IDEMPOTENTS FOR THE FREE LIE ALGEBRA AND q-ENUMERATION

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Abstract. The n^{th} homogeneous component of the free Lie algebra over an alphabet A may be expressed as the linear span of polynomials obtained by multiplying words of length n by a fixed idempotent of the group algebra of S_n. We construct here a number of such idempotents and prove some of their properties by means of identities obtained from the theory of P-partitions. We also construct an idempotent whose coefficients give the Taylor expansion of the reciprocal of the cyclotomic polynomial. As a by-product we obtain a new proof of the dimension formula for the free Lie algebra. The paper ends with the proof of a conjecture of R. Stanley concerning certain representations of S_n introduced by Reutenauer (Springer L.N. # 1234, pp. 267–293).

Introduction. Let $A = \{a_1, a_2, \cdots, a_N\}$ be an alphabet with N letters and let A^* and A^n, as customary, denote the collections of all A-words and respectively all A-words with n letters. The (non-commutative) algebra of all polynomials

$$f = \sum_{w \in A^*} f_w w$$

with rational coefficients will be denoted here by $Q[A^*]$. Multiplication in $Q[A^*]$ is carried out by means of the familiar concatenation product of words in A^*. The subspace of $Q[A^*]$ spanned by the words of length n will be denoted by $Q[A^n]$. The *bracket* of two polynomials f, g is defined by setting

$$[f, g] = fg - gf.$$

We shall deal here with vector subspaces of $Q[A^*]$ spanned by families of polynomials which are obtained by successive bracketings of elements of $Q[A^*]$. To be precise, given a complete binary tree T with n terminal nodes and a word $w = b_1 b_2 \cdots b_n$ of A^n, we shall let the pair (w, T) represent the configuration obtained by appending the letters of w, successively and from left to right, upon the leaves of T. Let T have T_1 and T_2 as left and right subtrees and let w_1 and w_2 be the portions of w respectively hanging from T_1 and T_2 in (w, T). This given, we recursively set

$$b(w, T) = [b(w_1, T_1), b(w_2, T_2)], \tag{I.1}$$

with the agreement that when T consists of a single node (that is if $n = 1$ and w is a letter of A) we must set

$$b(w, T) = w. \tag{I.2}$$

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The subspace of $Q[A^*]$ spanned by the polynomials $b(w,T)$ will be denoted here by LIE$[A]$ and referred to as the *Free Lie Algebra* on the Alphabet A. The subspace of LIE$[A]$ spanned by the $b(w,t)$ with w of length n will be denoted by LIE$[A^n]$. We shall also denote by

$$\mathrm{LIE}[a_1 a_2 \cdots a_n]$$

the subspace of LIE$[A]$ spanned by the polynomials $b(w,T)$ when w is restricted to vary among words which are permutations of the letters

$$a_1 a_2 \cdots a_n.$$

We shall make use here of the action of the symmetric group S_n on words of A^n. More precisely, if

$$\sigma = \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix} \in S_n$$

and

$$w = b_1 b_2 \cdots b_n \in A^n$$

then we let

$$w\sigma = b_{\sigma_1} b_{\sigma_2} \cdots b_{\sigma_n}.$$

It is convenient to identify LIE$[a_1 a_2 \cdots a_n]$ with a subspace of the group algebra $A(S_n)$ by identifying the word

$$a_{\sigma_1} a_{\sigma_2} \cdots a_{\sigma_n}$$

with the permutation

$$\sigma_1 \sigma_2 \cdots \sigma_n.$$

In this vein, as we shall see, we may represent the subspace LIE$[a_1 a_2 \cdots a_n]$ as a left ideal in $A(S_n)$. More precisely LIE$[a_1 a_2 \cdots a_n]$ may be written in the form

$$\mathrm{LIE}[a_1 a_2 \cdots a_n] = A(S_n)\rho, \tag{I.3}$$

with ρ an idempotent element of $A(S_n)$. Here and after, such an element will be referred to as a *Lie idempotent*.

In this paper we shall present a number of Lie idempotents with remarkable combinatorial properties. It develops that the free Lie Algebra is closely related to the q-enumeration of permutations by the major index statistic. Surprisingly, a study of this statistic leads to a new family of idempotents which involve in a curious way the coefficients of the cyclotomic polynomials. This also leads to some very interesting combinatorial and representation theoretical questions. One of the by-products of our work is a rather direct calculation of the dimension of the finely homogeneous components of the free Lie algebra.

Reutenauer in [8] studied certain idempotents $\rho_n^{(k)}$, which induce representations of S_n with dimension given by the Stirling numbers $s(n,k)$. Since these representations are closely related to the Free Lie Algebra, it is natural, in the present context, to ask for a combinatorial explanation of their dimension. At the end of this paper we give a proof of a conjecture of R. Stanley suggesting a certain method of constructing them which does throw some light into this question.

There is extensive literature on the free Lie algebra, written with many different outlooks. We adopt here the combinatorial point of view. Such an approach to the free Lie algebra, including all the background material that will be needed here, may be found in [4].

1. The standard bracketing idempotents. We shall denote here by L_n an R_n the standard totally *leftist* and *rightist* trees on n leaves. In pictures

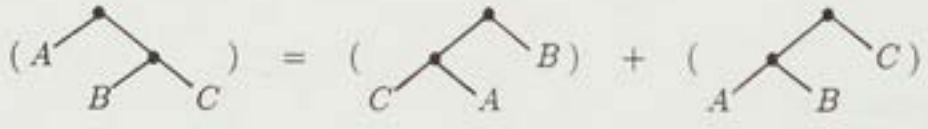
$L_4 =$  and $R_4 =$ 

We see that if $w = b_1b_2b_3b_4$ then

$$b(w, L_4) = [[[b_1, b_2], b_3], b_4] \quad \text{and} \quad b(w, R_4) = [b_1, [b_2, [b_3, b_4]]].$$

We shall refer to $b(w, L_n)$ and $b(w, R_n)$ as the standard *left* and *right* bracketings of words of length n.

By a successive application of the Jacobi identity in the form



we can derive (by an induction argument that every polynomial $b(w,T)$ may be expressed as a linear combination of standard left bracketings of words of A^*. A similar result holds for the standard right bracketings. Here and after we shall mostly be concerned with standard left bracketings.

It is not difficult to see that there is an element θ_n of the group algebra of S_n such that the standard left bracketing of a word $w = b_1b_2\cdots b_n$ is given by the expression

$$b(w, L_n) = \begin{bmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} \theta_n \tag{1.1}$$

Reutenauer in [8] points out that 1.1 actually holds with

$$\theta_n = (1-\gamma_2)(1-\gamma_3)\cdots(1-\gamma_n)$$

where γ_i (for $i = 2, \ldots, n$) denotes the *backwards* i-cycle that is

$$\gamma_i = (i, i-1, \ldots, 2, 1).$$

To see this, let us assume by induction that the result is true for $n-1$ (it is trivially true for $n=2$). We then have

$$b(a_1a_2,\dots a_n,L_n) = b(a_1a_2\dots a_{n-1},L_{n-1})a_n - a_n b(a_1a_2\dots a_{n-1},L_{n-1})$$

and this is equivalent to

$$\theta_n = \sum_{\sigma\in S_{n-1}} \theta_{n-1}(\sigma)\left(\begin{bmatrix}1 & 2 & 3 & \dots & n-1 & n\\ \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_{n-1} & n\end{bmatrix} - \begin{bmatrix}1 & 2 & 3 & \dots & n\\ n & \sigma_1 & \sigma_2 & \dots & \sigma_{n-1}\end{bmatrix}\right).$$

From which we easily derive that

$$\theta_n = \theta_{n-1}(1-\gamma_n)$$

which completes the induction.

Let us recall that the *descent set* of a permutation σ is the set

$$D(\sigma) = \{i : \sigma_i \geq \sigma_{i+1}\}.$$

If S is a subset of $[1,2,\dots,n]$ let us denote by $D_{=S}$ the formal sum of all the permutations of S_n whose descents are *exactly* the elements of S. Similarly, we let $D_{\subseteq S}$ denote the sum of the permutations whose descents are *contained* in S. In other words we have

$$D_{\subseteq S} = \sum_{T\subseteq S} D_{=T} \tag{1.2}$$

and, by inclusion exclusion

$$D_{=S} = \sum_{T\subseteq S}(-1)^{|S-T|}D_{\subseteq T}. \tag{1.3}$$

Now it develops that we have the following very useful identity

THEOREM 1.1.

$$\theta_n = \sum_{k=0}^{n-1}(-1)^k D_{=[1,k]}, \tag{1.4}$$

where here $[a,b]$ is a shorthand for the interval of integers $\{a,a+1,\dots,b\}$.

Proof. We shall only outline the proof here, details may be found in [4]. We clearly have

$$\theta_n = \sum_{S\subseteq[2,n]} (-1)^{|S|}\Big(\prod_{i\in S}\gamma_i\Big) \tag{1.5}$$

On the other hand, it is easily seen that if $S = \{i_1 < i_2 < \cdots < i_k\}$ and $[1,n] - S = \{1 < j_2 < j_3 < \cdots < j_h\}$ then

$$(1.6) \qquad (\prod_{i\in S} \gamma_i) = \begin{bmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & n-1 & n \\ i_k & i_{k-1} & \dots & i_1 & 1 & j_2 & \cdots & j_{h-1} & j_h \end{bmatrix}.$$

This given, grouping the terms in 1.5 with $|S| = k$, we get that

$$\sum_{|S|=k} \prod_{i\in S} \gamma_i = D_{=[1,k]},$$

which yields 1.4. □

Formula 1.3 can now be substituted in 1.4 and obtain

$$(1.7) \qquad \theta_n = \sum_{k=0}^{n-1} \sum_{T\subseteq[1,k]} (-1)^{|T|} D_{\subseteq T}.$$

This expression can be further transformed. To do this we need some notation.

If u and v are words of lengths k and $n-k$ respectively, let

$$u \sqcup\!\sqcup v$$

denote the formal sum of all the words of length n obtained by shuffling in all possible ways the letters of u with those of v. For instance

$$12 \sqcup\!\sqcup 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312.$$

If f is an element of the group algebra of S_n it is convenient to set

$$\downarrow f = \sum_{\sigma\in S_n} f(\sigma)\sigma^{-1}.$$

Note then that for $|T| = h$ we may write

$$(1.8) \qquad \downarrow D_{\subseteq T} = E_1 \sqcup\!\sqcup E_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_{h+1}$$

where E_i for $i = 1, 2, \dots, h+1$ are words obtained by cutting the word

$$12356\cdots n$$

precisely at the positions corresponding to the elements of T. For example, say $n = 9$ and $T = \{2,4,7\}$. Then

$$E_1 = 12,\ E_2 = 34,\ E_3 = 567,\ E_4 = 89$$

and

$$\downarrow D_{\subseteq\{2,4,7\}} = 12 \sqcup\!\sqcup 34 \sqcup\!\sqcup 567 \sqcup\!\sqcup 89.$$

Let us now introduce a scalar product on $Q[A^*]$ by setting for each pair of words $u, v \in A^*$

$$(1.9) \qquad \langle u, v\rangle = \chi(u = v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$H_n(u \sqcup\!\sqcup v)$$

be the subspace of $Q[A^n]$ consisting of polynomials which are linear combination of shuffles of pairs of non empty words.

This given, the following remarkable result holds true

THEOREM 1.2. *The four conditions below are equivalent for any element of* $Q(A^n)$:

(i). $f \in \mathrm{LIE}[A]$,

(ii). f *orthogonal to* $H_n(u \sqcup v)$ *with respect to the scalar product 1.9,*

(iii). $fD_{=S} = (-1)^{|S|} f$ *for all subsets* $S \subseteq [1, n-1]$,

(iv). $f\theta_n = nf$.

Proof.

The equivalence of (i), (ii) is classical and quite easy to show (see [4], [7]). Since (iv) $\rightarrow$ (i) trivially, it suffices to show the implications (ii) $\rightarrow$ (iii) $\rightarrow$ (iv).

(ii) $\rightarrow$ (iii)

Formula 1.8 yields that for any $w \in A^n$ and $T \neq \varnothing$

$$w \downarrow D_{\subsetneq T} \in H_n(u \sqcup v).$$

Thus (ii) implies that

$$\langle f, w \downarrow D_{\subsetneq T} \rangle = 0. \tag{1.10}$$

On the other hand, it is easy to see that for any $f, g \in Q(A^n)$ and any element ρ of the group algebra of S_n we have

$$\langle f\rho, g \rangle = \langle f, g \downarrow \rho \rangle.$$

Thus 1.10 may be rewritten as

$$\langle fD_{\subsetneq T}, w \rangle = 0$$

Since w is arbitrary we must necessarily have that

$$fD_{\subsetneq T} = 0$$

and formula 1.3 gives

$$fD_{=S} = \sum_{T \subseteq S} (-1)^{|S-T|} fD_{\subsetneq T} = (-1)^{|S|} f$$

as desired.

(iii) $\rightarrow$ (iv)

From 1.4 we get that

$$f\theta_n = \sum_{k=0}^{n-1} (-1)^k fD_{=[1,k]}.$$

Thus (iii) for $T = [1, k]$ gives

$$f\theta_n = \sum_{k=0}^{n-1}(-1)^k f\ (-1)^k = nf,$$

as asserted. □

Remark 1.1. The implication (i) → (iv) of Theorem 1.2 yields in particular that for any word $w \in A^n$

$$w\theta_n\theta_n = nw\theta_n.$$

Setting $w = a_1 a_2 \cdots a_n$ yields the identity

$$\theta_n\theta_n = n\theta_n.$$

In other words

$$\frac{1}{n}(1-\gamma_2)(1-\gamma_3)\cdots(1-\gamma_n)$$

is an *idempotent* element of the algebra of S_n.

This is a fact that is not obvious at all. A purely combinatorial proof of it, based on the identity 1.4. was found by M. Wachs [11].

It is not difficult to see that an independent proof of Theorem 1.2. follows from the purely combinatorial identity

$$\theta_n D_{=S} = (-1)^{|S|}\ \theta_n \tag{1.11}$$

which itself is a special case of (iii). Thus a simple direct proof of this identity would certainly be of interest here.

2. The Klyachko idempotent. In a little known pioneering paper [6], Klyachko brought himself to introduce, precisely in the Lie Algebra context, a remarkable element of the group algebra of S_n. We shall denote it by $\mathcal{K}_n$, in deference to Klyachko, it may be defined as follows:

$$\mathcal{K}_n = \frac{1}{n}\sum_{\sigma\in S_n}\omega^{\mathrm{maj}(\sigma)}\sigma$$

where $\omega = e^{2\pi i/n}$ is the primitive n^{th} root of unity, and *maj* denotes the major index statistics. That is for a permutation

$$\sigma = \sigma_1\sigma_2\cdots\sigma_n, \tag{2.1}$$

we set

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{n-1} i\ \chi(\sigma_i > \sigma_{i+1}).$$

It is convenient to introduce here the same statistic for sets. That is for $S \subseteq \{1,2,\dots\}$ we let

$$\mathrm{maj}(S) = \sum_{i \in S} i.$$

This given, we see that $\mathcal{K}_n$ can be rewritten in the form

$$\mathcal{K}_n = \frac{1}{n} \sum_{S \subseteq \{1,2,\dots,n-1\}} \omega^{\mathrm{maj}(S)} D_{=S}. \tag{2.2}$$

For an element ρ of the group algebra of S_n let

$$Q[A^n]\rho$$

denote the subspace of $Q[A^*]\rho$ spanned by the polynomials $w\rho$ as w varies among all A-words of length n.

Klyachko in [6] shows that

$$\mathrm{LIE}[A^n] = Q[A^n]\mathcal{K}_n. \tag{2.3}$$

Unfortunately, the proof of this result given in [6] is unnecessarily complicated. We shall see here that 2.3 may be easily obtained directly from the identities of section 1 combined with a simple q-enumeration result of [5].

Let us say that two elements ρ_1, ρ_2 of $A(S_n)$ are *right equivalent* if and only if

$$\begin{cases} a) & \rho_1\rho_2 = \rho_1 \\ b) & \rho_2\rho_1 = \rho_2 \end{cases} \tag{2.4}$$

Note that these conditions force ρ_1 and ρ_2 to be idempotents. Moreover, we see that a) and b) imply that the corresponding left ideals in $A(S_n)$ are identical. Likewise we also have

$$Q[A^n]\rho_1 = Q[A^n]\rho_2 \tag{2.5}$$

Thus to show the Klyachko result we need only verify that θ_n/n and $\mathcal{K}_n$ are right equivalent. Now, the identity

$$\theta_n\mathcal{K}_n = n\theta_n \tag{2.6}$$

is immediate. In fact, from 1.11 we get

$$\theta_n\mathcal{K}_n = \sum_{S \subseteq [1,n-1]} \omega^{\mathrm{maj}(S)}\theta_n D_{=S} = \sum_{S \subseteq [1,n-1]} \omega^{\mathrm{maj}(S)}(-1)^{|S|}\theta_n$$

which may be rewritten as

$$\theta_n\mathcal{K}_n = (1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})\theta_n = n\theta_n.$$

The latter equality being a consequence of the fact that ω is a primitive root of unity of order n.

At this point it may be good to observe that this argument actually establishes a stronger result. To state it we need to introduce some notation. Let $\phi_n(q)$ denote the n^{th} cyclotomic polynomial, as usual $\phi(n)$ will denote the Euler ϕ-function. Let also $r_m(q)$ denote the unique polynomial of degree strictly less than $\phi(n)$ which is a solution of

$$q^m \equiv r_m(q) \qquad (\mathrm{mod}\ \phi_n(q)). \tag{2.7}$$

Finally, let us set

$$\pi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n} r_{\mathrm{maj}(\sigma)}(q)\sigma$$

and

$$\mathcal{K}_n(q) = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)}\sigma.$$

Thus given, we have

THEOREM 2.1.

$$\theta_n \mathcal{K}_n(q) = (1-q)(1-q^2)\cdots(1-q^{n-1})\theta_n$$

and

$$\theta_n \pi_n(q) = \theta_n. \tag{2.8}$$

Proof. We need only observe that the polynomial

$$(1-q)(1-q^2)\cdots(1-q^{n-1})$$

is equal to n modulo $\phi_n(q)$.

To complete our proof of the Klyachko result we need to verify the identity

$$\mathcal{K}_n \theta_n = n\mathcal{K}_n. \tag{2.9}$$

In view of Theorem 1.1 this is equivalent to the statement that $\mathcal{K}_n \in \mathrm{LIE}[A^n]$. Again from Theorem 1.1 we see that this is equivalent to showing the identity

$$\mathcal{K}_n D_{=S} = (-1)^{|S|}\mathcal{K}_n, \tag{2.10}$$

better yet, using 1.3, we could just show that

$$\mathcal{K}_n D_{\subseteq S} = 0, \qquad (\text{ for all } S \neq \varnothing). \tag{2.11}$$

It should be noted however, that even the special case

$$\mathcal{K}_n D_{=[1,k]} = (-1)^k \mathcal{K}_n, \tag{2.12}$$

of 2.10 is sufficient to yield us 2.9. Indeed, 1.4 gives

$$\mathcal{K}_n\theta_n = \sum_{k=0}^{n-1}(-1)^k\mathcal{K}_n D_{[1,k]}$$

and 2.12 gives

$$\mathcal{K}_n\theta_n = \sum_{k=0}^{n-1}(-1)^k\mathcal{K}_n\,(-1)^k = n\mathcal{K}_n.$$

In summary we can obtain 2.9 either via 2.11 or via 2.12. We shall follow both these paths because each of them leads to interesting combinatorial questions. As a matter of fact, the underlying combinatorics is better understood if we work with $\pi_n(q)$ rather than $\mathcal{K}_n$. In this light we shall obtain 2.11 as a specialization of

$$\pi_n(q)D_{\subseteq S} = 0. \qquad (\text{for all } S \neq \varnothing) \tag{2.13}$$

Note first that the coefficient of a permutation α in the left-hand side of 2.13, modulo the cyclotomic polynomial $\phi_n(q)$ (except for a factor $1/n$), is given by

$$\mathcal{K}_n(q)D_{\subseteq S}|_\alpha = \sum_{\tau\in D_{\subseteq S}} q^{\mathrm{maj}(\alpha\tau^{-1})} = \sum_{\sigma\in\alpha\downarrow D_{\subseteq S}} q^{\mathrm{maj}(\sigma)}. \tag{2.14}$$

The meaning of the right-hand side of this identity becomes clearer through an example. Let $S = \{3,5\}$ with $n = 8$. Then

$$\downarrow D_{\{3,5\}} = 123 \sqcup 45 \sqcup 678,$$

and for

$$\alpha = b_1 b_2 \cdots b_8$$

we get

$$\alpha \downarrow D_{\{3,5\}} = b_1b_2b_3 \sqcup b_4b_5 \sqcup b_6b_7b_8.$$

Thus we see that in the general case the right-hand side of 2.14 is

$$\sum_{\sigma\in\alpha^{(1)}\sqcup\alpha^{(2)}\sqcup\cdots\sqcup\alpha^{(k+1)}} q^{\mathrm{maj}(\sigma)}, \tag{2.15}$$

where $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(k+1)}$ are the words obtained by segmenting α according to the elements of S. We have thus been naturally lead to a problem of q-enumeration of a family of perturbations by the major index. In fact, the following formula for 2.15 has already been given in [5]:

$$\sum_{\sigma\in\alpha^{(1)}\sqcup\cdots\sqcup\alpha^{(k+1)}} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(\alpha^{(1)})+\cdots+\mathrm{maj}(\alpha^{(k+1)})}\begin{bmatrix} n \\ p_1 \quad p_2 \quad \cdots \quad p_{k+1}\end{bmatrix}, \tag{2.16}$$

where p_i is the length of the word $\alpha^{(i)}$ and the expression in brackets is the usual q-multinomial coefficient.

But now we immediately see that the right-hand side of 2.16 is equal to zero because all these multinomial coefficients vanish modulo the cyclotomic polynomial $\phi_n(q)$. We have thus established the following fact

THEOREM 2.2.

$$\pi_n(q)D_{=S} = (-1)^{|S|}\pi_n(q), \tag{2.17}$$

and

$$\pi_n(q)\theta_n = n\pi_n(q). \tag{2.18}$$

Proof. The argument given above establishes the equality in 2.17, modulo $\phi_n(q)$. But in fact, exact equality must hold as well, since the coefficients of permutations in both sides of 2.17 are polynomials of degree strictly less than $\phi(n)$. Formula 2.18 follows from 2.17 by Theorem 1.2. □

Remark 2.1. Computer exploration with MAPLE, carried out in connection with this proof, suggested the following beautiful evaluation of 2.16:

$$\sum_{\sigma\in\alpha^{(1)}\ \sqcup\ \cdots\ \sqcup\ \alpha^{(k+1)}} q^{\mathrm{maj}(\sigma)} \equiv_{\bmod q^n-1} q^{\mathrm{maj}(\sigma)} \begin{bmatrix} & & & n & \\ p_1 & p_2 & \cdots & p_{k+1} \end{bmatrix}.$$

this identity was later shown to be true by M. Wachs (personal communication).

Theorems 2.1 and 2.2 have surprising consequences. For convenience let us set

$$\pi_n(q) = \delta_0 + \delta_1 q + \cdots + \delta_f q^f,$$

where $f = \phi(n) - 1$.

THEOREM 2.3. *Each of the δ_i is a Lie element. Moreover, we have the identities*

$$\delta_i\delta_j = \begin{cases} \delta_i, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.19}$$

In particular, $\pi_n(q)$ is a Lie idempotent for any value of q. More generally, the expression

$$\delta_0 + c_1\delta_1 + \cdots + c_f\delta_f, \tag{2.20}$$

always gives a Lie idempotent.

Proof. By equating coefficients of the powers of q in 2.8 we derive that

$$\theta_n\delta_0 = \theta_n \quad \text{and} \quad \theta_n\delta_i = 0 \quad (\text{for } i = 1, 2, \ldots, f). \tag{2.21}$$

Doing the same with 2.18 gives

$$\delta_i\theta_n = n\delta_i \qquad (\text{for } i = 0, 1, \ldots, f). \tag{2.22}$$

Combining these identities we get

$$\delta_i\delta_0 = \left(\frac{\delta_i\theta_n}{n}\right)\delta_0 = \frac{\delta_i}{n}\theta_n\delta_0 = \frac{\delta_i}{n}\theta_n = \delta_i.$$

This gives the first part of 2.19. A similar calculation using part two of 2.21 yields the second part of 2.19. The idempotency of $\pi_n(q)$ and that of expression 2.20 is an immediate consequence of 2.19. □

For any permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ let us set as in [6]

$$\mathrm{ind}(\sigma) = \sum_{i=1}^{n} \chi(\sigma_i > \sigma_{i+1}),$$

with the convention that $\sigma_{n+1} = \sigma_1$.

Clearly, this statistic only depends on the circular rearrangement class of σ. We shall refer to it as the *circular descent* index. Let us also denote by γ the n-cycle

$$\gamma = (1, 2, 3, \ldots, n)$$

Now it develops that *maj* and *ind* are related in a surprising way, namely

LEMMA 2.1. *(Klyachko)*

For any $\sigma \in S_n$:

$$(2.23) \qquad \begin{array}{ll} (a) & \mathrm{maj}(\sigma\gamma) = \mathrm{maj}(\sigma) - \mathrm{ind}(\sigma) \pmod{n} \\ (b) & \mathrm{maj}(\gamma\sigma) = \mathrm{maj}(\sigma) - 1 \pmod{n} \end{array}$$

Proof. The proof of these identities can be found in [6] and is quite straightforward. □

For a further study of the elements δ_i we need some properties of the polynomials $r_m(q)$ defined in 2.7.

LEMMA 2.2. *The sequence of polynomials* $r_m(q)$ *is periodic of period* n *and their generating function is given by the identity*

$$(2.24) \qquad \sum_{m\geq 0} r_m(q)x^m = \frac{1}{1-xq} - \frac{\phi_n(q)}{1-xq}\,\frac{x^{1+f}}{\phi_n(x)}.$$

Proof. The periodicity is immediate since the trivial congruence

$$q^{m+n} \equiv q^m \pmod{\phi_n(q)}$$

implies the congruence

$$r_{m+n} \equiv r_m \pmod{\phi_n(q)}$$

but the latter must reduce to an equality since both polynomials have degree less that the degree of $\phi_n(q)$.

It follows immediately from the definition of the polynomials $r_m(q)$ that there is a constant c such that

$$qr_m(q) = c\phi_n(q) + r_{m+1}(q).$$

Setting $q = 0$ and recalling that the constant term in $\phi_n(q)$ is equal to 1, we get

$$c = -r_{m+1}(0).$$

For convenience let us set

$$c_m = r_m(0).$$

We may thus write

$$r_{m+1}(q) = qr_m(q) + c_{m+1}\phi_n(q). \tag{2.25}$$

Multiplying this identity by x^{m+1} and summing gives

$$\sum_{m\geq 0} r_m(q)x^m = \frac{1}{1-xq} + \frac{\phi_n(q)}{1-xq}\sum_{m\geq 1} c_m x^m.$$

Setting $xq = t$, multiplying by $1-t$ and letting $t \to 1$ yields

$$0 = 1 + \phi_n(q)\sum_{m\geq 1} c_m q^{-m}, \tag{2.26}$$

The passage to the limit is justified for $q > 1$, and the fact that left hand side reduces to zero is an easy consequence of the periodicity of the sequence $\{r_m\}$.

Since the cyclotomic polynomial satisfies the identity.

$$x^{1+f}\phi_n(\frac{1}{x}) = \phi_n(x). \tag{2.27}$$

Formula 2.24 must hold true as asserted. □

THEOREM 2.3.

$$\gamma\pi_n(q) \equiv q\pi_n(q) \qquad (\mathrm{mod}\ \phi_n(q)) \tag{2.28}$$

Proof. Note that we can write

$$\gamma\pi_n(q) = \frac{1}{n}\sum_{\alpha\in S_n} r_{\mathrm{maj}(\gamma^{-1}\alpha)}(q)\,\alpha.$$

Using the Klyachko identity 2.23 b) and the periodicity of the r_m's we obtain

$$\gamma\pi_n(q) = \frac{1}{n}\sum_{\alpha\in S_n} r_{1+\mathrm{maj}(\alpha)}(q)\,\alpha.$$

Substituting 2.25 we finally get

$$\gamma\pi_n(q) = q\pi_n(q) + \frac{1}{n}\phi_n(q)\sum_{\alpha\in S_n} c_{1+\mathrm{maj}(\alpha)}\,\alpha \tag{2.29}$$

which yields 2.28. □

Remark 2.2. Setting $q = 0$ in 2.24 we get

$$\sum_{m\geq 0} r_m(0)x^m = \sum_{m\geq 0} c_m x^m = 1 - \frac{x^{1+f}}{\phi_n(x)}. \tag{2.30}$$

On the other hand the definition of $\pi_n(q)$ gives

$$\delta_0 = \frac{1}{n}\sum_{\sigma\in S_n} c_{\mathrm{maj}(\sigma)}\sigma. \tag{2.31}$$

Thus we see that this remarkable idempotent is intimately related to the *inverse* of the cyclotomic polynomial.

Theorem 2.3 has an interesting corollary. For convenience let us write the n^{th} cyclotomic polynomial in the form

$$\phi_n(q) = q^{f+1} + 1 - a_1 q - a_2 q^2 - \cdots - a_f q^f. \tag{2.32}$$

This given we have

THEOREM 2.4. *For* $i = 1, \ldots, f$

$$\delta_i = \gamma^{-i}(1 - a_1\gamma - a_2\gamma^2 - \cdots - a_i\gamma^i)\delta_0. \tag{2.33}$$

Proof.

Replacing q by $1/q$ in 2.29, multiplying by q^{1+f} and setting $q = 0$ (using 2.27) we get

$$0 = \delta_f + \frac{1}{n}\sum_{\alpha\in S_n} c_{1+\mathrm{maj}(\alpha)}\alpha.$$

So we can rewrite 2.29 as

$$\begin{aligned}\gamma\pi_n(q) &= q\pi_n(q) - \phi_n(q)\delta_f \\ &= q\pi_n(q) - (q^{1+f} + 1 - a_1 q - \cdots - a_f q^f)\delta_f.\end{aligned} \tag{2.34}$$

Equating coefficients of the successive powers of q we easily get the recurrences

$$\begin{aligned}\gamma\delta_0 &= -\delta_f, \\ \gamma\delta_1 &= \delta_0 + a_1\delta_f, \\ \gamma\delta_2 &= \delta_1 + a_2\delta_f, \\ \cdots &= \cdots \\ \gamma\delta_f &= \delta_{f-1} + a_f\delta_f,\end{aligned} \tag{2.35}$$

The i^{th} equation may be multiplied by γ^{i-1} to give

$$\gamma^i\delta_i = \gamma^{i-1}\delta_{i-1} - a_i\gamma^i\delta_0$$

from which our desired expression can be easily derived. □

Another corollary of Theorem 2.3 is the following remarkable fact

THEOREM 2.5. *For each* $i = 0, 1, \ldots, f$

$$\phi(\gamma)\delta_i = 0. \tag{2.36}$$

and moreover

$$\mathrm{LIE}[A^n] = Q[A^n]\delta_i \tag{2.37}$$

That is, not only the idempotent δ_0, *but the nilpotent elements* δ_i *also give* $\mathrm{LIE}[A^n]$.

Proof. Equation 2.28 gives that

$$\phi(\gamma)\pi_n(q) \equiv 0 \qquad (\mathrm{mod}\ \phi_n(q))$$

However, this congruence must in fact be an equality, since the left hand side is of degree at most f in q. This gives 2.36. Now the relation 2.33 may be inverted. Indeed, the polynomial

$$1 - a_1 q - a_2 q^2 - \cdots - a_i q^i$$

is invertible modulo $\phi_n(q)$. Let $P_i(q)$ be its inverse. Multiplying 2.33 by $P_i(\gamma)$, and using 2.36 we get

$$\delta_0 = \gamma^i P_i(\gamma)\delta_i.$$

This gives the inclusion

$$\mathrm{LIE}[A^n] \subseteq Q[A^n]\delta_i.$$

The other inclusion follows in the same manner from 2.33. □

3. q-Enumeration and the dimension of $\mathrm{LIE}[A]$. As we shall see, the crucial step in the calculation of the dimension of the free Lie Algebra, is the proof that the Klyachko idempotent is a Lie element. As we pointed out in the last section, to prove the latter, we do not have to establish 2.10 in full, but rather only

$$\mathcal{K}_n D_{=[1,k]} = (-1)^k \mathcal{K}_n \tag{3.1}$$

A direct verification of this identity leads to some interesting q-enumeration problems. Let us work again with $\mathcal{K}_n(q)$ and write the analogue of 2.14 for $D_{=[1,k]}$. Namely

$$\mathcal{K}_n(q) D_{=[1,k]} \mid_\alpha = \sum_{\sigma \in \alpha \downarrow D_{=[1,k]}} q^{\mathrm{maj}(\sigma)}. \tag{3.2}$$

Note that the generic element of $D_{=[1,k]}$ is of the form

$$\tau = i_1 > i_2 > \cdots > i_k > 1 < j_1 < j_2 < \cdots < j_k.$$

thus its inverse consists of $k+1$ followed by a shuffle of the words $k\,k-1\cdots 2\,1$ and $k+2\,k+3\cdots n$. This gives

$$\downarrow D_{=[1,k]} = k+1 \bullet (k\,k-1\cdots 1) \sqcup\!\sqcup (k+2\,k+3\cdots n) \tag{3.3}$$

where here the symbol $\bullet$ denotes concatenation product. Consequently

$$\alpha \downarrow D_{=[1,k]} = \alpha_{k+1} \bullet (\alpha_k \alpha_{k-1} \cdots \alpha_1) \sqcup\!\sqcup (\alpha_{k+2} \alpha_{k+3} \cdots \alpha_n)$$

Our identity 3.2 reduces then to

$$\mathcal{K}_n(q) D_{=[1,k]} \mid_\alpha = \sum_{\sigma \in \alpha_{k+1} \bullet (\alpha_k \alpha_{k-1} \cdots \alpha_1) \sqcup\!\sqcup (\alpha_{k+2} \alpha_{k+3} \cdots \alpha_n)} q^{\mathrm{maj}(\sigma)} \tag{3.4}$$

We are thus led to a special case of the general problem of q-enumeration by the major index of a set of permutations obtained by reading the labels of a tree. To be precise let T be a tree and α be a labeling of its nodes by the integers $1, 2, \ldots, n$. Let $L(T)$ denote the set of all linear extensions of the natural order of the elements T. Then the general problem is that of calculating the sum

$$\sum_{\sigma \in \alpha L(T)} q^{\mathrm{maj}(\sigma)} \tag{3.5}$$

where $\alpha L(T)$ denotes the collection of permutations described by

$$\alpha(\sigma) = \alpha(\sigma_1)\alpha(\sigma_2) \cdots \alpha(\sigma_n)$$

as σ varies in $L(T)$.

The summation in 3.4 is the special case when T consists of two total orders, of respective lengths $k+1$ and $n-k-1$, joined at their smallest elements. Unfortunately, we do not know of any simple expression, for the sum 3.5 when the tree order is *away from the root* (that is when *up* is as in the natural trees). However, if the order is *towards the root*, (that is if *up* is as in the Computer Science trees) then a *hook formula* is available. Namely, we have

THEOREM 3.1. *(Bjorner-Wachs [1])*

$$\sum_{\sigma \in \alpha L(T)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}\,\alpha(T)} \frac{[n]!}{\prod_{x \in T} [h_x]}. \tag{3.6}$$

Where, $[n]!$ is the q-analogue of $n!$, h_x denotes the number of nodes in the subtree below x ($[h_x]$ is its q-analogue) and finally,

$$\mathrm{maj}\,\alpha(T) = \sum_{x \in T} h_x \chi(\alpha(x) > \alpha(p(x))) \tag{3.7}$$

here $p(x)$ denotes the parent of x.

Upturning our trees we get two total orders joined by their top elements. We are thus replacing the labelled tree on the left in the picture by the one on the right.

The left tree corresponds to the sum in 3.4 while the right tree corresponds to the sum

$$(3.8) \qquad \sum_{\sigma \in (\alpha_1 \alpha_2 \cdots \alpha_k) \sqcup\!\sqcup (\alpha_n \alpha_{n-1} \cdots \alpha_{k+2}) \bullet \alpha_{k+1}} q^{\mathrm{maj}(\sigma)}.$$

The relationship between the permutation in 3.4 and those in 3.8 is very simple, the former are the reverse of the latter. More precisely, denoting by ω_0 the reversing permutation, that is

$$\omega_0 = \begin{bmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{bmatrix},$$

we can rewrite 3.2 in the form

$$(3.9) \qquad \mathcal{K}_n(q) D_{=[1,k]} \mid_\alpha = \sum_{\sigma \in (\alpha_1 \alpha_2 \cdots \alpha_k) \sqcup\!\sqcup (\alpha_n \alpha_{n-1} \cdots \alpha_{k+2}) \cdot \alpha_{k+1}} q^{\mathrm{maj}(\sigma \omega_0)}.$$

Now, simple manipulations yield the identity

$$(3.10) \qquad \mathrm{maj}(\sigma \omega_0) = \binom{n}{2} - n d(\sigma) + \mathrm{maj}(\sigma),$$

where $d(\sigma)$ denotes the number of descents of σ.

On the other hand, the expression in 3.8, by means of the hook formula 3.6, evaluates to

(3.11)

$$\sum_{\sigma \in (\alpha_1 \alpha_2 \cdots \alpha_k) \sqcup\!\sqcup (\alpha_n \alpha_{n-1} \cdots \alpha_{k+2}) \bullet \alpha_{k+1}} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(\alpha_1 \alpha_2 \cdots \alpha_k) + \mathrm{maj}(\alpha_n \alpha_{n-1} \cdots \alpha_{k+1})} \frac{[n]!}{[n][k]![n-k-1]!}$$

Substituting 3.10 and 3.11 in 3.9 we obtain

$$(3.12) \qquad \mathcal{K}_n(q) D_{=[1,k]} \mid_\alpha \equiv_{\bmod q^n - 1} q^{\binom{n}{2} + \mathrm{maj}(\alpha_1 \alpha_2 \cdots \alpha_k) + \mathrm{maj}(\alpha_n \alpha_{n-1} \cdots \alpha_{k+1})} \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Since

$$\mathrm{maj}(\alpha_1 \cdots \alpha_n) = \mathrm{maj}(\alpha_1 \cdots \alpha_k) + \mathrm{maj}(\alpha_{k+1} \cdots \alpha_n) + k d(\alpha_{k+1} \cdots \alpha_n)$$

and

$$\mathrm{maj}(\alpha_n \alpha_{n-1} \cdots \alpha_{k+1}) = \mathrm{maj}(\alpha_{k+1} \alpha_{k+2} \dots \alpha_n) + \binom{n-k}{2} - (n-k) d(\alpha_{k+1} \alpha_{k+2} \cdots \alpha_n),$$

we deduce that

$$\binom{n}{2} + \mathrm{maj}(\alpha_1 \alpha_2 \cdots \alpha_k) + \mathrm{maj}(\alpha_n \alpha_{n-1} \cdots \alpha_{k+1}) \equiv_{\bmod n} \binom{k+1}{2} + \mathrm{maj}(\alpha_1 \alpha_2 \cdots \alpha_n).$$

Which reduces 3.12 to our final identity

$$\mathcal{K}_n(q)D_{=[1,k]}\,|_\alpha \equiv_{\bmod q^n-1} q^{\binom{k+1}{2}+\mathrm{maj}(\alpha_1\alpha_2\cdots\alpha_n)}\begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{3.13}$$

Multiplying both sides by α and summing over S_n yields

$$\mathcal{K}_n(q)D_{=[1,k]}\,|_\alpha \equiv_{\bmod q^n-1} q^{\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}\mathcal{K}_n(q). \tag{3.14}$$

To get 3.1 we are led to evaluate the factor of $\mathcal{K}_n(q)$, on the right side, modulo the cyclotomic polynomial. However, this is easily done since the generating function of these factors (by the q-binomial theorem) may be written as

$$\sum_{k=0}^{n-1}(-t)^k q^{\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix} = (1-tq)(1-tq^2)\cdots(1-tq^{n-1}).$$

However, modulo the cyclotomic polynomial the right hand side of this identity reduces to

$$\frac{1-t^n}{1-t} = 1+t+t^2+\cdots+t^{n-1},$$

thus

$$q^{\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix} \equiv_{\bmod \phi_n(q)} (-1)^k.$$

this gives not only 3.1 but also

$$\pi_n(q)D_{=[1,k]} = (-1)^k\pi_n(q)$$

Remark 3.1. The identity in 3.14 gives a little bit more. Indeed, multiplying both sides by $(-1)^k$, summing and using the q-binomial theorem we get

$$\mathcal{K}_n(q)\theta_n \equiv_{\bmod q^n-1} (1-q)(1-q^2)\cdots(1-q^{n-1})\mathcal{K}_n(q).$$

which is a stronger identity than 2.18.

Remark 3.2. Curiously 3.13 is actually an equality when α is the identity permutation! For in this case we can write

$$\mathcal{K}_n(q)D_{=[1,k]}\,|_{id} = \sum_{2\le i_1<i_2<\cdots<i_k\le n} q^{i_1-1+i_2-1+\cdots+i_k-1}.$$

Now the generating function of these polynomials is

$$\sum_{k=0}^{n-1} t^k\,\mathcal{K}_n(q)D_{=[1,k]}|_{id} = (1+tq)(1+tq^2)\cdots(1+tq^{n-1})$$

and the q-binomial theorem gives

$$\mathcal{K}_n(q)D_{=[1,k]}\,|_{id} = q^{\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

This circumstance makes one wonder if for general α there is a similar exact evaluation of the left hand side of 3.13.

Remark 3.3. We should point out that if we are willing to use Stanley's theory of P-partitions, the hook formula 3.6 has a simpler derivation than that given in [1]. For sake of completeness we give here a sketch of this alternate argument.

The point of departure is Stanley's ω, P-partition formula. We are given a partially ordered set P and a labelling ω of its elements by the integers $1, 2, \ldots, n$. We let then $F_\omega(P)$ be the set of all integer valued functions f which are weakly decreasing with respect to the partial order of P and which decrease strictly when ω decreases in P. Stanley's [10] identity (see [5] for a proof) states that

$$\sum_{f \in F_\omega(P)} q^{|f|} = \frac{\sum_{\sigma \in \omega L(P)} q^{\mathrm{maj}(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{3.15}$$

again here $L(P)$ denotes the set of linear extensions of P and $\omega L(P)$ denotes the set of permutations obtained by reading the labelling ω according to these linear extensions. Of course n gives the cardinality of P and

$$|f| = \sum_{x \in P} f(x)$$

Now for the case of a Computer Science tree the summation on the left of 3.15 can be very easily evaluated. The basic fact that allows this evaluation is that, for CS-trees, each $f \in F_\omega(T)$ can be uniquely decomposed as a linear combination of indicator functions of *principal* lower order ideals. Since a lower order ideal below an element x of a CS-tree T is simply the subtree x and its cardinality is h_x, we can easily derive the formula

$$\sum_{f \in F_\omega(T)} q^{|f|} = q^{\mathrm{maj}\,\omega(T)} \prod_{x \in T} \frac{1}{1-q^{h_x}}. \tag{3.16}$$

The factor

$$q^{\mathrm{maj}\,\omega(T)}$$

represents the contribution of the smallest f in $F_\omega(T)$. When this function is subtracted from the general element of $F_\omega(T)$ we are left with a remainder which is just weakly decreasing in T. The second factor in 3.16 accounts for the contributions of all these remainders. Combining 3.15 and 3.16 we get the hook formula 3.6.

Before proceeding we need to recall the notions of *primitive words*, *primitive classes* and *Lyndon words* [2]. Briefly, a word is said to be primitive if all its circular rearrangements are distinct. A circular rearrangement class is called primitive if it is generated by a primitive word. Finally, a word is said to be *Lyndon* if it is lexicographically strictly smaller than all its circular rearrangements. It is not difficult to show that Lyndon words give a set of distinct representatives of the primitive circular rearrangements classes. It will be convenient to let $L(A^n)$ denote the collection of all Lyndon words of length n in the alphabet A.

The introduction of the Klyachko idempotent gives what is perhaps the simplest justification of the dimension formula for the Free Lie Algebra. This classical theorem may be stated as follows.

THEOREM 3.2. *The dimension of* LIE$[A^n]$ *is equal to the number of primitive circular rearrangement classes of words of length* n *in the alphabet* A. *Moreover, a basis for this space is given by the set of Lie polynomials*

$$\{u\mathcal{K}_n\}_{u\in L(A^n)} \tag{3.17}$$

Proof. Since we have all the ingredients to prove the result we shall include a sketch of the argument. Clearly, it is sufficient to show the second part of statement. We shall start by showing that the polynomials in 3.17 span. To this end note that, since $\mathcal{K}_n$ is a Lie idempotent, a trivial spanning set for LIE$[A^n]$ can be obtained by taking all the polynomials

$$u\mathcal{K}_n \qquad (u \in A^n). \tag{3.18}$$

On the other hand, 2.28 evaluated at $q = \omega$ gives that for any word $u \in A^n$

$$u\gamma^s\mathcal{K}_n = \omega^s u\mathcal{K}_n$$

This implies that $\mathcal{K}_n$ kills all non-primitive words and that words in the same circular rearrangement class yield essentially the same polynomial when acted upon on the right by $\mathcal{K}_n$. Thus a spanning set if obtained by letting u in 3.18 vary in any set of distinct representatives of primitive classes.

The shortest path to independence is given by Klyachko's original line of reasoning. Simple manipulations which use 2.23 (a) yield the remarkable identity

$$\mathcal{K}_n\tau = \tau \tag{3.19}$$

where

$$\tau = \frac{1}{n}\sum_{s=0}^{n-1}\omega^{-s}\gamma^s.$$

Now, it is very easy to see that the set of polynomials

$$\{u\tau\}_{u\in L(A^n)} \tag{3.20}$$

is independent. This is because if u_1 and u_2 are in different primitive classes the two polynomials $u_1\tau$ and $u_2\tau$ do not vanish and have disjoint support. However, 3.19 shows that right multiplication by τ sends the set in 3.17 onto the set in 3.20, consequently the set in 3.17 must be independent as well. This completes the proof of the theorem. □

4. The Reutenauer representations. The group algebra element

$$\rho_n = \frac{1}{n}\sum_{d=0}^{n-1}\frac{(-1)^d}{\binom{n-1}{d}}D_{=d}$$

(here $D_{=d}$ is the sum of the permutations with d descents) was shown (indirectly) by Solomon [9] and directly by Reutenauer [8] to be a Lie idempotent. In Reutenauer's

proof ρ_n is the first term of a sequence $\rho_n^{(k)}$ of idempotents that are closely connected to the Free Lie Algebra. A remarkable result proved in [8] is that the dimension of the representation induced by the action of S_n on the left ideal $A(S_n)\rho_n^{(k)}$ is given by the Stirling number of the first kind $s(n,k)$. In [8] Reutenauer asks for a combinatorial explanation of this fact. R. Stanley constructed representations with the same dimensions and conjectured (personal communication) that they might have the same character as those introduced by Reutenauer. In this section we shall give a proof of Stanley's conjecture.

The simplest way to introduce the $\rho_n^{(k)}$'s is through their generating function. More precisely, we let $\rho_n^{(k)}$ for $k = 1, 2, \ldots, n$ be the elements of the group algebra $A(S_n)$ satisfying the relation

$$\sum_{k=1}^{n} \rho_n^{(k)} x^k = \frac{1}{n!} \sum_{\sigma \in S_n} (x - d(\sigma)) \uparrow^n \sigma, \tag{4.1}$$

where $(x) \uparrow^n$ denotes the upper factorial polynomial. The proof that this sequence is the same as that studied by Reutenauer is given in [4]. We shall also need here some auxiliary facts concerning the $\rho_n^{(k)}$'s, this material including all of the necessary background may be found in [4].

Let us recall that the Frobenius image of a character χ of S_n is the symmetric polynomial

$$F\chi = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) \psi_1^{p_1(\sigma)} \psi_2^{p_2(\sigma)} \cdots \psi_n^{p_n(\sigma)}$$

where ψ_k is the k^{th} power symmetric function and $p_k(\sigma)$ denotes the number of cycles of length k in σ. It may also be shown that the Frobenius images of the character of a module $A(S_n)\rho$ (when ρ is an idempotent) is given by the polynomial

$$P_\rho = \sum_{\sigma \in S_n} \rho(\sigma)\, \psi_1^{p_1(\sigma)}\, \psi_2^{p_2(\sigma)} \cdots \psi_n^{p_n(\sigma)}$$

We shall use these symmetric polynomials in our proof of Stanley's conjecture.

We begin with Stanley's construction. For a given set of non-negative integers $p_1, p_2, \cdots, p_n$ satisfying

$$\begin{aligned} p_1 + p_2 + \cdots + p_n &= k \\ 1p_1 + 2p_2 + \cdots + np_n &= n \end{aligned} \tag{4.2}$$

we choose and fix an arbitrary permutation σ_p with p_m cycles of length m. Let G_p denote the stabilizer of σ_p, that is

$$G_p = \{\alpha : \alpha \sigma_p \alpha^{-1} = \sigma_p\}$$

Next, we pick a one dimensional character $\chi^{(p)}$ of G_p and induce from G_p to S_n. It is seen that the dimension of the resulting representation is $n!/|G_p|$, and this is precisely the number of permutations of cycle type $1^{p_1} 2^{p_2} \cdots n^{p_n}$. Thus the direct

sum of these representations over all p satisfying 4.2 (k and n remaining fixed) has dimension $s(n,k)$. Clearly the resulting character is

$$\chi^S = \sum_{\substack{p_1+p_2+\cdots+p_n=k \\ 1p_1+2p_2+\cdots+np_n=n}} \chi^{(p)} \uparrow_{G_p}^{S_n}.$$

Stanley conjectures that there is a choice of the $\chi^{(p)}$'s which makes χ^S equal to the Reutenauer character.

It appears that there is such a choice and it may be constructed as follows. We first observe that G_p is essentially the cartesian product of the wreath products $S_{p_m}[C_m]$. We shall decompose the set of integers $1,2,\cdots,n$ into successive blocks of sizes $p_m \times m$ and shall let $S_{p_m}[C_m]$ act only on the integers in its corresponding block.

The standard realization of the wreath product $S_p[C_m]$ acting on the integers $1,2,\ldots,p\times m$ has elements of the form

$$\xi = (\sigma;\gamma^{s_1},\gamma^{s_2},\cdots,\gamma^{s_p}) \tag{4.3}$$

where σ is an arbitrary element of S_p, γ is the cycle $(1,2,\ldots,m)$ and the exponents s_i vary between 0 and $m-1$. If we arrange the integers $1,2,\ldots,p\times m$ in rows of length m with $(i-1)m+1,(i-1)m+2,\cdots,im$ occupying the i^{th} row. Then the element ξ in 4.3 acts on these integers by first circularly shifting each row (the i^{th} row by s_i) and then bodily permuting the resulting rows according to the permutation σ. It is not difficult to show that the map defined by setting

$$\chi^{p,m}(\xi) = \omega^{s_1}\omega^{s_2}\cdots\omega^{s_p},$$

(where $\omega = e^{2\pi i/m}$) gives a one dimensional representation (or character) of $S_p[C_m]$. This given, our choice of $\chi^{(p)}$ will simply be the "product" of the $\chi^{p_m,m}$'s.

We shall next study the character of the Reutenauer representation. To this end we need to give a closer look at the idempotents $\rho_n^{(k)}$. It was shown in [8] that each $w\in A^n$ has the decomposition

$$w = w\rho_n^{(1)} + w\rho_n^{(2)} + \cdots w\rho_n^{(n)}. \tag{4.4}$$

It develops that the polynomial $w\rho_n^{(k)}$ gives the projection of w into a certain subspace $HS_{=k}$ of $Q[A^*]$. The nature of the components $w\rho_n^{(k)}$ was derived in [4] by combining the contents of [6] and [9]. Since this is crucial in our calculation of the Frobenius image of the Reutenauer character, we shall give a brief review of the contents of [4]. The reader is referred to [4] for proofs and further details.

Starting from any basis $\{b[u]\}$ indexed by Lyndon words (in particular the one in 3.17) we let $HS_{=k}$ be the subspace of $Q[A^*]$ spanned by the polynomials

$$s[u_1,u_2,\ldots,u_k] = \frac{1}{k!}\sum_{\sigma\in S_k} b[u_{\sigma_1}]b[u_{\sigma_2}]\cdots b[u_{\sigma_k}] \tag{4.5}$$

(It may be shown that $HS_{=k}$ is independent of the choice of the basis $\{b[u]\}$). The polynomials in 4.5 will actually give a basis for $HS_{=k}$ if we restrict the k-tuples $u_1, u_2, \cdots, u_k$ to be in some specific order, say lexicographically weakly decreasing.

We can define an action of the general linear group $GL(N)$ on $Q[A^*]$ as follows. For a matrix $T = \|t_{ij}\|$ and a letter $a_j \in A$ we set

$$Ta_j = \sum_{i=1}^{N} a_i t_{ij}.$$

This action is extended to words multiplicatively, that is for $w = b_1 b_2 \cdots b_n$ we let

$$Tw = (Tb_1)(Tb_2)\cdots(Tb_n)$$

and finally we can extend it to the whole of $Q[A^*]$ by linearity.

Clearly, this action restricted to $Q[A^n]$ is the n^{th} tensor power of the defining representation of $GL(N)$. It is not difficult to show that each of the subspaces $HS_{=k}$ is invariant under this action. In view of the definition 4.5 we see that the action on $HS_{=k}$ is the k^{th} symmetric power of the action on $HS_{=1}$ which is LIE$[A^*]$ itself. This implies (see for instance [4]) that the character of the action on $HS_{=k}$ is given by the plethysm of the homogeneous symmetric function h_k with the character of LIE$[A^*]$, that is the formal series

$$L(X) = \sum_{n \geq 1} \frac{1}{n} \sum_{p|n} \mu(p) \psi_p^{\frac{n}{p}}. \tag{4.6}$$

The n^{th} homogeneous component of $HS_{=k}$ (that is the linear span of the polynomials $s[u_1, u_2, \ldots, u_k]$ in 4.5 with $u_1, u_2, \ldots, u_k$ of combined length n) is clearly also invariant under this action. Let us denote it by $HS_{=k} \mid_n$. Its character is thus given by the n^{th} homogeneous component of the plethysm

$$h_k[L(X)]. \tag{4.7}$$

Let us denote this polynomial by

$$h_k[L(X)] \mid_n. \tag{4.8}$$

It may be shown (see for instance [4]) that if ρ is an idempotent of the group algebra $A(S_n)$ then the Frobenius image of the character of the action of S_n on the left ideal $A(S_n)\rho$ is the same symmetric polynomial as that giving the character of the action of $GL(N)$ on the subspace $Q[A^*]\rho$ of $Q[A^*]$ spanned by the polynomials $w\rho$ for $w \in A^n$. Since the Reutenauer idempotent $\rho_n^{(k)}$ projects a word $w \in A^n$ into $HS_{=k}|_n$, it follows that

$$HS_{=k} \mid_n = Q[A^n]\rho_n^{(k)}.$$

We thus obtain that the Frobenius image of the character of the Reutenauer representation must be given by the polynomial in 4.8. That is

$$P_{\rho_n^{(k)}} = h_k[L(X)] \mid_n. \tag{4.9}$$

We are left with the calculation of the Frobenius image of the Stanley character. To do this it is best to proceed combinatorially. Clearly, the product

$$\chi^{(p)} = \prod_{m=1}^{n} \chi^{p_m, m}$$

is an idempotent of S_n. Now it develops that the polynomial $P_{\chi^{(p)}}$ has an interesting combinatorial interpretation. To see this, we shall visualize words of length n with their letters written out into the cells of the sequence of $p_m \times m$ Ferrers' diagrams *(blocks)*. The words obtained by reading one of the rows of one of the blocks will be referred to as the *row words* of w. This given, it can be shown [4] that the polynomial $w\chi^{(p)}$ is equal to zero if any of the row words of w is not primitive. Let us now call a word $w \in A^n$ *p-Lyndon* if each of the row words is Lyndon, and within a block the row words are in lexicographically decreasing order. It is straightforward to show that the polynomials $w\chi^{(p)}$ as w describes all p-Lyndon words give a basis for the space $Q[A^*]\chi^{(p)}$. This implies [4] that the polynomial $P_{\chi^{(p)}}$ (which by the way is also the Frobenius image of $\chi^{(p)} \uparrow_{G_p}^{S_n}$) is the Polya enumerator of the set of p-Lyndon words. Consequently, the Frobenius image of Stanley's character (with this choice of $\chi^{(p)}$) is the Polya enumerator of all words which may be factored into a product of k Lyndon words of weakly increasing lengths with factors of equal length in weakly decreasing lexicographic order.

To complete our proof of Stanley's conjecture we need only give a combinatorial interpretation for the polynomial in 4.9. However, this is almost immediate since it is well known (see for instance [3]) that the plethysm $P_1[P_2]$ of two Polya enumerators P_1 and P_2 is the Polya enumerator of the patterns enumerated by P_1 *colored* by the patterns enumerated by P_2. This implies that

$$h_k[L(X)]$$

is the enumerator of lexicographically weakly decreasing k-tuples of Lyndon words. In particular,

$$h_k[L(X)] \mid_n$$

enumerates those k-tuples that are of combined length n.

Clearly then the patterns enumerated by $F\chi^S$ are simply rearrangements of the patterns enumerated by $P_{\rho_n^{(k)}}$. Thus these two polynomials are identical and our proof is complete.

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