Some old methods in the Theory of m-Quasi-Invariants

by
A.M. Garsia and N. Wallach

ABSTRACT

We show here that some representation theoretical methods developed in the early nineties for the study of the Garsia-Haiman modules may successfully be used in the theory of m-Quasi-Invariants. Using the non-degeneracy of the fundamental bilinear m-form we show here that the Cohen Macauliness of the ring $\mathcal{QI}_m$ of m-Quasi-Invariants may be derived with the greatest of ease. In fact, in addition, this approach directly yields that the quotient of $\mathcal{QI}_m$ by the ideal generated by the elementary symmetric functions affords a graded version of the left regular representation. These results had been previously obtained by combining work of Etingof-Ginzburg [], Feigin-Veselov [] and Felder-Veselov []. Our presentation here focuses on the S_n case and it is completely self-contained except in what concerns the bilinear m-form. These methods are also applicable in the case of general reflection groups.

Introduction

1. The orbit Ring

We begin by selecting a point

$$a = (a_1, a_2, \ldots, a_n)$$

with distinct components and denote by $[a]_{S_n}$, (or briefly by $[a]$) the S_n orbit of a. In symbols

$$[a\,] \;=\; \{a\sigma \,:\, \sigma \in S_n\}$$

Next we construct a polynomial $\phi_a(x)$ such that

$$\phi_a(x) \;=\; \begin{cases} 1 & \text{if } x = a \\ & \\ 0 & \text{if } x = b \text{ with } b \in [a\,] \text{ and } b \neq a \end{cases} \qquad 1.1$$

To get the best results it is important that we are as economical as possible with the degree of $\phi_a(x)$. To this end note that the polynomial

$$(x_1 - a_2)(x_1 - a_3)\cdots(x_1 - a_n)$$

vanishes at all points $a\sigma = (a_{\sigma_1} a_{\sigma_2}, \ldots, a_{\sigma_n})$ where $\sigma_1 \neq 1$. It is clear that once we place this factor in our $\phi_a(x)$, the only points of the orbit we need to worry about are those where $\sigma_1 = 1$. But on those we can use the same trick to force $\sigma_2 = 2$ by placing in ϕ_a the also the factor

$$(x_2 - a_3)(x_2 - a_4)\cdots(x_2 - a_n)$$

Proceeding in this manner we can easily see that the polynomial

$$\phi_a(x) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \frac{(x_i - a_j)}{(a_i - a_j)}$$

will satisfy the requirement in (1). We should note that we have

$$degree(\phi_a(x)) \; = \; \binom{n}{2}$$

It can be shown that this is the least degree needed to satisfy 1.1.

Now for each $b \in [a\,]$ with $b = a\sigma$ we set

$$\Phi_b(x) \; = \; \phi_a(x\sigma^{-1}) \,, \qquad \epsilon(b) \; = \; sign(\sigma) \,. \tag{1.2}$$

Note that from (1) it immediately follows that we have

$$\phi_b(x) = \begin{cases} 1 & \text{if } x = a\sigma \\ \\ 0 & \text{if } x = b\sigma \text{ with } b \in [a\,] \text{ and } b\sigma \neq a\sigma \end{cases}$$

this may be rewritten as

$$\phi_b(x) = \begin{cases} 1 & \text{if } x = b \\ \\ 0 & \text{if } x = b' \text{ with } b' \in [a\,] \text{ and } b' \neq b \end{cases} \tag{1.3}$$

Next we let $\mathcal{J}_{[a]}(m)$ denote the ideal of m-quasi-invariants that vanish on $[a\,]$. In symbols

$$\mathcal{J}_{[a]}(m) \; = \; \big\{ P(x) \in \mathcal{QI}_m[X_n] \; : \; P(b) = 0 \;\; \forall \;\; b \in [a\,] \big\} \tag{1.4}$$

This given, it follows that

Proposition 1.1

The quotient ring

$$\mathbf{R}_{[a\,]} \; = \; \mathcal{QI}_m[X_n]/\mathcal{J}_{[a]}(m) \tag{1.5}$$

has dimension $n!$ and affords the left regular representation of S_n

Proof

Let $\Pi(x)$ denote the Vandermonde determinant in $x_1, x_2, \ldots, x_n$, and set

$$\psi_b(x) \; = \; \phi_b(x) \tfrac{\Pi(x)^{2m}}{\Pi(a)^{2m}} \,.$$

Note that since $\Pi(x)^{2m}$ is S_n-invariant we derive from (3) that

$$\psi_b(x) = \begin{cases} 1 & \text{if } x = b \\ \\ 0 & \text{if } x = b' \text{ with } b' \in [a\,] \text{ and } b' \neq b \end{cases} \tag{1.6}$$

moreover we also have

$$\psi_b(x) \in \mathcal{QI}_m[X_n] \quad \forall \quad b \in [a\,]\,. \tag{1.7}$$

Now note that for any $P \in \mathcal{QI}_m[X_n]$ 1.6 gives

$$P(x) \,-\, \sum_{b \in [a\,]} P(b)\,\psi_b(x) \in \mathcal{J}_{[a\,]}(m)\,.$$

It will be convenient to express this relation by writing

$$P(x) \,\cong_{[a]} \sum_{b \in [a\,]} P(b)\,\psi_b(x)\,. \tag{1.8}$$

Thus the collection

$$\big\{\psi_b(x)\big\}_{b \in [a\,]} \tag{1.9}$$

is a basis for the quotient $\mathbf{R}_{[a\,]}$.

Since the ideal $\mathcal{J}_{[a\,]}$ is S_n-invariant it immediately follows that S_n acts on $\mathbf{R}_{[a\,]}$. Thus to complete our proof we only need to compute the character of this action. To this end note that for each $\sigma \,\&\, \beta \in S_n$ from 1.8 we derive that

$$\begin{aligned} \sigma\psi_{a\beta}(x) \,=\, \psi_a(x\sigma\beta^{-1}) \,&\cong_{[a\,]} \sum_{\alpha \in S_n} \psi_a(a\alpha\sigma\beta^{-1})\psi_{a\alpha}(x) \\ &\cong_{[a\,]} \sum_{\alpha \in S_n} \chi\big(\alpha\sigma\beta^{-1} = id\big)\psi_{a\alpha}(x) \\ &\cong_{[a\,]} \sum_{\alpha \in S_n} \chi\big(\alpha\sigma = \beta\big)\psi_{a\alpha}(x) \end{aligned}$$

Thus the action of S_n on the basis $\big\{\psi_b(x)\big\}_{b \in [a\,]}$ is given by the matrix $A(\sigma) = \|\chi\big(\alpha\sigma = \beta\big)\|_{\alpha,\beta \in S_n}$. It follows that the character of the S_n action on $\mathbf{R}_{[a\,]}$ is given by

$$trace\, A(\sigma) \,=\, \sum_{\alpha \in S_n} \chi\big(\alpha\sigma = \alpha\big) \,=\, \begin{cases} n! & \text{if } \sigma = id \\ \\ 0 & \text{if } \sigma \neq id \end{cases}$$

and this is the character of the left regular representation of S_n precisely as asserted.

Remark 1.1

The ring $\mathbf{R}_{[a\,]}$ is not graded but it has a filtration given by the subspaces

$$\mathcal{H}_{\leq k}(\mathbf{R}_{[a\,]}) \,=\, \mathcal{L}_{[a\,]}\big[P \,:\, P \in \mathcal{H}_{\leq k}\big(\mathcal{QI}_m[X_n]\big)\big] \tag{1.10}$$

where "$\mathcal{L}_{[a\,]}$" denotes "*Linear Span*" modulo $\mathcal{J}_{[a\,]}$ and $\mathcal{H}_{\leq k}\big(\mathcal{QI}_m[X_n]\big)$ is the subspace of $\mathcal{QI}_m[X_n]$ spanned by its elements of degree $\leq k$. Now note that since by construction we have

$$degree\, \psi_b(x) \,=\, (2m+1)\tbinom{n}{2} \tag{1.11}$$

it immediately follows from the expansion in 1.8 that

$$\mathbf{R}_{[a\,]} \;=\; \mathcal{H}_{\le d_{m,n}}\big(\mathbf{R}_{[a\,]}\big) \tag{1.12}$$

where for convenience we have set

$$d_{m,n} \;=\; (2m+1)\tbinom{n}{2} \tag{1.13}$$

2. The graded version of $\mathbf{R}_{[a\,]}$

For each polynomial P we shall here and after denote by $h(P)$ the homogeneous component of highest degree in P. This given, we let $\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}\,]}$ be the ideal in $\mathcal{QI}_m[X_n]$ generated by the highest degree components of elements of $\mathcal{J}_{[a\,]}$ in symbols

$$\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}\,]} \;=\; \big(h(P)\,:\,P\in\mathcal{J}_{[a\,]}\,\big)_{\mathcal{QI}_m[X_n]}\,. \tag{2.1}$$

This brings us to define the "*graded version* " of $\mathbf{R}_{[a\,]}$ as the quotient

$$\mathbf{grR}_{[\mathbf{a}\,]} \;=\; \mathcal{QI}_m[X_n]/\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}\,]}\,. \tag{2.2}$$

Since $\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}\,]}$ is generated by homogeneous polynomials, $\mathbf{grR}_{[\mathbf{a}\,]}$ is necessarily a graded ring. Note further that if P is any homogeneous polynomial of degree $> d_{m,n}$ then the fact that

$$P(x)\;-\;\sum_{b\in[a\,]}P(b)\,\psi_b(x)\in J_{[a\,]}$$

immediately implies that $P\in\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}\,]}$. Thus it follows that $\mathbf{grR}_{[\mathbf{a}\,]}$ has the direct sum decomposition

$$\mathbf{grR}_{[\mathbf{a}\,]} \;=\; \bigoplus_{k=0}^{d_{m,n}}\mathcal{H}_{=k}(\mathbf{grR}_{[\mathbf{a}\,]}) \tag{2.3}$$

where $\mathcal{H}_{=k}(\mathbf{grR}_{[\mathbf{a}\,]})$ denotes the subspace of $\mathbf{grR}_{[\mathbf{a}\,]}$ spanned by its homogeneous elements of degree k. It will be convenient to choose once and for all a basis $\mathcal{A}^{=k}$ for each subspace $\mathcal{H}_{=k}(\mathbf{grR}_{[\mathbf{a}\,]})$. This given, we have the following useful fact

Proposition 2.1

The collection

$$\mathcal{B}^{\le k} \;=\; \mathcal{A}^{=0}\;+\;\mathcal{A}^{=1}\;+\;\cdots\;+\;\mathcal{A}^{=k}$$

is also a basis for the subspace $\mathcal{H}_{\le k}(\mathbf{R}_{[a\,]})$. In particular $\mathcal{B}_{\le d_{m,n}}$ is a basis of $\mathbf{R}_{[a\,]}$

Proof

Clearly the result holds true for $k=0$ since $\mathcal{A}_{=0}$ reduces to the single constant $\mathbf{1}$. So we may proceed by induction on k. Let us then assume that each $P\in\mathcal{H}_{\le k-1}(\mathbf{R}_{[a\,]})$ has an expansion in terms of $\mathcal{B}^{\le k-1}$. This given, note that any $P\in\mathcal{H}_{\le k}(\mathcal{QI}_m[X_n])$, viewed as a representative of an element of $\mathbf{grR}_{[\mathbf{a}\,]}$, may be expanded in terms of the basis $\mathcal{B}^{\le k}$. In other words we will have constants c_ϕ such that

$$P \;=\; \sum_{\phi\in\mathcal{B}^{\le k}}c_\phi\,\phi\;+\;R \tag{2.4}$$

for a suitable $R \in \mathbf{gr}\,\mathcal{J}_{[\mathbf{a}]}$. Now the definition of $\mathbf{gr}\,\mathcal{J}_{[\mathbf{a}]}$ yields that there must be elements $f_i \in J_{[a]}$ and $A_i \in \mathcal{QI}_m[X_n]$ giving

$$R \;=\; \sum_i A_i\, h(f_i) \tag{2.5}$$

Since R is necessarily of degree $\leq k$ we see that there is no loss in assuming that we have

$$degree\, A_i \;\leq\; k - degree\, h(f_i) \tag{2.6}$$

Using 2.5 we may rewrite 2.4 in the form

$$P \;=\; \sum_{\phi\in\mathcal{B}^{\leq k}} c_\phi\, \phi \,+\, \sum_i A_i\, f_i \;-\; \sum_i A_i\,(f_i - h(f_i))\,. \tag{2.7}$$

and this implies that

$$P \;-\; \sum_{\phi\in\mathcal{B}^{\leq k}} c_\phi\, \phi \;\cong_{[a]}\; -\sum_i A_i\,(f_i - h(f_i))\,. \tag{2.8}$$

now 2.6 and the definition of $h(f_i)$ yield that

$$degree \sum_i A_i\,(f_i - h(f_i)) \;\leq\; k-1 \tag{2.9}$$

thus our inductive hypothesis yields that we have constants d_ϕ giving

$$\sum_i A_i\,(f_i - h(f_i)) \;\cong_{[a]} \sum_{\phi\in\mathcal{B}^{\leq k-1}} d_\phi\, \phi$$

and this combined with 2.8 completes the induction. This proves the first assertion. The second assertion follows from the identity in 1.12.

Proposition 2.1 has the following remarkable corollary.

Theorem 2.1

The ring $\mathbf{grR}_{[\mathbf{a}]}$ yields a graded version of the regular representation. In fact for every $0 \leq k \leq d_{m,n}$ we have the character relation

$$char\; \mathcal{H}_{=k}(\mathbf{grR}_{[\mathbf{a}]}) \;=\; char\; \mathcal{H}_{\leq k}(\mathbf{R}_{[a]}) \;-\; char\; \mathcal{H}_{\leq k-1}(\mathbf{R}_{[a]}) \tag{2.10}$$

In particular the characters of the subspaces $\mathcal{H}_{\leq k}(\mathbf{R}_{[a]})$ are related to the graded character of $\mathbf{grR}_{[\mathbf{a}]}$ by the following identity

$$\sum_{k=0}^{d_{m,n}} q^k char\; \mathcal{H}_{=k}(\mathbf{grR}_{[\mathbf{a}]}) \;=\; q^{d_{m,n}} char\; \mathbf{R}_{[a]} \;+\; (1-q) \sum_{k=0}^{d_{m,n}-1} q^k\; char\; \mathcal{H}_{\leq k}(\mathbf{R}_{[a]}) \tag{2.11}$$

Proof

Set $\mathcal{B}^{\leq k} = \{\phi_i\}_{i=1}^{m_k}$ so that $\mathcal{A}^{=k} = \{\phi_i\}_{m_{k-1}<i\leq m_k}$ and let $A(\sigma) = \|a_{i,j}(\sigma)\|_{i,j=1}^{n!}$ be the matrix expressing the action of S_n on the basis $\mathcal{B}^{\leq d_{m,n}}$ as elements of $\mathbf{R}_{[a\,]}$. Since $\mathcal{H}_{\leq k}(\mathbf{R}_{[a\,]})$ is S_n invariant it follows that for any $j \leq m_k$ we have the expansion

$$\sigma\phi_j \;=\; \sum_{i=1}^{m_k} \phi_i\, a_{i,j}(\sigma) \,+\, R \qquad\qquad (\text{with } R \in \mathcal{J}_{[a\,]} \text{ of degree} \leq k) \qquad\qquad 2.12$$

This means that the action of S_n on the subspace $\mathcal{H}_{\leq k}(\mathbf{R}_{[a\,]})$ induces a induces a representation given by the matrix

$$A^{\leq k}(\sigma) \;=\; \big\|a_{i,j}(\sigma)\big\|_{1<i,j\leq m_k} \qquad\qquad 2.13$$

Note further that when $m_{k-1} < j \leq m_k$, then ϕ_j is homogeneous of degree k and equating homogeneous terms of degree k in 2.12 we get

$$\sigma\phi_j \;=\; \sum_{m_{k-1}<i\leq m_k} \phi_i\, a_{i,j}(\sigma) \,+\, \begin{cases} h(R) & \text{if } \textit{degree } R = k \\ 0 & \text{if } \textit{degree } R < k \end{cases} \qquad\qquad (\text{with } R \in \mathcal{J}_{[a\,]}\ ).$$

In either case we derive that

$$\sigma\phi_j \;\cong\; \sum_{m_{k-1}<i\leq m_k} \phi_i\, a_{i,j}(\sigma) \qquad\qquad (\text{modulo } \mathbf{gr}\, \mathcal{J}_{[\mathbf{a}\,]})$$

This shows that the action of S_n on the subspace $\mathcal{H}_{=k}\big(\mathbf{grR}_{[\mathbf{a}\,]}\big)$ induces a representation given by the matrix

$$A^{=k}(\sigma) \;=\; \big\|a_{i,j}(\sigma)\big\|_{m_{k-1}<i,j\leq m_k}$$

Thus

$$\textit{char}\ \mathcal{H}_{=k}\big(\mathbf{grR}_{[\mathbf{a}\,]}\big) \;=\; \textit{trace}\ A^{=k} \;=\; \textit{trace}\ A^{\leq k} \,-\, \textit{trace}\ A^{\leq k-1}\,.$$

This proves 2.10. This given, multiplying 2.10 by q^k and summing for $0 \leq k \leq d_{m,n}$, the identity in 1.12 yields 2.11.

3. The orbit m-Harmonics

For a given orbit $[a]$ we set

$$\mathbf{H}_{[a]}(m) \;=\; \big\{P \in \mathcal{QI}_m[X_n] \,:\, \big\langle P\,,\, Q\big\rangle_m \;=\; 0 \quad \forall Q \in \mathbf{gr}\, \mathcal{J}_{[\mathbf{a}\,]}\big\}$$