \Rightarrow F^{o}$ is *not* linear. Nevertheless F^{o} and ${}^{o}f$ do have other properties in common with differentiation in addition to appearing in a formula such as I.18. It can be shown of course that as $q \to 1$

$$F^{o}(z) \to F'(z),\ {}^{o}f(z) \to f'(z).$$

Moreover there is a q-analogue of the chain rule. For instance if F(z) is the ordinary compositional inverse of f(z) we have

$$\text{I.19}\quad F'(z)f'(F(z)) = 1$$

and indeed we can show that in our case we have

$$\text{I.20}\quad F^{o}(z)\,{}^{o}f(\underline{F}_1(z)) = 1$$

where F_1 is the *right* inverse of f. Of course, for $q = 1$, F and F_1 coincide and I.20 reduces to I.19.

There are further familiar formulas as we shall see. However here it is more appropriate to state some of those which (like I.15) have no counterpart in the case $q = 1$. For instance we have the following remarkable identities

$$\text{I.21}\quad \Phi(\underline{f}) = \frac{(\Phi(z)R^{*}(z))^{\wedge}}{\hat{R}^{*}(z)},$$

$$\text{I.22}\quad \Phi(\overline{F}) = \frac{(\Phi(z)*r(z))^{\vee}}{{}^{\vee}_{*}r(z)}.$$

A number of classical identities are special cases of I.21 and I.22. For instance, let F(z) be the limit of the Rogers-Ramanujan continued fraction

$$F(z) = \cfrac{z}{1 - \cfrac{zq}{1 - \cfrac{zq^2}{1 - \cdots}}}.$$

Clearly F(z) can also be defined by the equation

I.23 $F(z) - \bar{F}(z)F(zq) = z.$

However this simply says that F(z) is the left inverse of

$$f(z) = z - z^2.$$

Thus in this case

$$r(z) = 1/1\text{-}z,\ {}^*r(z) = \Sigma_{n\geqslant 0} \frac{(-z)^n q^{\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}$$

and I.22 for $\Phi(z) = z$ gives

$$F(z) = z \frac{\Sigma_{n\geqslant 0} \frac{(-z)^n q^{n(n+1)}}{(1-q)\cdots(1-q^n)}}{\Sigma_{n\geqslant 0} \frac{(-z)^n q^{n^2}}{(1-q)\cdots(1-q^n)}}.$$

This is a well known result [21].

There are further curious identities. For instance Gessel in [18] expresses his basic result in terms of a formal power series $e_q(z)$ which for q = 1 reduces to

I.24 $e_1(z) = 1/1\text{-}zR'(f(z)).$

It is not difficult to show that this expression is actually equal to

$$e_1(z) = r(z)f'(z).$$

The situation for $q \neq 1$ is entirely analogous. We show in Section 2 that

I.25 $e_q(z) = r(z)\ {}^{o}f(z).$

It is clear that very curious things are going on here. It is also apparent that we have only scratched the tip of an iceberg and that further investigation is required to see the range of applicability of these methods.

We should perhaps point out that in [17] some of this material is developed within the context of polynomial differential operators very much in the same spirit of [15] and [16]. It is shown there that a very natural (from the combinatorial point of view) q-analogue of the Mullin-Rota theory of polynomials of binomial type is intimately related to the q-analogue of the Lagrange inversion problem studied here. In the third section of the paper we show how this work is related to the present development.

Our starring and roofing operations turn out to be useful in a more general context. In the last section we illustrate some further uses they can be put to by

deriving some partition identities. This leads to a new proof of some identities of Sylvester [29], Rogers-Ramanujan [25], Rogers-Selberg [25], [28], Bressoud [10].

Finally we wish to acknowledge our gratitude to G. Andrews who provided us with a wealth of information on the subject of q-series both directly by personal conversation and indirectly by making some of his work available to us. We are also indebted to J. Remmel who lent an indefatigable ear to our endless discussions on the subject and contributed countless improvements in the presentation of the material.

1. The matrix inversion result. We start with a given formal power series

$$1.1 \quad F(z) = \Sigma_{n\geqslant 1} F_n z^n, \quad F_1 \neq 0,$$

and let $f(z) = \Sigma_{n\geqslant 1} f_n z^n$ be the formal power series whose coefficients are recursively obtained by equating coefficients of z^n in the equation

$$1.2 \quad z = F_1 f(z) + F_2 f(z)f(z/q) + \cdots.$$

We set

$$1.3 \quad \beta_{nk} = F(z)\cdots F(zq^{k-1})|_{z^n}, \ \alpha_{nk} = f(z)\cdots f(z/q^{k-1})|_{z^n}.$$

This given we have the following result

THEOREM 1.1. *The matrices* $\alpha = \|\alpha_{nk}\|$, $\beta = \|\beta_{nk}\|$ *are inverses of each other.*

PROOF. For convenience let $\tilde{\beta} = \|\tilde{\beta}_{nk}\|$ denote the inverse of the matrix α. Set also

$$\beta_k(z) = \Sigma_{n\geqslant k} z^n \beta_{nk}, \ \tilde{\beta}_k(z) = \Sigma_{n\geqslant k} z^n \tilde{\beta}_{nk}.$$

We are to show that $\tilde{\beta}_k(z) = \beta_k(z)$. To this end note first that by definition we have

$$\Sigma_{k\geqslant 1} \alpha_{nk} \tilde{\beta}_{k1} = \chi(n = 1).$$

(Here and in the following if A is a statement $\chi(A)$ is to be taken equal to 1 or 0 according as A is true or not.) Multiplying by z^n and summing we get

$$1.4 \quad \Sigma_{k\geqslant 1} f(z)\cdots f(z/q^{k-1})\tilde{\beta}_{k1} = z.$$

Note now that 1.2, given f(z), determines the sequence $\{F_n\}$ uniquely. Thus 1.4 implies $F_k = \tilde{\beta}_{k1}$ for all $k \geqslant 1$. This gives

$$1.5 \quad \tilde{\beta}_1(z) = \beta_1(z).$$

Replacing z by z/q^k in 1.2 we get

$$z/q^k = F_1 f(z/q^k) + F_2 f(z/q^k) f(z/q^{k+1}) + \cdots.$$

Thus

$$(z/q^k) f(z) \cdots f(z/q^{k-1}) = F_1 f(z) \cdots f(z/q^k) + F_2 f(z) \cdots f(z/q^{k+1}) + \cdots.$$

Equating coefficients of z^n of both sides gives

$$\alpha_{n-1,k} = q^k \{F_1 \alpha_{n,k+1} + F_2 \alpha_{n,k+2} + \cdots\}.$$

Multiplying by $\tilde{\beta}_{m\,n}$ and summing with respect to n,

$$\Sigma_n \tilde{\beta}_{m,n} \alpha_{n-1,k} = q^k \Sigma_{h \geqslant 1} F_h \chi\,(m = k+h) = q^k F_{m-k}.$$

(We make the convention that $\alpha_{n,k}$, $\beta_{n,k}$ and $\tilde{\beta}_{n,k}$ are zero when $k > n$. This is consistent with our definitions.) Multiplying on the right by $\tilde{\beta}_{k,s}$ and summing with respect to k,

$$\tilde{\beta}_{m,s+1} = \Sigma_n \tilde{\beta}_{m,n} \chi\,(n-1 = s) = \Sigma_k F_{m-k} q^k \tilde{\beta}_{k,s}.$$

In other words we have

$$\tilde{\beta}_{s+1}(z) = F(z) \tilde{\beta}_s(qz).$$

Combining this identity with 1.5 we recursively derive

$$\tilde{\beta}_k(z) = F(z)F(zq) \cdots F(zq^{k-1}) = \beta_k(z) \text{ (for } k \geqslant 1).$$

This completes the proof of the theorem.

We are now in a position to establish the existence and uniqueness of a formal power series $F^o(z)$ which makes formula I.18 true for all Φ. Uniqueness is immediate: Setting $\Phi \equiv 1$ in I.18 we get

$$1.6 \quad \frac{F^o(zq^n)}{F(z)F(zq) \cdots F(zq^n)}\Big|_{z^{-1}} = \chi\,(0 = n)$$

or equivalently

$$1.7 \quad \frac{F^o(z)}{F(z)F(z/q) \cdots F(z/q^n)}\Big|_{z^{-1}} = \chi\,(0 = n).$$

Replacing F(z) by z/R(z) we can rewrite this equation in the form

$$1.8 \quad F^o(z)R(z)R(z/q) \cdots R(z/q^n)\Big|_{z^n} = \chi\,(0 = n).$$

Thus setting $F^o(z)|_{z^n} = F^o_n$, formulas 1.6 or 1.7 are equivalent to the equations

$$1.9 \quad F^o_0 = 1/R_0,\ F^o_n = -\frac{1}{R^n_o} \Sigma^{n-1}_{k=0} F^o_k F(z)R(z/q) \cdots R(z/q^n)\Big|_{z^{n-k}}.$$

This determines $F^{o}(z)$ completely. But we can reverse the argument and obtain

THEOREM 1.2. *Let* $F^{o}(z)$ *be the formal power series whose coefficients are recursively determined from the equations in 1.9, then for all formal power series* $\Phi(z)$ *we have*

$$1.10\quad \Phi(\underline{f})|_{z^n} = \frac{q^n F^{o}(zq^n)\Phi(z)}{F(z)F(zq)\cdots F(zq^n)}|_{z^{-1}}.$$

PROOF. Theorem 1.1 implies that we have

$$\Phi(\underline{f}) = \Sigma_{k\geqslant 0} c_k(\Phi) z^k$$

if and only if

$$\Phi(z) = \Sigma_{k\geqslant 0} c_k(\Phi) F(z)F(zq)\cdots F(zq^{k-1}).$$

Thus

$$1.11\quad \frac{q^n F^{o}(zq^n)\Phi(z)}{F(z)F(zq)\cdots F(zq^n)}|_{z^{-1}} = \Sigma_{k=0}^{n} c_k(\Phi) q^n F^{o}(zq^n) \frac{F(z)F(zq)\cdots F(zq^{k-1})}{F(z)F(zq)\cdots F(zq^n)}|_{z^{-1}}.$$

The summation stops at n here since the remaining terms yield no residue. Now the coefficient of $c_k(\Phi)$ can be rewritten as

$$\frac{q^n F^{o}(zq^n)}{F(zq^k)\cdots F(zq^n)}|_{z^{-1}} = q^n q^{-n} \frac{F^{o}(z)}{F(z)F(z/q)\cdots F(z/q^{n-k})}|_{z^{-1}}.$$

However the latter expression, in view of 1.7, is equal to zero for $k < n$ and equal to one for $k = n$. Using this fact in 1.11 yields 1.10 as asserted.

Formula I.15 as well as other identities giving F^{o} will be established in later sections. It is natural at this point to ask whether or not we can put together reasonable q-analogues of formulas I.5(b) and I.7. There are indications that this may be possible since some special cases of I.7 can be extended almost verbatim. We shall now present these formulas since they give us a rather interesting closed expression for F_k^{o} in terms of f(z). To see how this comes about observe that I.7, reversing the roles of F and f, becomes

$$\beta_{n,k} = \frac{k}{n}\alpha_{-k,-n}.$$

Thus in particular we have

$$(n+1)\beta_{n+1,1} = \alpha_{-1,-n-1}.$$

However this simply says that

$$F'(z)|_{z^n} = 1/(f(z))^{n+1}|_{z^{-1}}.$$

An entirely analogous result holds for $q \neq 1$.

THEOREM 1.3.

1.12 $$F^o(z)|_{z^k} = 1/f(z)f(z/q)\cdots f(z/q^k)|_{z^{-1}}.$$

PROOF. From Theorem 1.1 we deduce that

$$z^{k+1} = \Sigma_{m\geqslant k} f(z)f(z/q)\cdots f(z/q^m)\beta_{m+1,k+1}.$$

This gives

$$\frac{1}{f(z)f(z/q)\cdots f(z/q^k)}|_{z^{-1}}$$

$$= \frac{z^{k+1}}{f(z)f(z/q)\cdots f(z/q^k)}|_{z^{-1}}$$

$$= \Sigma_{m\geqslant k} f(z/q^{k+1})\cdots f(z/q^m)\beta_{m+1,k+1}|_{z^k}$$

$$= q^{-(k+1)k}\Sigma_{m\geqslant k} f(z)f(z/q)\cdots f(z/q^{m-k-1})\beta_{m+1,k+1}|_{z^k}.$$

Using formula 1.10 we can write this in the form

$$\frac{1}{f(z)f(z/q)\cdots f(z/q^k)}|_{z^{-1}} = q^{-(k+1)k}\frac{q^k F^o(zq^k)}{F(z)F(zq)\cdots F(zq^k)}\Sigma_{m\geqslant k} z^{m-k}\beta_{m+1,k+1}|_{z^{-1}}$$

$$= q^{-k^2}F^o(zq^k)\frac{1}{z^{k+1}}|_{z^{-1}}$$

$$= q^{-k^2}F^o(zq^k)|_{z^k} = F^o(z)|_{z^k}.$$

This establishes 1.12.

We terminate this section by relating some of Gessel's formulas to ours. In the introduction, using Theorem 1.1, we have shown that the solution $f(z)$ of

$$\Sigma_{n\geqslant 1} F_n f(z)f(z/q)\cdots f(z/q^{n-1}) = z$$

(when $F(z) = z/R(z)$) satisfies also the equation

1.13 $$f(z) = z/q\ \Sigma_{n\geqslant 0} R_n/q^{n-1} f(z)f(z/q)\cdots f(z/q^{n-1}).$$

Now Gessel (Theorem 6.9 of [18]) shows that there is a formal power series $e_q(z)$ such that the solution $f(z)$ of 1.13 satisfies the following identities

$$1.14\quad e_q(z)f(z)\cdots f(z/q^{k-1})|_{z^n} = q^{-\binom{n}{2}}R(z)\cdots R(zq^{n-1})|_{z^{n-k}}$$

$$= q^{-\binom{n}{2}}z^k R(z)\cdots R(zq^{n-1})|_{z^n}.$$

Thus multiplying by Φ_k and summing with respect to k we see that Gessel's result is equivalent to the formula

$$1.15\quad e_q(z)\Phi(\underline{f})|_{z^n} = q^{-\binom{n}{2}}\Phi(z)R(z)\cdots R(zq^{n-1})|_{z^n}$$

where $\Phi(z)$ is an arbitrary formal power series.

Now setting k = 0 in 1.14 we obtain

$$1.16\quad e_q(z)|_{z^n} = q^{-\binom{n}{2}}R(z)\cdots R(zq^{n-1})|_{z^n} = \frac{1}{F(z)\cdots F(zq^{n-1})}|_{z^0}.$$

Our goal is to give a relation between $e_q(z)$ and $F^{o}(z)$. Before we state the result it is good to examine the situation for q = 1. We know from formula 1.5 with $\Phi(z)$ replaced by $e_1(F(z))\Phi(z)$ that

$$e_1(z)\Phi(f(z))|_{z^n} = \frac{F'(z)e_1(F(z))\Phi(z)}{(F(z))^{n+1}}|_{z^{-1}} = F'(z)e_1(F(z))\Phi(z)(R(z))^{n+1}|_{z^n}.$$

On the other hand formula 1.15 for q = 1 becomes

$$e_1(z)\Phi(f(z))|_{z^n} = \Phi(z)(R(z))^n|_{z^n}.$$

Comparing these two formulas we get

$$F'(z)e_1(F(z)) = 1/R(z) = F(z)/z$$

and this relation is equivalent to

$$1.17\quad F'(f(z))e_1(z) = z/f(z).$$

In other words

$$1.18\quad e_1(z) = \frac{1}{F'(f(z))}z/f(z) = f'(z)r(z).$$

We shall give q-analogues for both 1.17 and 1.18. The curious fact is that the q-analogue of 1.17 is quire "tangled." Indeed we have

THEOREM 1.4.

$$1.19\quad \Sigma_{k\geqslant 0}\frac{F_k^o}{q^k}f(z)f(zq)\cdots f(zq^k)e_q(zq^k) = z.$$

PROOF. We start by manipulating formula 1.6 as follows

$$\chi(0=n) = \frac{F^o(zq^n)}{F(z)F(zq)\cdots F(zq^n)}\Big|_{z^{-1}} = \Sigma_{k\geqslant 0} F_k^o q^{nk} \frac{1}{F(z)F(zq)\cdots F(zq^n)}\Big|_{z^{-1-k}}$$

$$= \Sigma_{k\geqslant 0} F_k^o q^{nk} \frac{R(z)R(zq)\cdots R(zq^n)}{q^{\binom{n+1}{2}}}\Big|_{z^{n-k}}.$$

Thus formula 1.14 with n,k replaced by n+1,k+1 gives

$$\chi(0=n) = \Sigma_{k\geqslant 0} F_k^o q^{nk} e_q(z)f(z)f(z/q)\cdots f(z/q^k)|_{z^{n+1}}$$

$$= \Sigma_{k\geqslant 0} \frac{F_k^o}{q^k} e_q(zq^k)f(z)f(zq)\cdots f(zq^k)|_{z^{n+1}}$$

and this is 1.19.

2. Roofing, starring and q-series identities. Here and in the following, by a q-series we mean a formal power series

2.1 $A(z) = \Sigma_{n\geqslant 0} A_n(q) z^n$

whose coefficients $A_n(q)$ are rational functions of q. We shall refer to the operation

$$\Downarrow A(z) = \Downarrow \Sigma_{n\geqslant 0} A_n(q) z^n = \Sigma_{n\geqslant 0} A_n(1/q) z^n$$

as the *basic involution*.

In this section we shall derive a number of identities concerning our q-analogue of functional composition. Our basic tools will be the roofing and starring operations given in the introduction. We shall start by establishing a few useful properties of these operations.

First of all, roofing comes up naturally in our context since we have

$$\underline{A(z)} = \Sigma_{n\geqslant 0} A_n z(z/q)(z/q^2)\cdots(z/q^{n-1}) = \Sigma_{n\geqslant 0} A_n z^n q^{-\binom{n}{2}} = \hat{A}(z).$$

Similarly

$$\overline{A(z)} = \check{A}.$$

Furthermore roofing can transform *tangled products* such as 1.19 into ordinary products. More precisely we have

PROPOSITION 2.1 *If* A(z) *and* B(z) *are* q-*series then*

2.2 $(\Sigma_{n\geqslant 0} A_n z^n B(zq^n))\hat{} = \hat{A}(z)\hat{B}(z)$,

2.3 $(\Sigma_{n\geqslant 0} A_n z^n B(z/q^n))\check{} = \check{A}(z)\check{B}(z)$.

PROOF. Using the identity

$$\binom{h+k}{2} = \binom{h}{2} + \binom{k}{2} + hk$$

we see that both sides of 2.2 are equal to the expression

$$\Sigma_{h\geqslant 0}\Sigma_{k\geqslant 0} A_h B_k z^{h+k} q^{-\binom{h+k}{2}+hk}$$

Formula 2.3 is established in a similar way.

NOTE. q-series identities come in pairs (such as 2.2 and 2.3) one derivable from the other by means of the basic involution. In the following to avoid repetitions we shall sometimes state only one identity of such a pair.

It is appropriate to re-examine here our starring operations. To this end note that when $A_0 = 1$ we can write

$$2.4 \quad A(z) = 1 + \Sigma_{n\geqslant 1} A_n z^n = e^{\Sigma_{p\geqslant 1}\psi_p z^p/p}$$

where $\{\psi_p\}$ is uniquely determined by $\{A_n\}$ and vice versa. Indeed the relations between these two sequences are given by Newton's formulas

$$A_n(\psi_1,\ldots,\psi_n) = \det\begin{bmatrix} \psi_1 & -1 & 0 & \cdots & 0 \\ \psi_2 & \psi_1 & -2 & & \cdot \\ \psi_3 & \psi_2 & \psi_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -n+1 \\ \psi_n & \psi_{n-1} & & \cdots & \psi_1 \end{bmatrix} \quad 1/n!,\ \psi_p(A_1,\ldots,A_p) = \det\begin{bmatrix} A_1 & 1 & 0 & \cdots & 0 \\ 2A_2 & A_1 & 1 & & \cdot \\ 3A_3 & A2 & A_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ pA_p & A_{p-1} & & \cdots & A_1 \end{bmatrix}$$

This given, it is more appropriate to define our starring operation by setting

$$2.5 \quad \theta^*(z) = e^{\Sigma_{p\geqslant 1}\psi_p(\theta)\frac{1}{1-q^p}z^p/p},$$

$$2.6 \quad {}^*\theta(z) = e^{\Sigma_{p\geqslant 1}\psi_p(\theta)\frac{1}{1-(1/q)^p}z^p/p}$$

We can easily see that these formulas are in formal agreement with I.13 and I.14. The fact that starring a q-series gives another q-series now follows immediately from Newton's formulas, moreover we have the following basic properties:

PROPOSITION 2.2.

$$2.7 \quad \theta(z) = A(z)/A(zq) \Leftrightarrow A(z) = \theta^*(z),$$

2.8 $\theta(z) = A(z)/A(z/q) \Leftrightarrow A(z) = {}^*\theta(z)$,

2.9 ${}^*\theta(z)\theta^*(zq) = 1$.

PROOF. The left equality in 2.7 is equivalent to

$$e^{\Sigma_{p\geqslant 1}\psi_p(\theta)z^p/p} = e^{\Sigma_{p\geqslant 1}\psi_p(A)(1-q^p)z^p/p}.$$

The uniqueness of the representation 2.4 then gives

$$\psi_p(A) = \psi_p(\theta)/1-q^p.$$

The other assertions can be verified in an entirely analogous manner.

NOTE. In the following we shall need to apply starring and roofing in succession. In order to avoid excessive uses of parentheses we shall agree, unless otherwise specified, that starring is to be performed before roofing. This means that

$$\hat{R}^*(z) = (R^*(z))\hat{}\,, \quad {}^*\check{r}(z) = ({}^*r(z))\check{}\,, \text{ etc. . .}$$

We are now in a position to state and prove our identities.

THEOREM 2.1. *If* $f(z) = z/r(z)$ *is the right inverse of* $F(z) = z/R(z)$ *then for all formal power series* $\Phi(z)$ *we have*

2.10 $\Phi(\underline{f}) = ((\Phi(z)R^*z))\hat{}\,/\hat{R}^*(z))$,

2.11 $\Phi(\bar{F}) = ((\Phi(z)^*r(z))\check{}\,/{}^*\check{r}(z))$.

PROOF. Let us set

$$\Phi(\underline{f}) = \Sigma_{n\geqslant 0}c_n z^n.$$

Then our matrix inversion result gives

$$\Sigma_{n\geqslant 0}c_n F(z)F(zq)\cdots F(zq^{n-1}) = \Phi(z).$$

This may be rewritten as

$$\Sigma_{n\geqslant 0}c_n \frac{z^n q^{\binom{n}{2}}}{R(z)R(zq)\cdots R(zq^{n-1})} = \Phi(z).$$

Multiplying both sides by $R^*(z)$ and using 2.7 gives

$$\Sigma_{n\geqslant 0}c_n z^n q^{\binom{n}{2}} R^*(zq^n) = \Phi(z)R^*(z).$$

Roofing both sides of this identity and using 2.2 gives

$$(\Sigma_{n\geqslant 0}c_n z^n)(R^*(z))\hat{} = (\Phi(z)R^*(z))\hat{}\,.$$

But this is 2.10.

Formula 2.11 may be established in an analogous manner or may be derived from 2.10 by means of the basic involution.

Theorem 2.1 has an important corollary which is perhaps one of the most unusual formulations of the Lagrange inversion formula.

THEOREM 2.2. *If* $f(z) = z/r(z)$ *is the right inverse of* $F(z) = z/R(z)$ *then*

2.12 $f(z) = z\,\hat{R}^*(z/q)/\hat{R}^*(z)$

or equivalently

2.13 $^*r(z) = \hat{R}^*(z)$.

PROOF. Formula 2.10 with $\Phi(z) = z$ gives

$$f(z) = \frac{\Sigma_{n\geqslant 0} R_n^* z^{n+1} q^{-\binom{n+1}{2}}}{\hat{R}^*(z)} = z\frac{\Sigma_{n\geqslant 0} R_n^* (z/q)^n q^{-\binom{n}{2}}}{\hat{R}^*(z)}.$$

This is 2.12. However we also obtain

$$r(z) = \hat{R}^*(z)/\hat{R}^*(z/q).$$

Thus the equivalence of 2.12 and 2.13 follows from 2.8.

NOTE. Arguments similar to those used in our derivation of formula 2.10 have been used by Hardy and Wright in the derivation of the formula for the Rogers-Ramanujan continued fraction ([21], §19.15). See also Hodel [22] and Gessel [18]. However these authors, lacking the matrix inversion result, were not in a position to realize the full scope of these methods.

In the case $q = 1$ taking the right inverse of a formal power series $F(z)$ twice in succession brings us back to $F(z)$. Of course this is not the case for $q \neq 1$. Nevertheless there are some rather remarkable identities.

THEOREM 2.3. *Let* $f(z) = z/r(z)$ *be the right inverse of* $F(z) = z/R(z)$ *and* $F_1(z) = z/R_1(z)$ *be the right inverse of* $f(z)$, *then*

2.14 $^*R_1(z) = (1/^*r(z/q))\hat{} = (1/\hat{R}^*(z/q))\hat{}$.

PROOF. Since F_1 is the right inverse of f formula 2.13 applies with r replaced by R_1 and R by r. In other words we have

$$^*R_1(z) = \hat{r}^*(z).$$

Formula 2.9 then gives

$$*R_1(z) = (1/*r(z/q))\hat{}\,.$$

This establishes the first of the equalities in 2.14 To obtain the other we simply use 2.13 itself.

We are now in a position to derive our formulas for $F^o(z)$.

THEOREM 2.4. *If* $F(z) = z/R(z)$ *then*

2.15 $F^o(z) = R^*(zq)(1/\hat{R}^*(z/q))\hat{} = R^*(zq)*R_1(z) = R^*(zq)/R_1^*(zq)$.

PROOF. Formula 1.10 with $\Phi(z) = 1/R^*(z)$ gives

$$\frac{1}{R^*}(\underline{f})|_{z^n} = q^{-\binom{n}{2}} F^o(zq^n)\frac{R(z)R(zq)\cdots R(zq^n)}{z^{n+1}R^*(z)}|_{z^{-1}}$$

$$= q^{-\binom{n}{2}}\frac{F^o(zq^n)}{R^*(zq^{n+1})}|_{z^n} = q^{\binom{n}{2}+n}\frac{F^o(z)}{R^*(zq)}|_{z^n}.$$

On the other hand, 2.10 yields

$$\frac{1}{R^*}(\underline{f}) = \frac{1}{\hat{R}^*(z)}.$$

Combining these last two formulas we get

$$\frac{F^o(z)}{R^*(zq)}|_{z^n} = q^{-\binom{n}{2}-n}\frac{1}{\hat{R}^*(z)}|_{z^n}.$$

But this is precisely our first equality in 2.15.

The other equalities follow from 2.14 and 2.9 respectively.

We terminate the section by proving formulas I.25 and I.10. We start with the formula relating our $F^o(z)$ with Gessel's $e_q(z)$.

THEOREM 2.5.

2.16 $e_q(z) = r(z)\,{}^o f(z)$.

PROOF. The idea is to use the roofing operation to untangle formula 1.19. To do this we first put 1.19 into the proper from by the substitution $f(z) = z\hat{R}^*(z/q)/\hat{R}^*(z)$ (this is 2.12) obtaining

$$\Sigma_{k\geq 0}\frac{F_k^o}{q^k}z^{k+1}q^{\binom{k}{2}+k}\frac{\hat{R}^*(z/q)}{\hat{R}^*(z)}\cdots\frac{\hat{R}^*(zq^{k-1})}{\hat{R}^*(zq^k)}e_q(zq^k) = z$$

and this can be rewritten as

$$\Sigma_{k\geqslant 0} F_k^o z^k q^{\binom{k}{2}} \frac{e_q(zq^k)}{\hat{R}^*(zq^k)} = \frac{1}{\hat{R}^*(z/q)}.$$

Roofing both sides gives

$$F^o(z)\left(\frac{e_q(z)}{\hat{R}^*(z)}\right)^{\wedge} = \left(\frac{1}{\hat{R}^*(z/q)}\right)^{\wedge}.$$

Substituting for $F^o(z)$ as given by the first equality in 2.15 and simplifying we get

$$\left(\frac{e_q(z)}{\hat{R}^*(z)}\right)^{\wedge} = \frac{1}{R^*(zq)}.$$

Substituting for $\hat{R}^*(z)$ according to 2.13 gives

$$\left(\frac{e_q(z)}{*r(z)}\right)^{\wedge} = \frac{1}{*r(zq)}.$$

Unroofing this identity yields

$$2.17 \quad e_q(z) = *r(z)\left(\frac{1}{*r(zq)}\right)^{\vee} = r(z)*r(z/q)\left(\frac{1}{*r(zq)}\right)^{\vee}.$$

Now we claim that we have

$$2.18 \quad {}^of(z) = *r(z/q)\left(\frac{1}{*r(zq)}\right)^{\vee}.$$

This can be obtained in the same manner as 2.15 or may be derived from 2.15 itself by successive uses of the basic involution. Substituting 2.18 in 2.17 gives 2.16.

Our next formula is a rather peculiar instance of the chain rule.

THEOREM 2.6. *If* f *is the right inverse of* F *and* F_1 *is the right inverse of* f *then*

$$2.19 \quad F^o(z)^of(F_1) = 1.$$

PROOF. There are several avenues to this identity. With the information at hand the shortest route is again through the identity 1.19. This time we make the substitution $f(z) = z/r(z)$ and obtain

$$\Sigma_{k\geqslant 0} F_k^o \frac{z^k q^{\binom{k}{2}}}{r(z)r(zq)\cdots r(zq^k)} e_q(zq^k) = 1.$$

Multiplying both sides by $r^*(z)$ we get

$$\Sigma_{k\geqslant 0} F_k^o z^k q^{\binom{k}{2}} r^*(zq^{k+1}) e_q(zq^k) = r^*(z).$$

Roofing then gives

$$F^{o}(z)(r^{*}(zq)e_q(z))\hat{} = \hat{r}^{*}(z).$$

Substituting for $e_q(z)$ according to 2.16 this can be rewritten as

2.20 $$F^{o}(z)\frac{(r^{*}(z)\,{}^{o}f(z))\hat{}}{\hat{r}^{*}(z)} = 1.$$

However formula 2.10 with Φ, f, R replaced by ${}^{o}f$, F_1, r respectively gives

$${}^{o}f(\underline{F}_1) = \frac{({}^{o}f(z)\,r^{*}(z))\hat{}}{\hat{r}^{*}(z)}$$

Using this in 2.20 gives 2.19 as asserted.

3. **Q-binomial type and the roofing operation.** In [16] it was shown that the Mullin-Rota theory of polynomials of binomial type [24], [27], is intimately related to the Lagrange inversion problem Given the present notion of q-substitution and following the approach adopted in [15], it is tempting to define as q-analogue of binomial type (q-binomial type for brief) a polynomial sequence $\{P_n(x)\}$ whose exponential generating function is of the form

3.1 $$\Sigma_{n\geqslant 0}\frac{P_n(x)}{n!}z^n = e^{x\underline{f}(z)} = \Sigma_{n\geqslant 0}\frac{\mathbf{x}^{\mathbf{n}}}{n!}f(z)f(z/q)\cdots f(z/q^{n-1})$$

where $f(z) = \Sigma_{n\geqslant 1} f_n z^n$ ($f_1 \neq 0$) is a given q-series.

It develops [17] that this notion is quite natural combinatorially as well as analytically. Moreover it is shown in [17] that the methods of [15], [16] and [27] can be adapted almost verbatim to the study of these sequences. It will be good to state here some of the basic results. To this end recall that L_o, M, D are operators which act linearly on polynomials and are defined by setting

3.2 (a) $L_o x^n = \chi(0 = n)$ (evaluation at the origin),

(b) $Mx^n = q^n x^n$ (replacing x by qx),

(c) $Dx^n = nx^{n-1}$ (differentiation).

This given we have

THEOREM 3.1. $\{P_n(x)\}$ *is of* q-*binomial type if and only if*

3.3 (1) $P_n(x)$ *is precisely of degree* n

(2) $L_o P_n(x) = \chi(0 = n)$

for some q-*series* $F(z) = \Sigma_{n\geqslant 1} F_n z^n$ ($F_1 \neq 0$)

3.4 (3) $MF(D)P_n(x) = nP_{n-1}(x)$

and in this case 3.1 will hold true with f(z) *equal to the right inverse of* F(z).

Of course we also have a q-analogue of the original definition of Mullin and Rota namely:

THEOREM 3.2. $\{P_n(x)\}$ *is of* q*-binomial type if and only if* $P_n(x)$ *is of degree* n *and for all* x,y

3.5 $P_n(x+y) = \Sigma_{k=0}^{n} \binom{n}{k} P_k(y) P_{n-k}(x/q^k)$.

We refer the reader to [17] for some of the proofs and further developments. Our purpose here is to show that there are results for q-binomial type whose beauty and simplicity have no counterpart in the case q = 1.

The basic additional ingredients here are the roofing operations. More precisely our developments are based on a remarkable transformational formula which as we shall see has a wide range of applications.

THEOREM 3.4. *For any two* q*-series* A(z),B(z) *we have*

3.7 $\hat{A}(z)\check{B}(zq)|_{z^n} = q^{-\binom{n}{2}} A(z)B(zq^n)|_{z^n}$.

PROOF. Note that if h+k = n we have

$$nk - \binom{n}{2} = \binom{k}{2} + k - \binom{h}{2}.$$

Thus we get

$$\begin{aligned} q^{-\binom{n}{2}} A(z)B(zq^n)|_{z^n} &= \Sigma_{h+k=n} A_h B_k q^{nk-\binom{n}{2}} \\ &= \Sigma_{h+k=n} A_h q^{-\binom{h}{2}} B_k q^{\binom{k}{2}+k} \\ &= \Sigma_{h+k=n} \hat{A}_h \check{B}_k q^k \end{aligned}$$

and this is 3.7.

This identity gives us a relatively simple recipe for putting together q-binomial sequences.

THEOREM 3.5. *Let* $\alpha(z) = 1 + \alpha_1 z + \alpha_2 z^2 + \cdots$, $\beta(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$ *be* q*-series related by the equation*

3.8 $\alpha(z)\beta(zq) = 1$.

Set for convenience

3.9 $A(z) = \overset{\vee}{\alpha}(z)$, $B(z) = \hat{\beta}(z)$.

Then the sequence of polynomials

3.10 $P_n(x) = q^{-\binom{n}{2}} B(q^n D)A(D)x^n$

is of q*-binomial type. Moreover they satisfy 3 4 and 3.1 with*

3.11 (a) $F(z) = z\dfrac{A(zq)}{A(z)}$

(b) $f(z) = z\dfrac{\alpha(z/q)}{\alpha(z)}$.

PROOF. From 3.8 and 3.9 we get

$$\hat{A}(z)\overset{\vee}{B}(zq) = 1.$$

Thus formula 3.7 gives

$$A(z)B(q^n z)|_{z^n} = \chi(0 = n).$$

In other words we get

$$L_0 A(D)B(q^n D)x^n = \chi(0 = n)$$

and this gives 3.3(2).

To check 3.4 we note that if $F(z)$ is given by 3.11 then

$$MF(D)P_n(x) = q^{-\binom{n}{2}} MB(q^n D)A(Dq)nx^{n-1}$$

$$= q^{-\binom{n}{2}} B(q^{n-1}D)A(D)nq^{n-1}x^{n-1} = nP_{n-1}(x)$$

(perhaps we should note that $MD = \frac{D}{q}M$). Finally formula 2.13 yields that the right inverse of $F(z) = zA(zq)/A(z)$ is given by $f(z) = z/r(z)$ with

$$*r(z) = [(A(z)/A(zq))^*]\hat{} = \hat{A}(z) = \alpha(z).$$

This implies that $f(z)$ is given by 3.11(b).

Using Theorem 2.4 with $R^*(z) = A(z)$ and formulas 3.8 and 3.9 we get that in this case

3.12 $F^o(z) = A(zq)\left(\dfrac{1}{\hat{A}(z/q)}\right)\hat{} = A(zq)\left(\dfrac{1}{\alpha(z/q)}\right)\hat{} = A(zq)B(z)$

and our inversion formula (1.10) becomes

3.13 $\Phi(\underline{f})|_{z^n} = q^{-\binom{n}{2}} A(zq^{n+1})B(zq^n)\dfrac{A(z)}{A(zq)}\dfrac{A(zq)}{A(zq^2)}\cdots\dfrac{A(zq^n)}{A(zq^{n+1})}\Phi(z)|_{z^n}$

$$= q^{-\binom{n}{2}} B(zq^n)A(z)\Phi(z)|_{z^n} = \frac{q^{-\binom{n}{2}}}{n!} L_0 \Phi(D)B(q^n D)A(D)x^n$$

$$= \frac{1}{n!} L_0 \Phi(D)P_n(x).$$

Setting $\Phi(D) = e^{yD}$ yields

$$e^{yf}|_{z^n} = \frac{1}{n!} P_n(y)$$

and this establishes that these polynomials are indeed given by 3.1. Thus the proof of the theorem is now complete.

REMARK 3.1. The q-binomial formula 3.5 can be established directly from 3.1 in the same spirit as is done in [15]. The derivation is based on a q-analogue of the "Cauchy formula" valid for any pair of formal power series Φ, Ψ

$$3.14 \quad \Phi\Psi(f)|_{z^n} = \Sigma_{k=0}^{n} \Phi(f)|_{z^k} \Psi(f/q^k)|_{z^{n-k}}.$$

Setting $\Phi(z) = e^{xz}$, $\Psi(z) = e^{yz}$ and using 3.1 gives 3.5 as asserted.

We terminate this section by deriving Carlitz' original example from the present theory. To begin with note that if we use $F(z) = z/(1-z)$, $f(z) = z/(1+z)$ as inverse pairs and apply the classical Lagrange inversion formulas (I.5(a) and (b)) we obtain the expansions

$$3.15 \quad \text{(a)} \quad \Phi(z) = \Sigma_{n\geq 0} \left(\frac{z}{1-z}\right)^n \frac{(1-t)^{n-1}}{t^{n+1}} \Phi(t)|_{t^{-1}},$$

$$\text{(b)} \quad \Phi(z) = \Sigma_{n\geq 0} \left(\frac{z}{1-z}\right)^n \frac{1}{n} \frac{(1-t)^n \Phi'(t)}{t^n}|_{t^{-1}}.$$

Carlitz in [12] gives essentially a q-analogue of 3.15(b) namely

$$3.16 \quad \Phi(z) = \Sigma_{n\geq 0} \frac{z^n}{(1-z)\cdots(1-zq^{n-1})} \frac{1}{1-q^n} \frac{(1-t)\cdots(1-tq^{n-1})}{t^n} \Delta\Phi(t)|_{t^{-1}}$$

where Δ is the q-difference operator

$$\Delta\Phi(t) = \frac{1}{t}[\Phi(t) - \Phi(tq)].$$

Let us then use our results with $F(z) = z/(1-z)$. In the notation of Theorem 3.5 this corresponds to the choice

$$A(z) = (1-z)^* = \Sigma_{n\geq 0} \frac{(-z)^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)}.$$

Thus

$$\alpha(z) = \hat{A}(z) = \Sigma_{n\geqslant 0} \frac{(-z)^n}{(1-q)\cdots(1-q^n)} = 1/(1-z)^*.$$

This gives

$$B(zq) = \hat{\beta}(zq) = [(1-z)^*]^{\wedge} = 1/(1-z)^*.$$

Formula 3.13 becomes

$$\Phi(f)|_{z^n} = q^{-\binom{n}{2}} \frac{(1-z)^*}{(1-zq^{n-1})^*}\Phi(z)|_{z^n} = q^{-\binom{n}{2}}(1-z)\cdots(1-zq^{n-2})\Phi(z)|_{z^n}.$$

We thus obtain the expansion

$$\Phi(f) = \Sigma_{n\geqslant 0} z^n q^{-\binom{n}{2}} \frac{(1-t)\cdots(1-tq^{n-2})}{t^{n+1}}\Phi(t)|_{t^{-1}}.$$

By the matrix inversion result this can be rewritten as

$$\Phi(z) = \Sigma_{n\geqslant 0} \frac{z^n}{(1-z)\cdots(1-zq^{n-1})} \frac{(1-t)\cdots(1-tq^{n-2})}{t^{n+1}}\Phi(t)|_{t^{-1}}.$$

To see that this is Carlitz' formula we need only observe that

$$\frac{(1-t)(1-tq)\cdots(1-tq^{n-2})}{t^{n+1}}\Phi(t)|_{t^{-1}} = -\frac{1}{1-q^n}\Phi(tq)\Delta\frac{(1-t)\cdots(1-tq^{n-1})}{t^n}|_{t^{-1}}$$

$$= \frac{1}{1-q^n}\frac{(1-t)\cdots(1-tq^{n-1})}{t^n}\Delta\Phi(t)|_{t^{-1}}.$$

The latter equality follows from the fact that the q-difference of a formal Laurent series has no residue.

4. Further applications. Expressing classical q-series identities in terms of the roofing and starring operations sometimes leads to surprising new viewpoints. We shall illustrate this fact by deriving a selection of results from the theory of partitions.

For the sake of completeness we shall assume as little as possible. This will give us an opportunity to present our methods in the simplest setting.

(a) *The exponential series.* For beginners in the theory of q-series the identity

$$4.1 \quad \Sigma_{n\geqslant 0} \frac{z^n}{(1-q)\cdots(1-q^n)} \Sigma_{n\geqslant 0} \frac{(-z)^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)} = 1$$

may appear surprising. However if we set

$$e(z) = \Sigma_{n\geqslant 0} \frac{z^n}{(1-q)\cdots(1-q^n)}$$

and note that

4.2 $e(z) = e(zq) + ze(z)$

then unroofing we get

$$\overset{\vee}{e}(z) = \overset{\vee}{e}(zq) + z\overset{\vee}{e}(zq) = (1+z)\overset{\vee}{e}(zq).$$

Formula 2.7 then yields the identities

4.4 $e(z) = 1/(1-z)^*$, $\overset{\vee}{e}(z) = (1+z)^*$

from which we conclude that

$$e(z)\overset{\vee}{e}(-z) = 1.$$

We might also observe that

$$\Sigma_{n\geqslant 0} \frac{(-z)^n q^{\binom{n}{2}}}{(1-q)\cdots(1-q^n)} = \Downarrow \Sigma_{n\geqslant 0} \frac{(zq)^n}{(1-q)\cdots(1-q^n)}.$$

The explanation for this lies in formula 2.9. Indeed the latter gives

$$(1-z)^* = \frac{1}{*(1-z/q)} = \Downarrow \frac{1}{(1-zq)^*}.$$

(b) *The binomial identities.* We have

4.5 $$\frac{1}{(1-z)\cdots(1-zq^k)} = \Sigma_{n\geqslant 0} \begin{bmatrix} n+k \\ n \end{bmatrix} z^n,$$

4.6 $$(1+z)\cdots(1+zq^{N-1}) = \Sigma_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} q^{\binom{n}{2}} z^n,$$

where

$$\begin{bmatrix} N \\ n \end{bmatrix} = \frac{[N]!}{[n]![N-n]!}, \quad [n]! = (1-q)\cdots(1-q^n).$$

By writing

$$\frac{1}{(1-z)\cdots(1-zq^k)} = \frac{1}{[k]!}\Delta^k \frac{1}{(1-z)}$$

we can easily derive 4.5. This is difficult to beat. However, once 4.5 is given, 4.6 follows immediately. Indeed formula 3.7 (combined with the identities in 4.4) yields

$$(1+z)^* \left| \frac{1}{(1-zq^N)^*} \right|_{z^n} = q^{\binom{n}{2}} \left[(1+z)^* \right]^{\wedge} \left[\frac{1}{(1+zq^{N-n+1})^*} \right]^{\vee} \Big|_{z^n}$$

$$= q^{\binom{n}{2}} \frac{1}{(1-z)^*} (1-zq^{N-n+1})^* \Big|_{z^n}.$$

(c) *The Cauchy identity.* It can be shown by combinatorial methods (see [7]) that

$$4.7\quad \Sigma_{n\geqslant 0}\frac{q^{n^2-n}z^n}{(1-q)\cdots(1-q^n)(1-z)(1-zq)\cdots(1-zq^{n-1})}=\Pi_{m\geqslant 0}\frac{1}{1-zq^m}.$$

This formula can be obtained as a particular case of our matrix inversion result. However an even more general principle is involved here. The idea is to rewrite 4.7 in the form

$$4.8\quad \Sigma_{n\geqslant 0}\frac{q^{n^2-n}z^n}{(1-q)\cdots(1-q^n)}(1-zq^n)^*=1,$$

and view the latter as a "tangled" product. When we untangle it by means of 2.2 we simply get 4.1. Indeed, roofing 4.8 gives

$$\Sigma_{n\geqslant 0}\frac{q^{\binom{n}{2}}z^n}{(1-q)\cdots(1-q^n)}[(1-z)^*]\hat{}=\hat{1}=1.$$

The general principle referred to above is that a variety of q-series identities owe their intricacy to the fact that they are tangled versions of simpler ones. Establishing them in their tangled form can be somewhat mysterious but once untangled by means of roofing (or unroofing) they turn out to be consequences of more elementary series-product identities.

Our next examples should better illustrate this point.

(d) *The Sylvester identity.*

$$4.9\quad \Sigma_{n\geqslant 0}(-z)^nq^{n(3n-1)/2}(1-zq^{2n})\frac{(1-z)\cdots(1-zq^{n-1})}{(1-q)\cdots(1-q^n)}=\Pi_{m\geqslant 0}(1-zq^m).$$

Obtained combinatorially by Sylvester [29], it implies the famed Pentagonal Theorem of Euler (see [7] for a very interesting historical and mathematical analysis of its impact on the theory of partitions). It was given a somewhat mysterious analytic proof by Cayley [13].

It develops here that roofing untangles it into a simple consequence of the binomial identities. To see this we first write it in the form

$$4.10\quad \Sigma_{n\geqslant 0}\frac{(-z)^nq^{3\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}(1-zq^{2n})\frac{1}{(1-zq^n)^*}=1.$$

Calling for a moment $\varphi(z)$ the left hand side of 4.10, roofing gives

$$\varphi(z) = [\Sigma_{n\geqslant 0}\frac{(-z)^n q^{\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}]^{\vee}\frac{\hat{1}}{(1-z)^*} - [\Sigma_{n\geqslant 0}\frac{(-z)^n q^{\binom{n}{2}+2n}}{(1-q)\cdots(1-q^n)}]^{\vee}\frac{\hat{z}}{(1-z)^*}$$

$$= [(1-zq)^*]^{\vee}\frac{\hat{1}}{(1-z)^*} - [(1-zq^2)^*]^{\vee}\frac{\hat{z}}{(1-z)^*}.$$

Thus from 3.7 and 4.5 we obtain (for $n \geqslant 1$)

$$q^{\binom{n}{2}}\hat{\varphi}(z)|_{z^n} = \frac{(1-zq^n)^*}{(1-z)^*}|_{z^n} - \frac{(1-zq^{n+1})^*}{(1-z)^*}|_{z^{n-1}} = \begin{bmatrix} n-1+n \\ n \end{bmatrix} - \begin{bmatrix} n+n-1 \\ n-1 \end{bmatrix} = 0.$$

(e) *The Rogers-Ramanujan identity.*

$$4.11\quad \Sigma_{n\geqslant 0}(-1)^n z^{2n} q^{n(5n-1)/2}(1-zq^{2n})\frac{(1-z)\cdots(1-zq^{n-1})}{(1-q)\cdots(1-q^n)}$$

$$= \Pi_{m\geqslant 0}(1-zq^m)\Sigma_{n\geqslant 0}\frac{z^n q^{n^2}}{(1-q)\cdots(1-q^n)}.$$

Both Rogers and Ramanujan give separate proofs of this identity in [25] and derive from it the famed Rogers-Ramanujan identities. To this date the treatment of 4.11 given in [25] in spite of many claims to the contrary (see for instance [14]) remains the simplest and most elementary. Nevertheless the roofing operation adds a new twist to the study of 4.11.

First of all our setting here suggests that we are to replace q by q^2 in 4.11 and rewrite it in the form

$$4.12\quad \Sigma_{n\geqslant 0}\frac{(-1)^n z^{2n} q^{2\binom{2n}{2}+2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})\Pi_{m\geqslant 0}(1-zq^{2n+2m})}(1-zq^{4n})$$

$$= \Sigma_{n\geqslant 0}\frac{z^n q^{2n^2}}{(1-q^2)\cdots(1-q^{2n})}.$$

Denoting for a moment the left hand side of this relation by $\varphi(z) = \Sigma_{n\geqslant 0}\varphi_n z^n$ we get

$$\hat{\varphi}(z) = [\Sigma_{n\geqslant 0}\frac{(-1)^n z^{2n} q^{2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})}]^{\vee}\frac{\hat{1}}{\Pi_{m\geqslant 0}(1-zq^{2m})}$$

$$- [\Sigma_{n\geqslant 0}\frac{(-1)^n z^{2n} q^{2\binom{n+1}{2}+2n}}{(1-q^2)\cdots(1-q^{2n})}]^{\vee}\frac{\hat{z}}{\Pi_{m\geqslant 0}(1-zq^{2m})}$$

that is

$$4.13\quad \hat{\varphi}(z) = \overset{\vee}{\Gamma}(z)\hat{\Delta}(z) - \overset{\vee}{\Gamma}(zq)z\hat{\Delta}(z)$$

where for convenience we have set

$$\Gamma(z) = \Sigma_{n\geqslant 0} \frac{(-1)^n z^{2n} q^{2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})} = \Pi_{m\geqslant 1}(1-z^2q^{2m}),$$

$$\Delta(z) = \Sigma_{m\geqslant 0} \frac{1}{1-zq^{2m}}.$$

We can thus apply 3.7 and obtain

4.14 $\varphi_n = \Delta(z)\Gamma(zq^{n-1}) - z\Delta(z)\Gamma(zq^n)|_{z^n}$.

Clearly to show 4.12 we need only verify the recursion

$$\varphi_n = \frac{q^{4n-2}}{1-q^{2n}}\ \varphi_{n-1},$$

for $n \geqslant 1$. We shall do this by showing that the identities

4.16 (a) $\Gamma(z) = (1-z^2q^2)\Gamma(zq)$,

(b) $\Delta(zq^2) = (1-z)\Delta(z)$,

lead to the recursion

$$q^{-2n}\varphi_n = \varphi_n + q^{2n-2}\varphi_{n-1}.$$

To this end observe first that, using 4.16(a) and (b), we can write 4.14 in the form

$$\varphi_n = \Delta(z)\Gamma(zq^n)(1-z^2q^{2n}-z)|_{z^n} = \Delta(zq^2)\Gamma(zq^n) - z^2q^{2n}\Delta(z)\Gamma(zq^n)|_{z^n}.$$

This gives (using 4.16(a),(b))

4.17 $q^{-2n}\varphi_n = \Delta(z)\Gamma(zq^{n-2}) - z^2\Delta(z)\Gamma(zq^n)|_{z^n}$

$= \Delta(z)(1-z^2q^{2n-2})\Gamma(zq^{n-1}) + z[\Delta(zq^2) - \Delta(z)]\Gamma(zq^n)|_{z^n}$

$= \Delta(z)\Gamma(zq^{n-1}) - z\Delta(z)\Gamma(zq^n)|_{z^n}$

$+ \Delta(zq^2)\Gamma(zq^n) - zq^{2n-2}\Delta(z)\Gamma(zq^{n-1})|_{z^{n-1}}$

$= \varphi_n + q^{2n-2}\varphi_{n-1}$

as desired.

(f) *The Rogers-Selberg identities.*

In the same paper [25] quoted above, Rogers actually derives 4.11 as a special

case of a whole family of identities. Somewhat later the same identities were established by Selberg in [28]. They acquired special attention since Andrews [2] discovered the remarkable fact that they can be used to derive the more recent partition identities of Basil Gordon [19], [20].

Changing only slightly from the notation of Rogers in [25] we set $V_0 = 0$ and for $i = 1,\ldots,k$

$$V_i(z) = V_i(z,q) = \Sigma_{n\geqslant 0}(-z^k)^n q^{kn^2+(n+1)-in}(1-z^i q^{2ni})\frac{(1-z)\cdots(1-zq^{n-1})}{(1-q)\cdots(1-q^n)}.$$

The identities in question are then

$$\frac{V_{i+1}(z)-V_i(z)}{1-z} = z^i V_{k-i}(zq).$$

Rogers establishes them in a few lines starting from scratch. This given the only real question remaining is what causes the existence of such identities.

Taking all this into account it should be of interest to see what roofing brings us in this case. The natural thing to do in our context is to set

$$\varphi_i(z) = \frac{V_i(z,q^k)}{\Pi_{m\geqslant 0}(1-zq^{km})}.$$

In other words

$$4.18\quad \varphi_i(z) = \Sigma_{n\geqslant 0}\frac{(-z^k)^n q^{k\binom{n+1}{2}+kn(1-i)+2\binom{kn}{2}}}{(1-q^k)\cdots(1-q^{kn})\Pi_{m\geqslant 0}(1-zq^{kn+km})}(1-z^i q^{2ink}).$$

The Rogers-Selberg identities can thus be written in the form

$$4.19\quad \varphi_{i+1}(z) = \varphi_i(z) + z^i\varphi_{k-i}(zq^k).$$

Now setting

$$\Gamma(z) = \Sigma_{n\geqslant 0}\frac{(-z^k)^n q^{k\binom{n+1}{2}}}{(1-q^k)\cdots(1-q^{kn})} = \Pi_{m\geqslant 1}(1-z^k q^{km}),$$

$$\Delta(z) = \Pi_{m\geqslant 0}\frac{1}{1-zq^{km}},$$

roofing 4.18 yields

$$4.20\quad \hat{\varphi}_i(z) = \overset{\vee}{\Gamma}(zq^{1-i})\hat{\Delta}(z) - \overset{\vee}{\Gamma}(zq)[z^i\hat{\Delta}(z)].$$

Our aim is to show that the identities

$$4.21\quad (a)\quad \Gamma(z) = (1-z^k q^k)\Gamma(zq),$$

(b) $(1-z)\Delta(z) = \Delta(zq^k)$,

lead to 4.19 *coefficientwise*, that is

4.22 $\varphi_{i+1}(z)|_{z^n} = \varphi_i(z)|_{z^n} + \varphi_{k-i}(zq^k)|_{z^{n-i}}$.

To this end, observe first that 3.7 applied to 4.20 gives

$$\varphi_i(z)|_{z^n} = \Delta(z)\Gamma(zq^{n-i}) - z^i\Delta(z)\Gamma(zq^n)|_{z^n}.$$

Thus, using 4.21(a) and (b), we get

$$\begin{aligned}
\varphi_{i+1}(z)|_{z^n} &= \Delta(z)\Gamma(zq^{n-i-1}) - z^{i+1}\Delta(z)\Gamma(zq^n)|_{z^n} \\
&= \Delta(z)(1-z^kq^{k(n-i)})\Gamma(zq^{n-i}) + z^i[\Delta(zq^k) - \Delta(z)]\Gamma(zq^n)|_{z^n} \\
&= \Delta(z)\Gamma(zq^{n-i}) - z^i\Delta(z)\Gamma(zq^n)|_{z^n} \\
&\quad + \Delta(zq^k)\Gamma(zq^n) - q^{k(n-i)}z^{k-i}\Delta(z)\Gamma(zq^{n-i})|_{z^{n-i}} \\
&= \varphi_i(z)|_{z^n} + q^{k(n-i)}\varphi_{k-i}(z)|_{z^{n-i}}
\end{aligned}$$

and this is 4.22.

(g) *An identity of Bressoud.*

It may happen that we start with an identity and after subjecting it to successive roofing or unroofing we may end up with an equivalent identity which may look deceivingly different from the original one. As an example in point we shall show that the Rogers-Ramanujan identity 4.11 and the identity

4.23 $\Sigma_{n\geqslant 0}\dfrac{(-z)^nq^{n^2}(1-z^2)\cdots(1-z^2q^{n-1})}{(1-q)\cdots(1-q^n)}(1-zq^n) = \Pi_{m\geqslant 0}(1-zq^m)(1-z^2q^{2m+1})$

given by Bressoud in [10] (Lemma 7) are equivalent.

To bring 4.23 into proper from we replace q by q^2 and divide it by $\Pi_{m\geqslant 0}(1-z^2q^{2m})$. This gives

4.24 $\Sigma_{n\geqslant 0}\dfrac{(-z)^nq^{2n^2}(1-zq^{2n})}{(1-q^2)\cdots(1-q^{2n})\Pi_{m\geqslant 0}(1-z^2q^{2n+2m})} = \Pi_{m\geqslant 0}\dfrac{1}{1+zq^{2m}}$.

Now let $\psi(z)$ for a moment denote the left hand side of 4.24 and $\varphi(z)$ denote the left hand side of 4.12. We claim that

4.25 $\psi(z) = \Downarrow\hat{\hat{\varphi}}(z)$.

To see this note first that as in the case of 4.12 we get

$$\hat{\psi}(z) = \check{\Gamma}_1(z)\hat{\Delta}_1(z) - \check{\Gamma}_1(zq)[z\Delta_1(z)]\hat{}$$

where now

$$\Gamma_1(z) = \Pi_{m\geqslant 1}(1-zq^{2m}),$$

$$\Delta_1(z) = \Pi_{m\geqslant 0}\frac{1}{1-z^2q^{2m}}.$$

Thus 3.7 again gives

$$4.26\quad \psi_n = \Delta_1(z)\Gamma_1(zq^{n-1}) - z\Delta_1(z)\Gamma_1(zq^n)|_{z^n}.$$

Comparing with 4.14 we can see that the roles of Γ,Δ are interchanged from one identity to the other. This observation leads naturally to a proof of 4.25.

Indeed starting from 4.14 we get

$$\varphi_n = q^{(n-1)n}[\Gamma(z)\Delta(z/q^{n-1}) - z\Gamma(z)\Delta(z/q^n)]|_{z^n}.$$

Thus

$$4.27\quad \Downarrow\varphi_n = q^{-2\binom{n}{2}}[\Downarrow\Gamma(z)(\Downarrow\Delta)(zq^{n-1}) - z\Downarrow\Gamma(z)(\Downarrow\Delta)(zq^n)]|_{z^n}.$$

However we see by 2.9 that

$$\Downarrow\Gamma(z) = \Pi_{m\geqslant 1}(1-z^2/q^{2m}) = \frac{1}{\Pi_{m\geqslant 0}(1-z^2q^{2m})} = \Delta_1(z),$$

$$\Downarrow\Delta(z) = \Pi_{m\geqslant 0}\frac{1}{1-z/q^{2m}} = \Pi_{m\geqslant 1}(1-zq^{2m}) = \Gamma_1(z),$$

and thus 4.27 is none other than

$$\Downarrow\varphi_n = q^{-2\binom{n}{2}}\psi_n$$

which is another way of writing 4.25.

It might be good to point out that aside from the similarity between 4.14 and 4.26 the present theory suggests the relation 4.25 from the simple fact that it *does* hold trivially for the right hand sides of 4.12 and 4.24. Indeed we have

$$\Downarrow\frac{(-1)^n}{(1-q^2)\cdots(1-q^{2n})} = \frac{q^{2n^2-2\binom{n}{2}}}{(1-q^2)\cdots(1-q^{2n})}.$$

In [10] Bressoud labels 4.23 an *asymmetric triple product* identity. From the

present point of view it may be more appropriate to label it a *dual Rogers-Ramanujan* identity. This given it is tempting to apply the above procedure to construct *dual Rogers-Selberg* identities. Indeed this can be done with only minor modifications. More precisely the idea is to start with the φ_i's as given by 4.18 and set

$$4.28 \quad \psi_i(z) = (\Downarrow\hat{\hat{\varphi}}_i)(z/q^{i-1}).$$

Then we get the following result.

THEOREM 4.1. *The q-series*

$$4.29 \quad \psi_i(z) = \Sigma_{n\geqslant 0}\frac{(-z)^n q^{k\binom{n+1}{2}+n(1-i)+2\binom{n}{2}}}{(1-q^k)\cdots(1-q^{kn})\Pi_{m\geqslant 0}(1-z^k q^{kn+km})}(1-z^i q^{2ni}) \ (\psi_0 = 0)$$

are related by the following recursions

$$4.30 \quad \psi_{i+1}(zq) = \psi_i(z) + z^i\psi_{k-i}(z).$$

PROOF. We recall that the φ_i's as defined by 4.18 have coefficients given by the equations

$$4.31 \quad \varphi_i(z)|_{z^n} = \Delta(z)\Gamma(zq^{n-i}) - z^i\Delta(z)\Gamma(zq^n)|_{z^n}$$

where

$$4.32 \quad \Gamma(z) = \Pi_{m\geqslant 1}(1-z^k q^{km}),$$

$$\Delta(z) = \Pi_{m\geqslant 0}\frac{1}{1-zq^{km}}.$$

Now, we can rewrite 4.31 in the form

$$\varphi_i(z)|_{z^n} = q^{n(n-i)}\{\Delta(z/q^{n-i})\Gamma(z) - z^i\Delta(z/q^n)\Gamma(z)\}|_{z^n}.$$

Thus

$$4.33 \quad \Downarrow\varphi_i(z) = q^{-n(n-i)}\{(\Downarrow\Delta)(zq^{n-i})\Downarrow\Gamma(z) - z^i(\Downarrow\Delta)(zq^n)\Downarrow\Gamma(z)\}|_{z^n}.$$

Now 4.28 yields that

$$\psi_i(z)|_{z^n} = q^{-n(i-1)}q^{n^2-n}\Downarrow\varphi_i(z)|_{z^n} = q^{n(n-i)}\Downarrow\varphi_i(z)|_{z^n}.$$

Comparing with 4.33 we obtain

$$4.34 \quad \psi_i(z)|_{z^n} = \Downarrow\Gamma(z)(\Downarrow\Delta)(zq^{n-i}) - z^i\Downarrow\Gamma(z)(\Downarrow\Delta)(zq^n)|_{z^n}$$

which, as we see, is of the same form as 4.31.

Formulas 4.32 and 2.9 then give

$$\Downarrow\Gamma(z) = \Pi_{m\geqslant 1}(1-z^k/q^{km}) = \Pi_{m\geqslant 0}\frac{1}{1-z^k q^{km}},$$

$$\Downarrow\Delta(z) = \Pi_{m\geqslant 0}\frac{1}{1-z/q^{km}} = \Pi_{m\geqslant 1}(1-zq^{km}).$$

Substituting this in 4.34 and reversing the steps that led to 4.31 from 4.18, yields that the ψ_i's defined by 4.28 have precisely the expressions given in 4.29.

To obtain the recursions 4.30 we apply the same procedure to 4.19. Indeed roofing 4.19 twice yields

$$\hat{\hat{\varphi}}_{i+1}(z) = \hat{\hat{\varphi}}_i(z) + z^i q^{-2\binom{i}{2}}\hat{\hat{\varphi}}_{k-i}(zq^{k-2i}),$$

thus

$$\Downarrow\hat{\hat{\varphi}}_{i+1}(z) = \Downarrow\hat{\hat{\varphi}}_i(z) + z^i q^{i(i-1)}(\Downarrow\hat{\hat{\varphi}}_{k-i})(zq^{2i-k}).$$

Replacing z by z/q^{i-1} this can be rewritten in the form

$$(\Downarrow\hat{\hat{\varphi}}_{i+1})(zq/q^i) = (\Downarrow\hat{\hat{\varphi}}_i)(z/q^{i-1}) + z^i(\Downarrow\hat{\hat{\varphi}}_{k-i})(z/q^{k-i-1}).$$

However, in view of 4.28, this is simply 4.30.

Andrews informs us that the series given in 4.29 can be obtained from formula 2.1 of [9] by a suitable specialization of the parameters. However, the recursions in 4.30 and the close connection of Theorem 4.1 to the Rogers-Selberg identities appear to be new.

All the identities we have studied here have the remarkable feature that their validity is a more or less direct consequence of the elementary series product identity of Euler. The reader may find that quite many identities in the theory of q-series are simply "tangle" products of various versions of the Euler Identity. Of course some of them lie deeper than others although on the surface they may look of equal complexity. The present point of view suggests that a crude measure of complexity for a given q-series identity may be defined in terms of how far removed it is (in terms of roofing steps) from a simple series-product identity, the identities of the $k+1^{st}$ level being tangle products of an Euler identity with an identity of the k^{th} level. This would put examples (a) and (b) at the 0^{th} level, Cauchy and Sylvester at the 1^{st}, Rogers-Ramanujan at the 2^{nd} together with Bressoud's and the Rogers-Selberg,

together with their dual at the k^{th} level. That is one way of interpreting Andrews' multiple series identities (see [4] formulas 7.38 and 7.39 page 112).

The reader may verify for himself that the following identities of Rogers [26] (see also [5] formulas 3.21 and 3.22) are only of the first level of complexity.

$$\Sigma_{n\geqslant 0}\frac{q^{n^2}z^n}{(1-q)\cdots(1-q^n)}=\Sigma_{n\geqslant 0}\frac{q^{n^2}z^n}{(1-q^2)\cdots(1-q^{2n})}\Pi_{m\geqslant 0}(1+zq^{2m+2n+2})$$

$$\Sigma_{n\geqslant 0}\frac{q^{4n^2}z^{2n}}{(1-q^4)\cdots(1-q^{4n})}=\Sigma_{n\geqslant 0}\frac{q^{n^2}z^n}{(1-q^2)\cdots(1-q^{2n})}\Pi_{m\geqslant 0}(1-zq^{2m+2n+1}).$$

The same holds true for the identity given by Andrews in [6],

$$\Sigma_{n\geqslant 0}\frac{q^{2n^2-n}(1-zq^{2n}+zq^{3n})z^n}{(1-q)\cdots(1-q^n)(1-z)\cdots(1-zq^{2n})}=\Pi_{m\geqslant 0}\frac{1}{1-zq^m}.$$

Indeed the latter affords a proof entirely analogous to that we gave the Sylvester identity.

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