Hall-Littlewood Operators
in
the Theory of Parking Functions and Diagonal Harmonics

by

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ABSTRACT. In a recent work Jim Haglund, Jennifer Morse and Mike Zabrocki proved a variety of identities involving Hall-Littlewood symmetric functions indexed by compositions. When they applied ∇ to these symmetric functions the resulting identities and computer data led them to some truly remarkable refinements of the shuffle conjecture. We prove here the symmetric function side of a recursion which combined with a recent parking function recursion of Angela Hicks [18] settles some some special cases of the Haglund-Morse-Zabrocki conjectures. Our main result of a compositional q, t-Catalan and Schröder theorem yields as a consequence surprisingly simple new proofs of the original q, t-Catalan and Schröder results.

Introduction

In this writing it is convenient to represent parking functions in the $n \times n$ lattice square as two line arrays

$$PF \;=\; \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \tag{I.1}$$

with $u_1, u_2, \ldots, u_n$ integers satisfying

$$u_1 = 0 \qquad \text{and} \qquad 0 \le u_i \le u_{i-1} + 1 \tag{I.2}$$

and $V = (v_1, v_2, \ldots, v_n)$ a permutation in S_n satisfying

$$u_i = u_{i-1} + 1 \implies v_i > v_{i-1} \ . \tag{I.3}$$

Here and after, we will denote by $\sigma(PF)$ the permutation obtained by successive right to left readings of the components of the vector $V = (v_1, v_2, \ldots, v_n)$ according to decreasing values of $u_1, u_2, \ldots, u_n$. More precisely if $k = max\{u_i : 1 \le i \le n\}$ we read first the v_i that are above k in I.1, then those that above $k-1$. etc ... ending with the v_i that are above 0. We will here and after refer to $\sigma(PF)$ as the *diagonal permutation* of PF. It will also be convenient to let $ides(PF)$ denote the descent set of the inverse of $\sigma(PF)$.

This given, each parking function is assigned the *weight*

$$w(PF) \;=\; t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X] \tag{I.4}$$

where for $S \subseteq \{1, 2, \cdots, n-1\}$, $Q_S[X]$ denotes the corresponding Gessel [12] fundamental quasi-symmetric function,

$$area(PF) \;=\; \sum_{i=1}^{n} u_i \tag{I.5}$$

and

$$dinv(PF) \;=\; \sum_{1\leq i<j\leq n} \chi(u_i = u_j \,\&\, v_i < v_j) + \sum_{1\leq i<j\leq n} \chi(u_i = u_j+1 \,\&\, v_i > v_j) \;. \tag{I.6}$$

Parking functions are endowed by a colorful history and a jargon (see for instance [14]) that is very helpful in dealing with them combinatorially as well as analytically. For us it is sufficient to be able to translate properties of the two line array in I.1 to visual properties of the corresponding tableau. A single example of this correspondence should be sufficient for our purposes.

In Figure 1 we have a parking function as it is usually depicted on the right together with the corresponding vector $U = (u_1, u_2, \ldots, u_n)$ in the bottom row and its corresponding vector $V = (v_1, v_2, \ldots, v_n)$ in the top row of the array.
The diagonal of shaded cells is usually referred to as *main diagonal* (or 0-diagonal) of PF. The numbers in the lattice cells are the *cars*. The path along whose vertical steps we have set the cars is the Dyck path *supporting* PF. This given, the components of $U = (u_1, u_2, \ldots, u_n)$ are none other than the orders of the diagonals containing the cars. In this case car 3 is in the third diagonal, 1 and 8 are in the second diagonal, $5, 7$ and 6 are in the first diagonal and 2 and 4 are in the main diagonal. We have purposely listed the cars by diagonals from right to left starting with the highest diagonal. This gives the diagonal permutation $\sigma(PF)$. It is clear from this imagery, that the sum in I.5 gives the total number of cells between the supporting Dyck path and the main diagonal. We also see that two cars in the same diagonal with the car on the left smaller than the car on the right will contribute a unit to $dinv(PF)$. The same holds true when a car on the left is bigger than a car on the right with the latter in the adjacent lower diagonal. It will be convenient to think that the parking functions, with a given Dyck path D in the $n \times n$ lattice square, are constructed by first placing circles along the of NORTH steps of D and then filling the circles with $1, 2, \ldots, n$ in a column increasing manner.

In the present development, an additional parameter plays a crucial role. This is the *diagonal composition* of a parking function, which we denote by $p(PF)$. This is simply the composition which gives the position of the zeros in the vector $U = (u_1, u_2, \ldots, u_n)$, or equivalently the lengths of the segments of the main diagonal between successive hits of its Dyck path. In summary, for the present example we have

$$p(PF) = (5, 3), \quad area(PF) = 10, \;\; dinv(PF) = 4, \;\; \sigma(PF) = 31857624, \;\; ides(PF) = \{2, 4, 6, 7\}$$

yielding

$$w(PF) \;=\; t^{10} q^4 Q_{\{2,4,6,7\}}[X].$$

Here and after, the vectors U and V in the two line representation will be also referred to as $U(PF)$ and $V(PF)$. It will also be convenient to denote by $\mathcal{PF}_n$ the collection of parking functions in the $n \times n$ lattice square.

This given, we are now finally in a position to state the main result of this paper.
To begin we should note that the so called *shuffle conjecture* (see [14] or [15]) is simply the identity

$$\sum_{PF\in\mathcal{PF}_n} w(PF) \;=\; DH_n[X; q, t] \tag{I.11}$$

where $DH_n[X;q,t]$ denotes the Frobenius characteristic of the Diagonal Harmonics polynomials. Now it was conjectured in [6] and proved by Mark Haiman in [16] that we have

$$DH_n[X;q,t] \;=\; \nabla e_n$$

with ∇ is the symmetric function operator introduced in [2]. Thus, as a symmetric function identity, I.11 can be simply stated as

$$\sum_{PF\in\mathcal{PF}_n} w(PF) \;=\; \nabla e_n \;.$$

In particular setting

$$F_U[X] \;=\; \sum_{U(PF)=U} w(PF)$$

we have the decomposition

$$\nabla e_n \;=\; \sum_U F_U[X] \;. \qquad \text{I.12}$$

Since it can be shown that each $F_U[X]$ is an LLT polynomial and therefore it is Schur positive (see [1],[14]). The shuffle conjecture suggests that each $F_U[X]$ should be the Frobenius characteristic of a bigraded sub-module of the Diagonal Harmonics of S_n and I.12 corresponds to a direct sum decomposition of Diagonal Harmonics. The identification of these submodules would be more accessible if we had an explicit expression for each $F_U[X]$ in terms of ∇. The Haglund-Morse-Zabrocki conjectures provide such expressions for appropriate sums of the polynomials $F_U[X]$.

More precisely

Conjecture I (Haglund-Morse-Zabrocki [17])

For each composition $p=(p_1,p_2,\cdots,p_k)$ we have

$$\nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,1 \;=\; \sum_{PF\in\mathcal{PF}_n} w(PF)\,\chi\big(p(PF)=p\big) \qquad \text{I.13}$$

with $\mathbf{C}_a$ the operator that acts on a symmetric function $F[X]$ according to the plethystic formula

$$\mathbf{C}_aF[X] \;=\; (-\tfrac{1}{q})^{a-1}F\big[X-\tfrac{1-1/q}{z}\big]\Omega[zX]\Big|_{z^a}, \qquad \text{I.14}$$

Recall that a composition p is a refinement of a composition $r=(r_1,r_2,\ldots,r_m)$ and we write $p\preceq r$ if and only if p is a concatenation $p=p^{(1)},p^{(2)},\ldots,p^{(m)}$ with $p^{(i)}$ a composition of r_i. This given we can state

Conjecture II (Haglund-Morse-Zabrocki [17])

For each composition $r=(r_1,r_2,\cdots,r_m)\models m$ we have

$$\nabla\,\mathbf{B}_{r_m}\cdots\mathbf{B}_{r_2}\mathbf{B}_{r_1}\mathbf{1}=\sum_{p\preceq r} q^{\sum_{i=1}^k (k-i)l(p^{(i)})}\sum_{PF\in\mathcal{PF}_n} w(PF)\;\chi\Big(p(PF)=p\Big) \qquad \text{I.15}$$

with $\mathbf{B}_a=\omega\widetilde{\mathbf{B}}_a\omega$ and

$$\widetilde{\mathbf{B}}_aP[X] \;=\; P\big[X-\tfrac{1-q}{z}\big]\Omega[zX]\Big|_{z^a} \;. \qquad \text{I.16}$$

Our main result here can be stated as follows

Theorem I.1

For all compositions $p=(p_1,p_2,\ldots,p_k)$ *and* $1\le r\le n$ *we have*

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\;e_rh_{n-r}\big\rangle=\begin{cases} t^{p_1-1}q^{k-1}\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{B}_{p_1-1}\,1\,,\;e_{r-1}h_{n-r}\big\rangle & \text{if } p_1>1\\ q^{k-1}\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{B}_{0}\,1\,,\;e_{r-1}h_{n-r}\big\rangle & \\ \qquad\qquad +\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\;e_{r}h_{n-1-r}\big\rangle & \text{if } p_1=1\end{cases}. \qquad \text{I.17}$$

In particular for $r=n$ we obtain the following recursion

Corollary I.1

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\;e_n\big\rangle \;=\; t^{p_1-1}q^{k-1}\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{B}_{p_1-1}\,1\,,\;e_{n-1}\big\rangle \qquad (\text{for all } p_1\ge 1)\;.$$

These identities combined with two beautiful parking function bijections proved by Angela Hicks in [18] yield the following combinatorial result.

Theorem I.2

For any composition $p=(p_1,p_2,\ldots,p_k)$ *and every* $1\le r\le n$ *we have*

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\;e_rh_{n-r}\big\rangle \;=\; \sum_{PF\in\mathcal{PF}_n(r)} t^{area(PF)}q^{dinv(PF)}\chi\big(p(PF)=p\big) \qquad \text{I.18}$$

where $\mathcal{PF}_n(r)$ *denotes the collection of parking functions with diagonal permutation a shuffle of* $n(n-1)\cdots(n-r+1)$ *and* $123\cdots(n-r)$, *or equivalently, the collection of parking functions with diagonal permutation a shuffle of* $r\cdots 321$ *and* $(r+1)\cdots(n-1)n$.

For a given subset $S\subseteq\{1,2,\ldots,n-1\}$ let Z_S denote the skew Schur function whose standard tableau words are the permutations with descent set S. It follows from a theorem of Gessel [12] that if

$$F[X] \;=\; \sum_{\sigma\in S_n} c_\sigma\, Q_{ides(\sigma)}[X]$$

is a symmetric function then

$$\big\langle F\,,\;Z_S\big\rangle \;=\; \sum_{\sigma\in S_n} c_\sigma \chi\big(ides(\sigma)=S\big)\;.$$

Using this fact it is not difficult to show that the identities in I.18 are all special cases of Conjecture I.

It is also shown in [17] that

$$\sum_{p\models n}\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1 \;=\; e_n. \qquad \text{I.19}$$

In particular from I.18 and I.19 it follows that

$$\big\langle \nabla e_n\,,\;e_rh_{n-r}\big\rangle \;=\; \sum_{PF\in\mathcal{PF}_n(r)} t^{area(PF)}q^{dinv(PF)}. \qquad \text{I.20}$$

This is the q,t-Catalan [4] result for $r = n$ and the Schröder Theorem [13] for all the other values of r. In fact, note that when $\sigma(PF) = n \cdots 321$ then all pairs of circles in the same diagonal contribute to $dinv(PF)$ and all pairs with the circle on the right in the adjacent lower diagonal contribute as well. Indeed, in the first case their contents from left to right will be increasing and in the second case decreasing. This means that when $\sigma(PF) = n \cdots 321$, I.6 reduces to

$$dinv(U) \;=\; \sum_{1 \le i < j \le n} \chi(u_i = u_j) + \sum_{1 \le i < j \le n} \chi(u_i = u_j + 1)$$

and this expression may be viewed as the "dinv" of the corresponding Dyck path. Thus in this case I.18 may be restated as

$$\big\langle \nabla \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k}\, 1\, ,\; e_n \big\rangle \;=\; \sum_{D \in \mathcal{D}_n} t^{area(D)} q^{dinv(D)} \chi\big(p(D) = (p_1, p_2, \ldots, p_k)\big) \qquad \text{I.21}$$

where the sum is over all Dyck paths in the $n \times n$ lattice square which touch the main diagonal according to $p = (p_1 p_2, \ldots, p_k)$.

When $r < n$ we obtain a similar result where the sum is over Schröder paths. More precisely if we denote by $\mathcal{SCH}_n(d)$ the collection of Schröder paths with d diagonal steps I.17 may be restated as

$$\big\langle \nabla \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k}, 1\, ,\; e_{n-d} h_d \big\rangle \;=\; \sum_{SCH \in \mathcal{SCH}_n(d)} t^{area(SCH)} q^{dinv(SCH)} \chi\big(p(SCH) = (p_1, p_2, \ldots, p_k)\big) \qquad \text{I.22}$$

This identity follows immediately from the second combinatorial interpretation of the right-hand side of I.17 in the case that $\sigma(PF)$ is restricted to be a shuffle of $d \cdots 321$ and $(d+1)(d+2)\cdots n$. This follows from a simple bijection given in [14] between parking functions with such a diagonal permutation and Schröder paths with d diagonal steps, as illustrated in Figure 2 where the example $\sigma(PF)$ is a shuffle of 5678 and 4321.

In fact the column increasing condition together with the fact that $5, 6, 7, 8$ are increasing, by decreasing diagonals and from right to left, forces $5, 6, 7, 8$ to be on the top of their columns. This given the bijection, in general, is simply to place a diagonal step across each of the increasing big numbers, as we have done in the above display. Of course, in I.22 as well as in I.21, the area as well as the diagonal composition are defined to be the same as in the original parking function.

The operators, $\mathbf{C}_a$ and $\mathbf{B}_a$ are both closely related to the Hall-Littlewood polynomials obtained by setting $t = 0$ in the version of Macdonald polynomials $\tilde{H}_\mu[X; q, t]$ introduced in [5]. In fact, it can be shown that we have for any $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$

$$\mathbf{C}_{\mu_1} \mathbf{C}_{\mu_2} \cdots \mathbf{C}_{\mu_k}\, 1 \;=\; (-1/q)^n \tilde{H}_{\mu'}[X; q, 0]$$

and

$$\widetilde{\mathbf{B}}_{\mu_1} \widetilde{\mathbf{B}}_{\mu_2} \cdots \widetilde{\mathbf{B}}_{\mu_k}\, 1 \;=\; (q)^{e_\mu} \tilde{H}_{\mu}[X; 1/q, 0]$$

for a suitable exponent e_μ.

Our strategy, in the proof of Theorem I.2 is a standard one in this area of algebraic combinatorics. We simply show that both sides of I.18 satisfy the same recursion and the same initial conditions. The recursion for the left hand side is given by Theorem I.1. The bijections of Angela Hicks [18], constructed for this purpose, give the recursion for the right hand side. This given, we are only left to verify the base cases.

As in the proofs of the q,t-Catalan and Schröder theorems the proof of I.17 is based on some identities governing the Macdonald polynomials $\tilde{H}_\mu[X;q,t]$ developed in [3] and [8]. We also use plethystic notation throughout, in our manipulations of symmetric functions. However, in contrast with the arguments in [4] and [13], where a variety of new and complex summation formulas had to be created, the arguments used here are quite straightforward and use only identities that can be found in [3] and [8].

The contents of this paper are divided into two sections. In section 1 we introduce some notation, including a brief introduction to plethystic notation and recall the basic identities of Macdonald Theory we will use in section 2. In section 2. we prove Theorem I.1 and then complete the proof of Theorem I.2 .

Our proof of I.17 did not come together all by itself. The variety of identities and methods that allowed us to converge to its present deceptive simplicity should be of interest to the practitioners of this type of research. However, in order to keep our developments here as simple as possible these additional findings will be the subject of the two forthcoming publications [10] and [11].

1. A Macdonald Polynomial tool kit.

The space of symmetric polynomials will be denoted Λ. The subspace of homogeneous symmetric polynomials of degree m will be denoted $\Lambda^{=m}$. It will also convenient to let $\Lambda^{\leq m}$ denote the subspace of symmetric polynomials that are of degree $\leq m$. We will seldom work with symmetric polynomials expressed in terms of variables but rather express them in terms of one of the six classical symmetric function bases

(1) *power* $\{p_\mu\}_\mu$, (2) *monomial* $\{m_\mu\}_\mu$, (3) *homogeneous* $\{h_\mu\}_\mu$,
(4) *elementary* $\{e_\mu\}_\mu$, (5) *forgotten* $\{f_\mu\}_\mu$, and (6) *Schur* $\{s_\mu\}_\mu$.

We recall that the fundamental involution ω may be defined by setting for the power basis indexed by $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$

$$\omega p_\mu \;=\; (-1)^{n-k} p_\mu \;=\; (-1)^{|\mu|-l(\mu)} p_\mu \tag{1.1}$$

where for any vector $v = (v_1, v_2, \cdots, v_k)$ we set $|v| = \sum_{i=1}^k v_i$ and $l(v) = k$.

In dealing with symmetric function identities, specially with those arising in the Theory of Macdonald Polynomials, we find it convenient and often indispensable to use plethystic notation. This device has a straightforward definition which can be verbatim implemented in MAPLE of MATHEMATICA for computer experimentation. We simply set for any expression $E = E(t_1, t_2, \ldots)$ and any power symmetric function p_k

$$p_k[E] \;=\; E(\,t_1^k, t_2^k, \ldots). \tag{1.2}$$

This given, for any symmetric function F we set

$$F[E] \;=\; Q_F(p_1, p_2, \ldots)\Big|_{p_k \rightarrow E(\,t_1^k, t_2^k, \ldots)} \tag{1.3}$$

where Q_F is the polynomial yielding the expansion of F in terms of the power basis. Note that in writing $E(t_1, t_2, \ldots)$ we are tacitly assuming that $t_1, t_2, t_3, \ldots$ are all the variables appearing in E and in writing $E(t_1^k, t_2^k, \ldots)$ we intend that all the variables appearing in E have been raised to their k^{th} power.

A paradoxical but necessary property of plethystic substitutions is that 1.1 requires

$$p_k[-E] \;=\; -p_k[E]. \tag{1.5}$$

This notwithstanding, we will still need to carry out ordinary changes of signs. To distinguish it from the *plethystic* minus sign, we will carry out the *ordinary* sign change by means of a new variables ϵ which outside of the plethystic bracket is simply replaced by -1. For instance, these conventions give for $X_k = x_1 + x_2 + \cdots + x_n$

$$p_k[-\epsilon X_n] \;=\; -\epsilon^k \sum_{i=1}^n x_i^k \;=\; (-1)^{k-1} \sum_{i=1}^n x_i^k \ .$$

In particular we get for $X = x_1 + x_2 + x_3 + \cdots$

$$\omega p_k[X] \;=\; p_k[-\epsilon X].$$

Thus for any symmetric function $F \in \Lambda$ and any expression E we have

$$\omega F[E] \;=\; F[-\epsilon E]\ . \tag{1.6}$$

In particular, if $F \in \Lambda^{=k}$ we may also rewrite this as

$$F[-E] \;=\; (-1)^k \omega F[E]. \tag{1.7}$$

The formal power series

$$\Omega \;=\; exp\Big(\sum_{k\geq 1} \frac{p_k}{k}\Big)$$

combined with plethysic substitutions will provide a powerful way of dealing with the many generating functions occurring in our manipulations.

Using 1.6, we can define in one step the operator $\mathbf{B}_a$ by setting, for every polynomial $P[X]$

$$\mathbf{B}_a P[X] \;=\; P\big[X + \epsilon \tfrac{1-q}{z}\big]\Omega[-\epsilon z X]\Big|_{z^a} \;. \tag{1.8}$$

Indeed since by definition $\mathbf{B}_a = \omega \widetilde{\mathbf{B}}_a \omega$, formula I.16 gives

$$\mathbf{B}_a P[X] \;=\; \omega \widetilde{\mathbf{B}}_a P[-\epsilon X] \;=\; \omega P\big[-\epsilon X + \epsilon \tfrac{1-q}{z}\big]\Omega[zX]\Big|_{z^a} \;=\; P\big[X + \epsilon \tfrac{1-q}{z}\big]\Omega[-\epsilon z X]\Big|_{z^a} \;.$$

Here and after we will use 1.8 to compute the action of $\mathbf{B}_a$.

For a given expression E we will set

$$\Omega[E] \;=\; exp\Big(\sum_{k\geq 1} \frac{p_k[E]}{k}\Big)$$

and since for any two expressions A, B 1.1 gives

$$p_k[A+B] \;=\; p_k[A] + p_k[B] \;. \tag{1.9}$$

We derive from this the fundamental formula

$$\Omega[A+B] \;=\; \Omega[A]\,\Omega[B] \;. \tag{1.10}$$

In particular when $A = \sum_{i=1}^n a_i$ and $B = \sum_{j=1}^m b_j$ we get

$$\Omega[tA - tB] = \frac{\prod_{j=1}^m (1 - t b_j)}{\prod_{i=1}^n (1 - t a_i)} \;. \tag{1.11}$$

Clearly, for any two expressions A, B we can also view $\Omega[t(A-B)]$ as the generating functions of the homogeneous symmetric functions plethystically evaluated at $A - B$

$$\Omega[t(A-B)] = \sum_{m\geq 1} t^m h_m[A-B] \;.$$

In particular, by equating coefficients of t^m on both sides of 1.11, 1.9 gives (using 1.8)

$$h_m[A-B] \;=\; \sum_{r=0}^m h_{m-r}[A] h_r[-B] \;=\; \sum_{r=0}^m h_{m-r}[A](-1)^r e_r[B] \tag{1.12}$$

which is an identity that will play a crucial role in many of our calculations.

To present our Macdonald polynomial kit, it is convenient to identify partitions with their (french) Ferrers diagram. Given a partition μ and a cell $c \in \mu$, Macdonald introduces four parameters $l = l_\mu(c)$, $l' = l'_\mu(c)$, $a = a_\mu(c)$ and $a' = a'_\mu(c)$ called *leg*, *coleg*, *arm* and *coarm* which give the number of lattice cells of μ strictly NORTH, SOUTH, EAST and WEST of c, (see Figure 3). Following Macdonald we will set

$$n(\mu) \;=\; \sum_{c\in\mu} l_\mu(c) \;=\; \sum_{c\in\mu} l'_\mu(c) \;=\; \sum_{i=1}^{l(\mu)} (i-1)\mu_i.$$

Denoting by μ' the conjugate of μ, the basic ingredients playing a role in the theory of Macdonald polynomials are

$$\begin{gathered} T_\mu = t^{n(\mu)} q^{n(\mu')}\ , \quad B_\mu(q,t) = \sum_{c\in\mu} t^{l'_\mu(c)} q^{a'_\mu(c)}\ , \quad \Pi_\mu(q,t) = \prod_{c\in\mu; c\neq(0,0)} (1 - t^{l'_\mu(c)} q^{a'_\mu(c)}), \\ w_\mu(q,t) = \prod_{c\in\mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}), \end{gathered} \tag{1.13}$$

together with a deformation of the Hall scalar product, which we call the $*$-scalar product, defined by setting for the power basis

$$\langle p_\lambda\ ,\ p_\mu \rangle_* \;=\; (-1)^{|\mu|-l(\mu)} \prod_i (1-t^{\mu_i})(1-q^{\mu_i})\ z_\mu\ \chi(\lambda=\mu), \tag{1.14}$$

where z_μ gives the order of the stabilizer of a permutation with cycle structure μ.

This given, the modified Macdonald Polynomials we will deal with here are the unique symmetric function basis $\{\tilde H_\mu[X;q,t]\}_\mu$ which satisfies the orthogonality condition

$$\langle \tilde H_\lambda\ ,\ \tilde H_\mu \rangle_* \;=\; \chi(\lambda=\mu) w_\mu(q,t) \tag{1.15}$$

and a triangularity condition with respect to the basis $s_\lambda[X/(t-1)]$.

The $*$-scalar product, is simply related to the ordinary Hall scalar product by setting for all pairs of symmetric functions f, g

$$\langle f\ ,\ g \rangle_* \;=\; \langle f\ ,\ \omega\phi g \rangle \tag{1.16}$$

where it has been customary to let ϕ be the operator defined by setting for any symmetric function f

$$\phi\, f[X] \;=\; f[MX] \tag{1.17}$$

with

$$M \;=\; (1-t)(1-q) \tag{1.18}$$

Note that the inverse of ϕ is usually written in the form

$$f^*[X] \;=\; f[X/M]\ . \tag{1.19}$$

In particular we also have for all symmetric functions f, g

$$\langle f\ ,\ g \rangle \;=\; \langle f, \omega g^* \rangle_*\ . \tag{1.20}$$

The orthogonality relations in 1.15 yield the Cauchy identity for our Macdonald polynomials in the form

$$\Omega\left[-\epsilon \tfrac{XY}{M}\right] \;=\; \sum_{\mu} \frac{\tilde H_\mu[X]\tilde H_\mu[Y]}{w_\mu} \tag{1.21}$$

which restricted to its homogeneous component of degree n in X and Y reduces to

$$e_n\left[\tfrac{XY}{M}\right] \;=\; \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]\tilde H_\mu[Y]}{w_\mu}\ . \tag{1.22}$$

In fact, from the definition in 1.14 it follows that the reproducing kernel for the $*$-scalar product is given the the sum

$$\sum_{\mu}(-1)^{|\mu|-l(\mu)}\frac{p_\mu[X]p_\mu[Y]}{p_\mu[M]} \;=\; \sum_{\mu}(-1)^{|\mu|-l(\mu)}p_\mu\left[\tfrac{XY}{M}\right] \;=\; \Omega\left[-\epsilon \tfrac{XY}{M}\right]$$

since the left hand side of this identity must be equal to the right hand side of 1.21 the equality in 1.21 must hold true as well. It will be convenient here and in the sequel to use the short hand notation

$$\widetilde\Omega\left[\tfrac{XY}{M}\right] \;=\; \Omega\left[-\epsilon \tfrac{XY}{M}\right]\ .$$

A crucial tool which provides many of the transformations we will need in the sequel is the so called the Macdonald-Koorwinder "reciprocity" formula (see [19] or [8]). For our version of the Macdonald polynomials this formula can be written in the following concise form

$$\frac{\tilde H_\alpha[1+u\,D_\beta]}{\prod_{c\in\alpha}(1-u\,t^{l'}q^{a'})} \;=\; \frac{\tilde H_\beta[1+u\,D_\alpha]}{\prod_{c\in\beta}(1-u\,t^{l'}q^{a'})} \qquad \text{(for all pairs } \alpha,\beta) \tag{1.23}$$

where for convenience we have set

$$D_\alpha(q,t) \;=\; MB_\alpha(q,t)-1\ . \tag{1.24}$$

We will use here several special evaluations of 1.23, To begin, canceling the common factor $(1-u)$ out of the denominators on both sides of 1.23 then setting $u=1$ gives

$$\frac{\tilde H_\alpha[MB_\beta]}{\Pi_\alpha} \;=\; \frac{\tilde H_\beta[MB_\alpha]}{\Pi_\beta} \qquad \text{(for all pairs } \alpha,\beta)\ . \tag{1.25}$$

On the other hand replacing u by $1/u$ and letting $u=0$ in 1.23 gives

$$(-1)^{|\alpha|}\frac{\tilde H_\alpha[D_\beta]}{T_\alpha} \;=\; (-1)^{|\beta|}\frac{\tilde H_\beta[D_\alpha]}{T_\beta} \qquad \text{(for all pairs } \alpha,\beta)\ . \tag{1.26}$$

Since for β the empty partition we can take $\tilde H_\beta=1$ and $D_\beta=-1$, 1.23 in this case reduces to

$$\tilde H_\alpha[1-u] \;=\; \prod_{c\in\alpha}(1-ut^{l'}q^{a'}) \;=\; (1-u)\sum_{r=0}^{n-1}(-u)^r e_r[B_\mu-1]\ . \tag{1.27}$$

This identity yields the coefficients of hook Schur functions in the expansion

$$\tilde{H}_\mu[X;q,t] \;=\; \sum_{\lambda\vdash|\mu|} s_\mu[X]\tilde{K}_{\lambda\mu}(q,t)\ . \tag{1.28}$$

In fact an application of 1.12 with $A = X$ and $B = uX$ gives

$$s_\mu[1-u] \;=\; \begin{cases} (-u)^r(1-u) & \text{if } \mu=(n-r,1^r) \\ 0 & \text{otherwise} \end{cases}\ . \tag{1.29}$$

Thus 1.28, with $X = 1-u$, combined with 1.27 gives for $\mu\vdash n$

$$\left\langle \tilde{H}_\mu\ ,\ s_{(n-r,1^r)}\right\rangle \;=\; e_r[B_\mu-1] \tag{1.30}$$

and the identity $e_rh_{n-r} = s_{(n-r,1^r)} + s_{(n-r-1,1^{r-1})}$ gives

$$\left\langle \tilde{H}_\mu\ ,\ e_rh_{n-r}\right\rangle \;=\; e_r[B_\mu]. \tag{1.31}$$

The case $\beta = (1)$ of 1.25 is also significant in that it reduces to the identity

$$\tilde{H}_\alpha[M] \;=\; MB_\alpha\Pi_\alpha. \tag{1.32}$$

Another crucial ingredient in our manipulations is the symmetric function operator ∇ which is defined by setting for the Macdonald basis

$$\nabla\tilde{H}_\mu(X;q,t) \;=\; T_\mu\tilde{H}_\mu(X;q,t)\ . \tag{1.33}$$

It was conjectured in [6] and proved in [16] that the bigraded Frobenius characteristic of the diagonal Harmonics of S_n is given by the symmetric function

$$DH_n[X;q,t] \;=\; \sum_{\mu\vdash n}\frac{T_\mu\tilde{H}_\mu(X;q,t)B_\mu(q,t)\Pi_\mu(q,t)(1-t)(1-q)}{w_\mu(q,t)}\ . \tag{1.34}$$

Surprisingly the intricate rational function on the right hand side is none other than ∇e_n. To see this we simply combine the relation in 1.32 with the degree n restricted Cauchy formula 1.22, obtaining

$$e_n[X]=e_n\left[\tfrac{XM}{M}\right] \;=\; \sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X]MB_\mu\Pi_\mu}{w_\mu}\ . \tag{1.35}$$

This discovery is precisely what led to the introduction of ∇ in the first place.

Our final ingredients we need, to carry out our proofs, are the coefficients $d_{\mu\nu}$ and $c_{\mu\nu}$ occurring in the Pieri formulas

$$a)\quad e_1\tilde{H}_\nu \;=\; \sum_{\mu\leftarrow\nu} d_{\mu\nu}\tilde{H}_\mu\ , \qquad\qquad b)\quad e_1^\perp\tilde{H}_\mu \;=\; \sum_{\nu\to\mu} c_{\mu\nu}\tilde{H}_\nu\ , \tag{1.36}$$

and their corresponding summation formulas (see [9],[3]) (†)

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases} \frac{tq}{M}\,h_{k+1}\big[D_\mu(q,t)/tq\big] & \text{if } k\geq 1\ , \\ B_\mu(q,t) & \text{if } k=0\ . \end{cases} \tag{1.37}$$

$$\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases} (-1)^{k-1}\,e_{k-1}\big[D_\nu(q,t)\big] & \text{if } k\geq 1\ , \\ 1 & \text{if } k=0\ . \end{cases} \tag{1.38}$$

Here $\nu\to\mu$ simply means that the sum is over ν's obtained from μ by removing a corner cell and $\mu\leftarrow\nu$ means that the sum is over μ's obtained from ν by adding a corner cell.

It will be useful to know that these two Pieri coefficients are related by the identity

$$d_{\mu\nu} \;=\; Mc_{\mu\nu}\tfrac{w_\nu}{w_\mu}\ . \tag{1.39}$$

2. Hall-Littlewood operators magics.

The proof of the q,t-Catalan result in [4] hinged on the discovery of the symmetric functions $E_{n,k}[X;q]$ defined through the plethystic identity

$$e_n\big[X\tfrac{1-x}{1-q}\big] \;=\; \sum_{k=1}^{n}\frac{(x,q)_k}{(q,q)_k}\;E_{n,k}[X;q] \tag{2.1}$$

where as customary we have set

$$(x,q)_k \;=\; (1-x)(1-xq)\cdots(1-xq^{k-1})\ . \tag{2.2}$$

These symmetric functions played a crucial role also in the proof of the q,t-Schröder Theorem [13]. Here they play a minor role since their contribution to the subject seems to be superseded by the Hall-Littlewood operators, as it is quite evident from the following beautiful identity.

Proposition 2.1

For every integer $n\geq 1$ we have

$$q\,\delta_q\,e_n\big[X\tfrac{1-x}{1-q}\big] \;=\; \sum_{a=1}^{n}\mathbf{C}_a e_{n-a}\big[X\tfrac{1-x}{1-q}\big] \tag{2.3}$$

where δ_q is the operator which acts on a polynomial $P(x)$ according to the formula

$$\delta_q\,P(x) \;=\; \frac{P(x)-P(x/q)}{x}\ . \tag{2.4}$$

In particular it follows that the $E_{n,k}$ can be obtained from the recursion

$$E_{n,k} \;=\; \sum_{a=1}^{n}\mathbf{C}_a E_{n-a,k-1}\ . \tag{2.5}$$

(†) Formula 1.37 is proved in [9] and a proof of 1.38 is given in [3]

Proof

To prove 2.3 we start by evaluating the left hand side of 2.3 using 2.4. This gives

$$q\delta_q\, e_n\big[X\tfrac{1-x}{1-q}\big] \;=\; \tfrac{q}{x}\Big(e_n\big[X\tfrac{1-x}{1-q}\big] \,-\, e_n\big[X\tfrac{1-x/q}{1-q}\big]\Big)\,. \qquad 2.6$$

Note next that we have

$$\begin{aligned} e_n\big[X\tfrac{1-x/q}{1-q}\big] \;&=\; e_n\big[X\tfrac{1-x}{1-q}+xX\tfrac{1-1/q}{1-q}\big] \;=\; e_n\big[X\tfrac{1-x}{1-q}-\tfrac{x}{q}X\big] \\ &=\; e_n\big[X\tfrac{1-x}{1-q}\big] \,+\, \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{x}{q})^a h_a[X]\,. \end{aligned}$$

Thus 2.6 reduces to

$$q\,\delta_q\, e_n\big[X\tfrac{1-x}{1-q}\big] \;=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{x}{q})^{a-1} h_a[X]\,. \qquad 2.7$$

To work on the right hand side of 2.3 we use I.14 to get

$$\begin{aligned} (-q)^{a-1}\mathbf{C}_a e_{n-a}\big[X\tfrac{1-x}{1-q}\big] \;&=\; e_{n-a}\big[\big(X-\tfrac{1-1/q}{z}\big)\tfrac{1-x}{1-q}\big]\Omega[zX]_{z^a} \\ &=\; e_{n-a}\big[X\tfrac{1-x}{1-q}\big]h_a[X] \,+\, \sum_{s=1}^{n-a} e_{n-a-s}\big[X\tfrac{1-x}{1-q}\big]e_s\big[-\tfrac{(1-x)(1-1/q)}{1-q}\big]h_{s+a}[X] \\ &=\; e_{n-a}\big[X\tfrac{1-x}{1-q}\big]h_a[X] \,+\, \sum_{s=1}^{n-a} e_{n-a-s}\big[X\tfrac{1-x}{1-q}\big]e_s\big[\tfrac{1-x}{q}\big]h_{s+a}[X]\,. \end{aligned} \qquad 2.8$$

But for $s\geq 1$ we have

$$e_s[\tfrac{1-x}{q}] \;=\; \tfrac{1}{q^s}(-x)^{s-1}(1-x) \;=\; \tfrac{1}{q}(1-x)(-\tfrac{x}{q})^{s-1}$$

and 2.8 becomes

$$(-q)^{a-1}\mathbf{C}_a e_{n-a}\big[X\tfrac{1-x}{1-q}\big] \;=\; e_{n-a}\big[X\tfrac{1-x}{1-q}\big]h_a[X] \,+\, \tfrac{1}{q}(1-x)\sum_{s=1}^{n-a} e_{n-a-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{x}{q})^{s-1}h_{s+a}[X]\,.$$

Using this, the right and side of 2.3 becomes

$$\begin{aligned} \sum_{a=1}^{n}\mathbf{C}_a e_{n-a}\big[X\tfrac{1-x}{1-q}\big] \;&=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}h_a[X] \,+\, \tfrac{1}{q}(1-x)\sum_{a=1}^{n}\sum_{s=1}^{n-a} e_{n-a-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}(-\tfrac{x}{q})^{s-1}h_{s+a}[X] \\ &=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}h_a[X] \,-\, (1-x)\sum_{a=1}^{n}\sum_{s=1}^{n-a} e_{n-a-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a+s-1}x^{s-1}h_{s+a}[X] \\ &=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}h_a[X] \,-\, (1-x)\sum_{a=1}^{n}\sum_{s=a+1}^{n} e_{n-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{s-1}x^{s-a-1}h_{s}[X] \\ &=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}h_a[X] \,-\, (1-x)\sum_{s=1}^{n} e_{n-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{s-1}h_s[X]\sum_{a=1}^{s-1}x^{s-a-1} \\ &=\; \sum_{a=1}^{n} e_{n-a}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{a-1}h_a[X] \,-\, \sum_{s=1}^{n} e_{n-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{1}{q})^{s-1}h_s[X](1-x^{s-1}) \\ &=\; \sum_{s=1}^{n} e_{n-s}\big[X\tfrac{1-x}{1-q}\big](-\tfrac{x}{q})^{s-1}h_s[X] \;=\; q\,\delta_q\, e_n\big[X\tfrac{1-x}{1-q}\big]\,. \quad \text{(by 2.6)} \end{aligned}$$

This completes our argument.

The identity in 2.3 has the following important corollary.

Theorem 2.1(Haglund-Morse-Zabrocki)

For all $1 \leq k \leq n$ *we have*

$$E_{n,k} \;=\; \sum_{p\models n\,;\, l(p)=k} \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,1 \tag{2.9}$$

$$e_n \;=\; \sum_{p\models n} \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,1\ . \tag{2.10}$$

Proof

Setting $x=q$ in 2.1 gives

$$e_n \;=\; \sum_{k=1}^{n} E_{n,k}\ . \tag{2.10}$$

Thus 2.10 follows from 2.9. On the other hand 2.9 itself is obtained by successive iterations of 2.5 together with the initial conditions

$$E_{n,1} \;=\; (-1/q)^{n-1}h_n[X] \;=\; \mathbf{C}_n 1\ . \tag{2.11}$$

In fact, setting $x=1/q$ in 2.1 gives

$$(-1/q)^n h_n[X] \;=\; e_n\big[X\tfrac{1-1/q}{1-q}\big] \;=\; \tfrac{1-1/q}{1-q}E_{n,1} \;=\; (-1/q)E_{n,1}\ .$$

This proves the first equality in 2.11. The second equality follows from the definition in I.14

Remark 2.1

We should mention that the original proof of the identities in 2.9 and 2.10 used a full blown repertoire of Macdonald polynomial identities. We gave this new proof since it reveals the intimate connections of the $E_{n,k}$ with the $\mathbf{C}_a$ operators and uses only elementary symmetric function identities.

For $E_1, E_2, \ldots, E_k$ given expressions and $P[X]$ a symmetric polynomial we set

$$P^{(r_1,r_2,\ldots,r_k)}[X] \;=\; P[X+E_1u_1+E_2u_2+\cdots+E_ku_k]\Big|_{u_1^{r_1}u_2^{r_2}\cdots u_k^{r_k}}\ .$$

The important property is that if

$$Q^{(r_1)}[X] \;=\; P[X+E_1u_1]\Big|_{u_1^{r_1}}$$

then

$$Q^{(r_1)}[X+E_2u_2]\Big|_{u_2^{r_2}} \;=\; P[X+E_1u_1+E_2u_2]\Big|_{u_1^{r_1}u_2^{r_2}} \;=\; P^{(r_1,r_2)}[X]\ .$$

Proposition 2.2

$$\big(q\,\mathbf{C}_b\mathbf{B}_a \,-\, \mathbf{B}_a\mathbf{C}_b\big)P[X] \;=\; (q-1)(-1)^{a+b-1}/q^{b-1}\times\begin{cases}0 & \text{if } a+b>0\\ P[X] & \text{if } a+b=0\\ \sum_{r_1+r_2=-(a+b)}P^{r_1,r_2}[X] & \text{if } a+b<0\end{cases} \tag{2.12}$$

Proof

Using I.16 we get

$$(-q)^{b-1}\mathbf{C}_b P[X] \;=\; \sum_{r_1=0}^{d} P^{(r_1)}[X]\tfrac{1}{z^{r_1}}\sum_{m\geq 0} z^m h_m[X]\Big|_{z^b} \;=\; \sum_{r_1=0}^{d} P^{(r_1)}[X]h_{b+r_1}[X]$$

and 1.8 gives

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_b P[X] \;&=\; \sum_{r_1=0}^{d} P^{(r_1)}\big[X+\epsilon\tfrac{1-q}{z_2}\big]h_{b+r_1}\big[X+\epsilon\tfrac{1-q}{z_2}\big]\Omega[-\epsilon z_2 X]\Big|_{z_2^a}\\
&=\; \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}(\tfrac{1}{z_2})^{r_2}\sum_{s=0}^{b+r_1} h_{b+r_1-s}[X]h_s\big[\epsilon\tfrac{1-q}{z_2}\big]\Omega[-\epsilon z_2 X]\Big|_{z_2^a}\\
&=\; \sum_{r_1,r_2=0}^{d}\sum_{s=0}^{b+r_1} P^{(r_1,r_2)}h_{b+r_1-s}[X]h_s\big[\epsilon(1-q)\big]\Omega[-\epsilon z_2 X]\Big|_{z_2^{a+r_2+s}}\\
&=\; \sum_{r_1,r_2=0}^{d}\sum_{s=0}^{b+r_1} P^{(r_1,r_2)}[X]h_{b+r_1-s}[X](-1)^s h_s\big[(1-q)\big]h_{a+r_2+s}[-\epsilon X]\;.\end{aligned}$$

Now note that 1.29 gives

$$h_s\big[(1-q)\big] \;=\; \begin{cases}1 & \text{if } s=0\\ 1-q & \text{if } s>0\end{cases}\;.$$

We can thus write

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_b P[X] \;&=\; \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{b+r_1}[X]h_{a+r_2}[-\epsilon X]+\\
&\quad+\;(1-q)\sum_{r_1,r_2=0}^{d}\sum_{s=1}^{b+r_1} P^{(r_1,r_2)}[X]h_{b+r_1-s}[X](-1)^s h_{a+r_2+s}[-\epsilon X]\end{aligned}$$

and the change of summation index $u=a+r_2+s$ gives

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_b P[X] \;&=\; \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{b+r_1}[X]h_{a+r_2}[-\epsilon X]+\\
&\quad+\;(1-q)\sum_{r_1,r_2=0}^{d}\sum_{u=a+r_2+1}^{a+b+r_1+r_2} P^{(r_1,r_2)}[X]h_{a+b+r_1+r_2-u}[X](-1)^{u-a-r_2}h_u[-\epsilon X]\;.\end{aligned}\tag{2.13}$$

Similarly, we get

$$\mathbf{B}_a P[X] \;=\; \sum_{r_2=0}^{d} P^{r_2}[X](\tfrac{1}{z_2})^{r_2}\sum_{u\geq 0} z_2^u h_u[-\epsilon z_2 X]\Big|_{z_2^a} \;=\; \sum_{r_2=0}^{d} P^{r_2}[X]h_{r_2+a}[-\epsilon X]\;.$$

Thus

$$
\begin{aligned}
(-q)^{b-1}\mathbf{C}_b\mathbf{B}_a P[X] &= \sum_{r_2=0}^{d} P^{r_2}\big[X-\tfrac{1-1/q}{z}\big]h_{r_2+a}\big[-\epsilon\big(X-\tfrac{1-1/q}{z_1}\big)\big]\,\Omega[z_1X]\Big|_{z_1^b}\\
&= \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X](\tfrac{1}{z_1})^{r_1}\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(\tfrac{1}{z_1})^{s}h_s\big[\epsilon(1-1/q)\big]\,\Omega[z_1X]\Big|_{z_1^b},\\
&= \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(-1)^s h_s\big[1-1/q\big]\,h_{r_1+s+b}[X]\ .
\end{aligned}
$$

Note that now 1.29 gives

$$
h_s\big[(1-1/q)\big] \;=\; \begin{cases} 1 & \text{if } s=0\\ 1-1/q & \text{if } s>0 \end{cases}\ .
$$

Thus

$$
\begin{aligned}
(-q)^{b-1}\mathbf{C}_b\mathbf{B}_a P[X] &= \big(1-(1-1/q)\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X]\\
&\qquad + \big(1-1/q\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(-1)^s\,h_{r_1+s+b}[X]\ .
\end{aligned}
$$

and the change of summation index $u=r_2+a-s$ gives

$$
\begin{aligned}
(-q)^{b-1}\mathbf{C}_b\mathbf{B}_a P[X] &= \frac{1}{q}\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X]\\
&\qquad + \big(1-1/q\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{u=0}^{a+r_2} h_{u}[-\epsilon X]\,(-1)^{r_2+a-u}\,h_{a+b+r_1+r_2-u}[X]\ .
\end{aligned}
$$

In summary, we get

$$
\begin{aligned}
(-q)^{b-1}\ \ q\,\mathbf{C}_b\mathbf{B}_a P[X] &= \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X]\\
&\qquad + \big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{u=0}^{a+r_2} h_{u}[-X]\,(-1)^{r_2}\,h_{a+b+r_1+r_2-u}[X]\ .
\end{aligned}
$$

On the other hand 2.13 can also be written as

$$
\begin{aligned}
(-q)^{b-1}\mathbf{B}_a\mathbf{C}_b P[X] &= \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X]+\\
&\qquad + (-1)^a(1-q)\sum_{r_1,r_2=0}^{d}\ \sum_{u=a+r_2+1}^{a+b+r_1+r_2} P^{(r_1,r_2)}[X]h_{a+b+r_1+r_2-u}[X](-1)^{r_2}h_u[-X]
\end{aligned}
$$

and thus subtraction gives

$$\begin{aligned}(-q)^{b-1}\ \big(q\,\mathbf{C}_b\mathbf{B}_a\ -\ \mathbf{B}_a\mathbf{C}_b\big)P[X] \ &=\ \big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d}P^{r_1,r_2}[X]\sum_{u=0}^{a+b+r_1+r_2}h_u[-X]\,(-1)^{r_2}\,h_{a+b+r_1+r_2-u}[X]\\ &=\ \big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d}P^{r_1,r_2}[X](-1)^{r_2}\,h_{a+b+r_1+r_2}[X-X]\ .\end{aligned}$$

Carrying out the summations and using the definition of $P^{r_1,r_2}[X]$ we finally obtain

$$(-q)^{b-1}\ \big(q\,\mathbf{C}_b\mathbf{B}_a\ -\ \mathbf{B}_a\mathbf{C}_b\big)P[X]\ =\ (q-1)(-1)^a\times\begin{cases}0 & \text{if } a+b>0\\ P[X] & \text{if } a+b=0\\ \sum_{r_1+r_2=-(a+b)}P^{r_1,r_2}[X] & \text{if } a+b<0\end{cases}$$

which is easily seen to be 2.12, completing the proof.

In particular we have shown that

Theorem 2.2(Haglund-Morse-Zabrocki)

For all $a+b>0$, our Hall-Littlewood operators have the following commutativity property

$$\mathbf{B}_a\,\mathbf{C}_b\ =\ q\,\mathbf{C}_b\,\mathbf{B}_a\ . \tag{2.14}$$

Remark 2.2

The original proof of 2.14, although much shorter, used manipulations which we found difficult to justify. Similar dubious manipulations occur throughout the literature involving Vertex operators. Our efforts to put these manipulations on a solid foundation led to our forthcoming papers [10] and [11]. It will be shown there that the partial fraction algorithm developed in [20] for the computation of constant terms, provides a natural and solid foundation for dealing with compositions of Vertex operators. Moreover, it will be shown in [11] how an application of this algorithm led us to the discovery of the Macdonald Polynomial identity that plays a major role in our proof of Theorem I.1.

Theorem 2.2 allows us a first reduction in our path to Theorem I.1:

Proposition 2.3

The identity

$$\big\langle\nabla\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,1\ ,\ e_rh_{n-r}\big\rangle=\begin{cases}t^{p_1-1}q^{k-1}\big\langle\nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\mathbf{B}_{p_1-1}\,1\ ,\ e_{r-1}h_{n-r}\big\rangle & \textit{if } p_1>1\\ q^{k-1}\big\langle\nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\mathbf{B}_0\,1\ ,\ e_{r-1}h_{n-r}\big\rangle & \\ \qquad\qquad+\big\langle\nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\,1\ ,\ e_rh_{n-1-r}\big\rangle & \textit{if } p_1=1\end{cases} \tag{2.15}$$

is valid for all compositions $p=(p_1,p_2,\ldots,p_k)$ and $1\le r\le n$ if and only if we have

$$\mathbf{C}_a^*\nabla h_r\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big]\ =\ \begin{cases}t^{a-1}\mathbf{B}_{a-1}^*\nabla h_{r-1}\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big] & \textit{if } a>1\\ \mathbf{B}_0^*\nabla h_{r-1}\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big]\ +\ \nabla h_r\big[\tfrac{X}{M}\big]e_{n-1-r}\big[\tfrac{X}{M}\big] & \textit{if } a=1\end{cases} \tag{2.16}$$

with $\mathbf{C}_a^*$ and $\mathbf{B}_b^*$ the operator adjoints of $\mathbf{C}_a$ and $\mathbf{B}_b$ with respect to the $*$-scalar product.

Proof

Using 2.14 we can start by rewriting the first equality in 2.15 as

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\; e_r h_{n-r}\big\rangle \;=\; t^{p_1-1}\big\langle \nabla \mathbf{B}_{p_1-1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\; e_{r-1}h_{n-r}\big\rangle\,. \qquad 2.17$$

This given, note that since for $(p_2,p_3,\ldots,p_k)$ a partition the expressions $\mathbf{C}_{p_2}\mathbf{C}_{p_3}\cdots\mathbf{C}_{p_k}1$ are essentially Hall-Littlewood polynomials and therefore constitute a symmetric function basis, it follows that 2.17 holds true of all $(p_2,p_3,\ldots,p_k)$ if and only if for any $a>1$ and any symmetric polynomial $P[X]$ we have

$$\big\langle \nabla \mathbf{C}_a P[X]\,,\; e_r h_{n-r}\big\rangle \;=\; t^{a-1}\big\langle \nabla \mathbf{B}_{a-1}P[X]\,,\; e_{r-1}h_{n-r}\big\rangle\,.$$

Changing from the customary Hall scalar product to the $*$-scalar product using 1.20 gives

$$\big\langle \nabla \mathbf{C}_a P[X]\,,\; h_r^* e_{n-r}^*\big\rangle_* \;=\; t^{a-1}\big\langle \nabla \mathbf{B}_{a-1}P[X]\,,\; h_{r-1}^* e_{n-r}^*\big\rangle_*\,. \qquad 2.18$$

But now the $*$-duality of the Macdonald bases $\tilde H_\mu$ and $\tilde H_\mu/w_\mu$ and the fact that ∇ is an eigen-operator for both $\tilde H_\mu$ and $\tilde H_\mu/w_\mu$ yields that ∇ is $*$-self adjoint. This allows us to rewrite 2.18 in the form

$$\big\langle P[X]\,,\; \mathbf{C}_a^*\nabla h_r^* e_{n-r}^*\big\rangle_* \;=\; t^{a-1}\big\langle P[X]\,,\; \mathbf{B}_{a-1}^*\nabla h_{r-1}^* e_{n-r}^*\big\rangle_*$$

and thus the equivalence of the first equalities in 2.15 and 2.16 follows from the arbitrariness of $P[X]$. The equivalence of the second equalities is shown in exactly the same manner,

Our next result provides explicit expressions for both $\mathbf{C}_a^*$ and $\mathbf{B}_{a-1}^*$.

Theorem 2.2

For all symmetric polynomials $P[X]$ we have

$$\mathbf{B}_a^* P[X] \;=\; P\big[X+\tfrac{M}{z}\big]\,\Omega\big[\tfrac{-zX}{1-t}\big]\Big|_{z^{-a}} \qquad 2.19$$

and

$$\mathbf{C}_a^* P[X] \;=\; (\tfrac{-1}{q})^{a-1} P\big[X-\tfrac{\epsilon M}{z}\big]\,\Omega\big[\tfrac{-\epsilon z X}{q(1-t)}\big]\Big|_{z^{-a}}\,. \qquad 2.20$$

Proof

Since $\tilde\Omega[\frac{XY}{M}]=\Omega[\frac{-\epsilon XY}{M}]$ is the reproducing kernel for the $*$-scalar product it follows that we must have

$$^{\frac{1}{x}}\mathbf{B}_a\tilde\Omega[\tfrac{XY}{M}] \;=\; {}^{\frac{1}{y}}\mathbf{B}_a^*\tilde\Omega[\tfrac{XY}{M}] \qquad 2.21$$

where we have prepended superscripts $\frac{1}{x}$ and $\frac{1}{y}$ to indicate on which alphabet the corresponding operator is supposed to act. Now 1.8 gives

$$\begin{aligned} ^{\frac{1}{x}}\mathbf{B}_a\tilde\Omega[\tfrac{XY}{M}] &= \tilde\Omega\Big[\tfrac{\big(X+\epsilon\frac{1-q}{z}\big)Y}{M}\Big]\,\Omega[-\epsilon zX]\Big|_{z^a} \\ &= \tilde\Omega[\tfrac{XY}{M}]\,\tilde\Omega\big[\tfrac{\epsilon Y}{z(1-t)}\big]\,\tilde\Omega[\tfrac{zXM}{M}]\Big|_{z^a} \end{aligned} \qquad 2.22$$

Since for any two formal power series $\Phi(u), \Psi(u)$ we have

$$\Phi(z)\Psi(\tfrac{1}{z})\Big|_{z^a} \;=\; \Phi(\tfrac{1}{z})\Psi(z)\Big|_{z^{-a}} \tag{2.23}$$

the identity in 2.22 may be rewritten as

$$\begin{aligned}
\tfrac{1}{x}\mathbf{B}_a\tilde\Omega[\tfrac{XY}{M}] &= \tilde\Omega[\tfrac{XY}{M}]\tilde\Omega\big[\tfrac{\epsilon zY}{(1-t)}\big]\tilde\Omega[\tfrac{XM/z}{M}]\Big|_{z^{-a}}\\
&= \tilde\Omega\Big[\frac{X\big(Y+\frac{M}{z}\big)}{M}\Big]\tilde\Omega\big[\tfrac{\epsilon zY}{(1-t)}\big]\Big|_{z^{-a}} \;=\; \tilde\Omega\Big[\frac{X\big(Y+\frac{M}{z}\big)}{M}\Big]\Omega\big[\tfrac{-zY}{(1-t)}\big]\Big|_{z^{-a}}.
\end{aligned}$$

Thus 2.19 follows from 2.21.

Similarly we must have

$$\tfrac{1}{x}\mathbf{C}_a\tilde\Omega[\tfrac{XY}{M}] \;=\; \tfrac{1}{y}\mathbf{C}^*_a\tilde\Omega[\tfrac{XY}{M}]\ . \tag{2.24}$$

Now I.14 gives

$$\begin{aligned}
(-q)^{a-1}\ \tfrac{1}{x}\mathbf{C}_a\tilde\Omega[\tfrac{XY}{M}] &= \tilde\Omega\Big[\frac{\big(X+\frac{1-q}{qz}\big)Y}{M}\Big]\Omega[zX]\Big|_{z^a}\\
&= \tilde\Omega[\tfrac{XY}{M}]\tilde\Omega\big[\tfrac{Y}{zq(1-t)}\big]\tilde\Omega[\tfrac{-\epsilon zXM}{M}]\Big|_{z^a}\\
\text{(using 1.17)}\quad &= \tilde\Omega[\tfrac{XY}{M}]\tilde\Omega\big[\tfrac{zY}{q(1-t)}\big]\tilde\Omega[\tfrac{-\epsilon XM/z}{M}]\Big|_{z^{-a}}\\
&= \tilde\Omega\Big[\frac{X\big(Y-\epsilon\frac{M}{z}\big)}{M}\Big]\tilde\Omega\big[\tfrac{zY}{q(1-t)}\big]\Big|_{z^{-a}} \;=\; \tilde\Omega\Big[\frac{X\big(Y-\epsilon\frac{M}{z}\big)}{M}\Big]\Omega\big[\tfrac{-\epsilon zY}{q(1-t)}\big]\Big|_{z^{-a}}\ .
\end{aligned}$$

Thus 2.20 follows from 2.24.

In view of 2.16 our next task is to compute the symmetric functions $\nabla h^*_r e^*_{n-r}$. The breakthrough is provided by the following simple result.

Proposition 2.4

For any $0 \le r \le n$ we have

$$h_r\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big] \;=\; \sum_{\mu\vdash n}\frac{\tilde H_\mu[X]}{w_\mu}\,e_r[B_\mu(q,t)]\ . \tag{2.25}$$

In particular we have

$$a)\quad e_n\big[\tfrac{X}{M}\big] \;=\; \sum_{\mu\vdash n}\frac{\tilde H_\mu[X]}{w_\mu} \qquad \textit{and} \qquad b)\quad h_n\big[\tfrac{X}{M}\big] \;=\; \sum_{\mu\vdash n}\frac{\tilde H_\mu[X]}{w_\mu}\,T_\mu \tag{2.26}$$

and more importantly

$$\nabla h_r\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big] \;=\; \Delta_{e_r}h_n[\tfrac{X}{M}] \tag{2.27}$$

with Δ_{e_r} the symmetric function operator defined by setting for all partitions μ

$$\Delta_{e_r}\tilde H_\mu[X;q,t] \;=\; e_r[B_\mu(q,t)]\ \tilde H_\mu[X;q,t]\ .$$

Proof

The $*$-duality of the Macdonald bases yields the expansion

$$\begin{aligned} h_r[\tfrac{X}{M}]e_{n-r}[\tfrac{X}{M}] &= \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]}{w_\mu} \big\langle \tilde H_\mu \, , \, h_r^* e_{n-r}^* \big\rangle_* \\ \text{(by 1.16)} &= \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]}{w_\mu} \big\langle \tilde H_\mu \, , \, e_r h_{n-r} \big\rangle \\ \text{(by 1.31)} &= \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]}{w_\mu}\, e_r[B_\mu(q,t)] \ . \end{aligned}$$

This proves 2.25. This given, 2.26 a) is the case $r = 0$ of 2.25 and 2.26 b) is the case $r = n$ since $e_n[B_\mu(q,t)] = T_\mu$ when $\mu \vdash n$. For 2.27 we simply note that 2.26 b) and 2.25 give

$$\Delta_{e_r} h_n[\tfrac{X}{M}] \;=\; \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]}{w_\mu} T_\mu\, e_r[B_\mu(q,t)] \;=\; \nabla h_r\big[\tfrac{X}{M}\big] e_{n-r}\big[\tfrac{X}{M}\big] \ .$$

The identity in 2.27 immediately yields us our second reduction in our path to Theorem I.1:

Proposition 2.5

The identity

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\, 1 \, , \, e_r h_{n-r}\big\rangle = \begin{cases} t^{p_1-1}q^{k-1}\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{B}_{p_1-1}\, 1 \, , \, e_{r-1}h_{n-r}\big\rangle & \textit{if } p_1>1 \\ q^{k-1}\big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{B}_{0}\, 1 \, , \, e_{r-1}h_{n-r}\big\rangle & \\ \qquad\qquad + \big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\, 1 \, , \, e_{r}h_{n-1-r}\big\rangle & \textit{if } p_1=1 \end{cases} \qquad 2.28$$

is valid for all compositions $p = (p_1, p_2, \ldots, p_k)$ *and* $1 \le r \le n$ *if and only if we have*

$$\mathbf{C}_a^* \Delta_{e_r} h_n[\tfrac{X}{M}] \;=\; \begin{cases} t^{a-1}\mathbf{B}_{a-1}^*\Delta_{e_{r-1}} h_{n-1}[\tfrac{X}{M}] & \text{if } a>1 \\ \mathbf{B}_0^*\Delta_{e_{r-1}} h_{n-1}[\tfrac{X}{M}] + \Delta_{e_r} h_{n-1}[\tfrac{X}{M}] & \text{if } a=1 \end{cases} \ . \qquad 2.29$$

Proof

The identity in 2.29 is simply 1.16 with multiple uses of 2.27 on the left on the right hand sides.

Now it happened that Glenn Tesler in [8] had derived plethystic formulas for all the operators Δ_{e_r} directly from the original Macdonald operators (see [8] Theorem 5.1). Forbidding as they appear in [8] these formulas should in principle yield a proof of 2.29 and thereby establish our result. Fortunately, computer explorations of the action Tesler's operators on h_n^*, made possible by the partial fraction constant term algorithm of [20], yielded a surprising discovery which led to a much simpler path to 2.29. We refer the reader to [11] for the methods and results that led to this discovery. Here we will directly obtain it from our tool kit. It may be stated as follows.

Theorem 2.4

The Δ'_r symmetric function operator defined by setting

$$\Delta'_r \tilde{H}_\mu[X;q,t) \;=\; e_r[B_\mu(q,t) - \tfrac{1}{M}]\tilde{H}_\mu[X;q,t] \tag{2.30}$$

acts on h^*_m according to the following identity

$$\Delta'_r h_m\big[\tfrac{X}{M}\big] \;=\; (-1)^r \sum_{\lambda\vdash r} \frac{T_\lambda h_m\big[X(\frac{1}{M} - B_\lambda)\big]}{w_\lambda}. \tag{2.31}$$

Proof

The definition in 2.30 applied to 2.26 b) gives

$$\Delta'_r\, h_m[\tfrac{X}{M}] \;=\; \sum_{\mu\vdash m} \frac{T_\mu \tilde{H}_\mu[X]}{w_\mu}\; e_r\big[B_\mu - \tfrac{1}{M}\big] \tag{2.32}$$

and using 2.26 a) we get

$$e_r\big[B_\mu - \tfrac{1}{M}\big] \;=\; e_r\Big[\tfrac{MB_\mu - 1}{M}\Big] \;=\; \sum_{\lambda\vdash r} \frac{1}{w_\lambda}\tilde{H}_\lambda[MB_\mu - 1].$$

Thus 2.32 becomes

$$\Delta'_r\, h_m[\tfrac{X}{M}] \;=\; \sum_{\mu\vdash m} \frac{T_\mu \tilde{H}_\mu[X]}{w_\mu} \sum_{\lambda\vdash r} \frac{1}{w_\lambda}\tilde{H}_\lambda[MB_\mu - 1] \;=\; \sum_{\lambda\vdash r}\frac{1}{w_\lambda}\sum_{\mu\vdash m}\frac{T_\mu \tilde{H}_\mu[X]\tilde{H}_\lambda[MB_\mu - 1]}{w_\mu}\ . \tag{2.33}$$

But now the reciprocity formula 1.26 gives

$$T_\mu \tilde{H}_\lambda[MB_\mu - 1] \;=\; (-1)^{m-r} T_\lambda\, \tilde{H}_\mu[MB_\lambda - 1] \tag{2.34}$$

and 2.33 becomes

$$\begin{aligned}
\Delta'_r\, h_m[\tfrac{X}{M}] \;&=\; (-1)^{m-r}\sum_{\lambda\vdash r}\frac{T_\lambda}{w_\lambda}\sum_{\mu\vdash m}\frac{\tilde{H}_\mu[X]\,\tilde{H}_\mu[MB_\lambda - 1]}{w_\mu}\\
(\text{ by } 1.22\text{ }) \;&=\; (-1)^{m-r}\sum_{\lambda\vdash r}\frac{T_\lambda}{w_\lambda}\;\; e_m\Big[\tfrac{X(MB_\lambda - 1)}{M}\Big]\\
(\text{ by } 1.7\text{ }) \;&=\; (-1)^{r}\sum_{\lambda\vdash r}\frac{T_\lambda}{w_\lambda}\;\; h_m\Big[-\tfrac{X(MB_\lambda - 1)}{M}\Big]\ .
\end{aligned} \tag{2.35}$$

This proves 2.31.

Our next and final reduction should reveal the significance of the identity in 2.31. .

Proposition 2.6

For $a \geq 1$ and $k \geq 1$ we have

$$\mathbf{C}^*_a \Delta'_k h_n[\tfrac{X}{M}] = \mathbf{B}^*_{a-1}\Delta'_{k-1}h_{n-1}[\tfrac{X}{M}] \,+\, \chi(a=1)\Delta'_k h_{n-1}[\tfrac{X}{M}] \tag{2.36}$$

if and only if

$$\mathbf{C}^*_a \Delta_{e_k} h_n[\tfrac{X}{M}] = \mathbf{B}^*_{a-1}\Delta_{e_{k-1}}h_{n-1}[\tfrac{X}{M}] \,+\, \chi(a=1)\Delta_{e_k} h_{n-1}[\tfrac{X}{M}]\ . \tag{2.37}$$

Proof

Note first that

$$e_k[B_\mu] \;=\; \sum_{i=0}^{k} e_{k-i}[B_\mu - \tfrac{1}{M}]e_i[\tfrac{1}{M}] \quad\Longrightarrow\quad \Delta_{e_k} \;=\; \sum_{i=0}^{k-1} \Delta'_{k-i}\, e_i[\tfrac{1}{M}] \,+\, e_k[\tfrac{1}{M}] \tag{2.38}$$

in particular we also have

$$\Delta_{e_{k-1}} \;=\; \sum_{i=0}^{k-1} \Delta'_{k-1-i}\, e_i[\tfrac{1}{M}]\;. \tag{2.39}$$

Now 2.38 gives

$$\begin{aligned}
\mathbf{C}^*_a\Delta_{e_k}h^*_n \;&=\; \sum_{i=0}^{k-1} e_i[\tfrac{1}{M}]\mathbf{C}^*_a\Delta'_{k-i}h^*_n \;+\; e_k[\tfrac{1}{M}]\mathbf{C}^*_a\, h^*_n \\
\text{(using 2.36)} \;&=\; \sum_{i=0}^{k-1} e_i[\tfrac{1}{M}]\Big(\mathbf{B}^*_{a-1}\Delta'_{k-1-i}h^*_{n-1} \,+\, \chi(a=1)\Delta'_{k-i}h_{n-1}[\tfrac{X}{M}]\Big) \;+\; e_k[\tfrac{1}{M}]\mathbf{C}^*_a\, h^*_n \\
&=\; \mathbf{B}^*_{a-1}\sum_{i=0}^{k-1} e_i[\tfrac{1}{M}]\Delta'_{k-1-i}h^*_{n-1} \,+\, \chi(a=1)\sum_{i=0}^{k-1} e_i[\tfrac{1}{M}]\Delta'_{k-i}h_{n-1}[\tfrac{X}{M}] \,+\, e_k[\tfrac{1}{M}]\mathbf{C}^*_a\, h^*_n\;.
\end{aligned} \tag{2.40}$$

However, from the definition in 2.20 we derive that

$$\begin{aligned}
\mathbf{C}^*_a h_n[\tfrac{X}{M}] \;&=\; (-\tfrac{1}{q})^{a-1}h_n\Big[\tfrac{X-\epsilon M/z}{M}\Big]\Omega\big[\tfrac{-\epsilon z X}{q(1-t)}\big]\Big|_{z^{-a}} \\
&=\; (-\tfrac{1}{q})^{a-1}h_n\Big[\tfrac{X}{M} - \epsilon/z\Big]\Omega\big[\tfrac{-\epsilon z X}{q(1-t)}\big]\Big|_{z^{-a}} \\
&=\; (-\tfrac{1}{q})^{a-1}\big(h_n[\tfrac{X}{M}] + h_{n-1}[\tfrac{X}{M}]\tfrac{1}{z}\big)\Omega\big[\tfrac{-\epsilon z X}{q(1-t)}\big]\Big|_{z^{-a}} \;=\; \chi(a=1)h_{n-1}[\tfrac{X}{M}]\;.
\end{aligned}$$

Thus 2.40 becomes

$$\begin{aligned}
\mathbf{C}^*_a\Delta_{e_k}h^*_n \;&=\; \mathbf{B}^*_{a-1}\sum_{i=0}^{k-1} e_i[\tfrac{1}{M}]\Delta'_{k-1-i}h^*_{n-1} \,+\, \chi(a=1)\sum_{i=0}^{k} e_i[\tfrac{1}{M}]\Delta'_{k-i}h_{n-1}[\tfrac{X}{M}] \\
\text{(by 2.39 and 2.38)} \;&=\; \mathbf{B}^*_{a-1}\Delta_{e_{k-1}}h^*_{n-1} + \chi(a=1)\Delta_{e_k}h^*_{n-1}\;.
\end{aligned}$$

This shows that 2.36 implies 2.37, which is the only side of this Proposition we will use. To show the reverse implication we start with the relations

$$e_k[B_\mu - \tfrac{1}{M}] = \sum_{i=0}^{k} e_{k-i}[B_\mu]e_i[-\tfrac{1}{M}] \quad\Longrightarrow\quad \Delta'_k = \sum_{i=0}^{k-1}\Delta_{e_{k-i}}\, e_i[-\tfrac{1}{M}] + e_k[-\tfrac{1}{M}]\,, \qquad \Delta'_{k-1} = \sum_{i=0}^{k-1}\Delta_{e_{k-1-i}}\, e_i[-\tfrac{1}{M}]$$

and carry out the same steps of in the preceding argument.

Given our three reductions (Propositions 2.3, 2.5 and 2.6), our next result completes proof of the identities in I.23 and I.24.

Theorem 2.5

For all $a \geq 1$ and $1 \leq r \leq n$ we have

$$\mathbf{C}^*_a \Delta'_r h_n[\tfrac{X}{M}] \;=\; t^{a-1}\mathbf{B}^*_{a-1}\Delta'_{r-1}h_{n-1}[\tfrac{X}{M}] \,+\, \chi(a=1)\Delta'_r h_{n-1}[\tfrac{X}{M}]\ . \qquad 2.41$$

Proof

In view of 2.31 we will use 2.19 to get

$$\begin{aligned}\mathbf{B}^*_{a-1}h_{n-1}\left[X(\tfrac{1}{M}-B_\mu)\right] &= h_{n-1}\left[(X+M/z)(\tfrac{1}{M}-B_\mu)\right]\Omega[-zX/(1-t)]\Big|_{z^{-a+1}}\\ &= \sum_{s=0}^{n-1} h_{n-1-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[M(\tfrac{1}{M}-B_\mu)\right]\tfrac{1}{z^s}\Omega[-zX/(1-t)]\Big|_{z^{-a+1}}\\ &= \sum_{s=0}^{n-1} h_{n-1-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[1-MB_\mu\right] h_{s-a+1}[-X/(1-t)]\ .\end{aligned}$$

Using this with 2.31 for $r \to r-1$ and $m \to n-1$ we may rewrite 2.41 as

$$\begin{aligned}&\mathbf{C}^*_a\Delta'_r h_n[\tfrac{X}{M}] - \chi(a=1)\Delta'_r h_{n-1}[\tfrac{X}{M}] \;=\\ &\quad= (-1)^{r-1}t^{a-1}\sum_{\mu\vdash r-1}\frac{T_\mu}{w_\mu}\sum_{s=0}^{n-1} h_{n-1-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[1-MB_\mu\right] h_{s-a+1}[-X/(1-t)]\\ &\quad= (-1)^{r-1}t^{a-1}\sum_{s=a-1}^{n-1} h_{s-a+1}[-X/(1-t)]\sum_{\mu\vdash r-1}\frac{T_\mu}{w_\mu} h_{n-1-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[1-MB_\mu\right]\ ,\end{aligned}$$

or better yet,

$$\begin{aligned}&\mathbf{C}^*_a\Delta'_r h_n[\tfrac{X}{M}] - \chi(a=1)\Delta'_r h_{n-1}[\tfrac{X}{M}] \;=\\ &\quad= (-1)^{r-1}t^{a-1}\sum_{s=a}^{n} h_{s-a}[-X/(1-t)]\sum_{\mu\vdash r-1}\frac{T_\mu}{w_\mu} h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_{s-1}\left[1-MB_\mu\right]\ .\end{aligned} \qquad 2.42$$

Our task is to prove this identity. Now 2.20 gives

$$\begin{aligned}(-q)^{a-1}\mathbf{C}^*_a h_n\left[X(\tfrac{1}{M}-B_\mu)\right] &= h_n\left[(X-\epsilon M/z)(\tfrac{1}{M}-B_\mu)\right]\Omega[-\epsilon zX/q(1-t)]\Big|_{z^{-a}}\\ &= \sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[-\epsilon M(\tfrac{1}{M}-B_\mu)\right]\tfrac{1}{z^s}\Omega[-\epsilon zX/q(1-t)]\Big|_{z^{-a}}\\ &= \sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right](-1)^s h_s\left[-1+MB_\mu\right](-1/q)^{s-a}h_{s-a}[-X/(1-t)]\ .\end{aligned}$$

Thus

$$\mathbf{C}^*_a h_n\left[X(\tfrac{1}{M}-B_\mu)\right] \;=\; (-q)\sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[-1+MB_\mu\right](1/q)^s h_{s-a}[-X/(1-t)]$$

and 2.31 for $m=n$ gives

$$\begin{aligned}
\mathbf{C}^*_a\Delta'_r h_n[\tfrac{X}{M}] &= (-q)(-1)^r\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}\sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[-1+MB_\mu\right](1/q)^s h_{s-a}[-X/(1-t)] \\
&= (-q)(-1)^r\sum_{s=a}^{n}(1/q)^s h_{s-a}[-X/(1-t)]\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] h_s\left[-1+MB_\mu\right] \ .
\end{aligned}$$

Now we use 1.37 in the form

$$h_s\left[-1+MB_\mu\right] \;=\; (tq)^{s-1}M\sum_{\nu\to\mu}c_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s-1} \;-\; \chi(s=1)$$

and obtain

$$\begin{aligned}
\mathbf{C}^*_a\Delta'_r h_n[\tfrac{X}{M}] &= (-1)^{r-1}\sum_{s=a}^{n}(1/q)^{s-1}h_{s-a}[-X/(1-t)]\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right]\Big((tq)^{s-1}M\sum_{\nu\to\mu}c_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s-1} \;-\; \chi(s=1)\Big) \\
&= (-1)^{r-1}\sum_{s=a}^{n}h_{s-a}[-X/(1-t)]\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right]t^{s-1}M\sum_{\nu\to\mu}c_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s-1}+ \\
&\qquad + \;\chi(a=1)(-1)^r\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}h_{n-1}\left[X(\tfrac{1}{M}-B_\mu)\right] \ .
\end{aligned}$$

Using 2.31 with $m=n-1$ this can be rewritten as

$$\begin{aligned}
&\mathbf{C}^*_a\Delta'_r h_n[\tfrac{X}{M}] \;-\; \chi(a=1)\Delta'_r h_{n-1}\big[\tfrac{X}{M}\big] \;= \\
&\qquad = (-1)^{r-1}\sum_{s=a}^{n}h_{s-a}[-X/(1-t)]\sum_{\mu\vdash r}\frac{T_\mu}{w_\mu}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right]t^{s-1}M\sum_{\nu\to\mu}c_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s-1} \\
&\qquad = (-1)^{r-1}\sum_{s=a}^{n}h_{s-a}[-X/(1-t)]t^{s-1}\sum_{\nu\vdash r-1}\frac{T_\nu}{w_\nu}\sum_{\mu\leftarrow\nu}Mc_{\mu\nu}\tfrac{w_\nu}{w_\mu}(\tfrac{T_\mu}{T_\nu})^{s}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] \\
&(\text{by } 1.39) = (-1)^{r-1}\sum_{s=a}^{n}h_{s-a}[-X/(1-t)]t^{s-1}\sum_{\nu\vdash r-1}\frac{T_\nu}{w_\nu}\sum_{\nu\leftarrow\mu}d_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] \ .
\end{aligned} \tag{2.43}$$

Next we split $B_\mu(q,t)$ into the sum $B_\mu(q,t)=B_\nu(q,t)+\frac{T_\mu}{T_\nu}$ to get

$$\begin{aligned}
\sum_{\mu\leftarrow\nu}d_{\mu\nu}(\tfrac{T_\mu}{T_\nu})^{s}h_{n-s}\left[X(\tfrac{1}{M}-B_\mu)\right] &= \sum_{u=0}^{n-s}h_{n-u-s}\left[X(\tfrac{1}{M}-B_\nu)\right]h_u[-X]\sum_{\mu\leftarrow\nu}d_{\mu\nu}\big(\tfrac{T_\mu}{T_\nu}\big)^{u+s} \\
(\text{by } 1.38) &= \sum_{u=0}^{n-s}h_{n-u-s}\left[X(\tfrac{1}{M}-B_\nu)\right]h_u[-X](-1)^{u+s-1}e_{u+s-1}\big[MB_\nu-1\big] \\
&= \sum_{v=s}^{n}h_{n-v}\left[X(\tfrac{1}{M}-B_\nu)\right]h_{v-s}[-X]h_{v-1}\big[1-MB_\nu\big] \ .
\end{aligned}$$

Using this in 2.43 gives

$$
\begin{aligned}
\mathbf{C}_a^*\Delta_r' h_n\big[\tfrac{X}{M}\big] - \chi(a=1)\Delta_r' h_{n-1}\big[\tfrac{X}{M}\big] \;\; &= \\
&= (-1)^{r-1}\sum_{s=a}^{n} h_{s-a}[-X/(1-t)]t^{s-1}\sum_{\nu\vdash r-1}\frac{T_\nu}{w_\nu}\sum_{v=s}^{n} h_{n-v}\big[X(\tfrac{1}{M}-B_\nu)\big]\, h_{v-s}[-X]h_{v-1}\big[1-MB_\nu\big] \\
&= (-1)^{r-1}t^{a-1}\sum_{v=a}^{n}\sum_{\nu\vdash r-1}\frac{T_\nu}{w_\nu} h_{n-v}\big[X(\tfrac{1}{M}-B_\nu)\big]\, h_{v-1}\big[1-MB_\nu\big]\sum_{s=a}^{v} h_{s-a}[-tX/(1-t)]h_{v-s}[-X] \\
&= (-1)^{r-1}t^{a-1}\sum_{v=a}^{n} h_{v-a}[-X/(1-t)]\sum_{\nu\vdash r-1}\frac{T_\nu}{w_\nu} h_{n-v}\big[X(\tfrac{1}{M}-B_\nu)\big]\, h_{v-1}\big[1-MB_\nu\big]
\end{aligned}
$$

and now we can clearly see, that except for some trivial notational changes, this is precisely the desired identity in 2.42. Thus 2.41 is valid precisely as asserted.

Our proof of Theorem I.1 is thus complete.

Note that setting $r=n$ in the lower equality in I.17 yields

$$\big\langle \nabla \mathbf{C}_1\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1, e_n\big\rangle \;=\; q^{k-1}\big\langle \nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}\mathbf{B}_0 1, e_{n-1}\big\rangle + \big\langle \nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1, e_n h_{-1}\big\rangle\,. \qquad 2.44$$

Since we can only interpret h_{-1} as zero, the second term must identically vanish and we see that Corollary I.1 is indeed a consequence of Theorem I.1. However, the vanishing of the second term in 2.44 can be made more explicit by a direct argument based on the identity in 2.37, which (by Proposition 2.6) is now a consequence of Theorem 2.5. To see this note that, passing to the $*$-scalar product, we can rewrite the left-hand side of 2.44 in the form

$$\big\langle \nabla \mathbf{C}_1\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1, e_n\big\rangle \;=\; \big\langle 1\,,\, \mathbf{C}_{p_k}^*\cdots\mathbf{C}_{p_2}^*\mathbf{C}_1^*\nabla h_n^*\big\rangle_* \;=\; \big\langle 1\,,\, \mathbf{C}_{p_k}^*\cdots\mathbf{C}_{p_2}^*\mathbf{C}_1^*\Delta_{e_n} h_n^*\big\rangle_* \qquad 2.45$$

but 2.37 for $a=1$ specializes to

$$\mathbf{C}_1^*\Delta_{e_n}h_n^* \;=\; \mathbf{B}_0^*\Delta_{e_{n-1}}h_{n-1}^* + \Delta_{e_n}h_{n-1}^*\,. \qquad 2.46$$

However, since all the eigenvalues $e_n[B_\nu]$ vanish identically for all $\nu\vdash n-1$ the second term in 2.46 will necessarily vanish giving

$$\mathbf{C}_1^*\Delta_{e_n}h_n^* \;=\; \mathbf{B}_0^*\Delta_{e_{n-1}}h_{n-1}^*\,.$$

Using this 2.45 becomes

$$
\begin{aligned}
\big\langle \nabla \mathbf{C}_1\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1, e_n\big\rangle &= \big\langle 1\,,\, \mathbf{C}_{p_k}^*\cdots\mathbf{C}_{p_2}^*\mathbf{B}_0^*\Delta_{e_{n-1}}h_{n-1}^*\big\rangle_* \\
&= \big\langle \Delta_{e_{n-1}}\mathbf{B}_0\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_s}1\,,\, h_{n-1}^*\big\rangle_* \\
\text{(by 2.14)}\quad &= q^{k-1}\big\langle \Delta_{e_{n-1}}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_s}\mathbf{B}_0 1\,,\, h_{n-1}^*\big\rangle_* \\
&= q^{k-1}\big\langle \Delta_{e_{n-1}}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_s}\mathbf{B}_0 1\,,\, e_{n-1}\big\rangle\,.
\end{aligned}
$$

Since the definition in 1.8 gives $\mathbf{B}_0 1=1$ and $\Delta_{e_{n-1}}=\nabla$ on homogeneous polynomials of degree n we obtain

$$\big\langle \nabla \mathbf{C}_1\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1, e_n\big\rangle \;=\; q^{k-1}\big\langle \nabla\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1\,,\, e_{n-1}\big\rangle \qquad 2.47$$

as desired.

The next identities convert Theorem I.1 into a recursion.

Proposition 2.7

For all compositions $p=(p_1,p_2,\ldots,p_k)\models n$ *we have for* $1\le r\le n$

$$\Big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_rh_{n-r}\Big\rangle \;=\; t^{p_1-1}q^{k-1}\sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{l(\alpha)})\models p_1-1}\Big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\mathbf{C}_{\alpha_1}\mathbf{C}_{\alpha_2}\cdots \mathbf{C}_{\alpha_{l(\alpha)}}\,1\,,\,e_{r-1}h_{n-r}\Big\rangle \qquad (\textit{for } p_1>1) \tag{2.48}$$

and for $p_1=1$

$$\Big\langle \nabla \mathbf{C}_{1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_rh_{n-r}\Big\rangle = q^{k-1}\Big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_{r-1}h_{n-r}\Big\rangle + \chi(r<n)\Big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_{r}h_{n-1-r}\Big\rangle\ . \tag{2.49}$$

Proof

Note that the definition in 1.8 for $a=p_1-1$ gives

$$\mathbf{B}_{p_1-1}\,1 \;=\; \Omega[-\epsilon zX]\Big|_{z^{p_1-1}}=e_{p_1-1}\ .$$

Thus the first equality in I.17 becomes

$$\Big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_rh_{n-r}\Big\rangle = t^{p_1-1}q^{k-1}\Big\langle \nabla \mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}e_{p_1-1}\,,\,e_{r-1}h_{n-r}\Big\rangle$$

and 2.48 then follows from the identity in 2.10. On the other hand 2.49 is exactly I.17, taking account of 2.47 and the fact that $\mathbf{B}_0 1=1$.

We are now finally in a position to interpret our polynomials as weighted sums over collections of parking functions. To this end it will be convenient to introduce the same notation as in the work of Angela Hicks. For a given $1\le r\le n$ let $Sh_{r,n}$ denote the family of parking functions with diagonal permutation a shuffle of $12\cdots n-r$ with $n(n-1)\cdots(n-r+1)$ and set for a composition $p=(p_1,p_2,\ldots,p_k)\models n$

$$S^p_{r,n}(q,t) \;=\; \sum_{PF\in Sh_{r,n}} t^{area(PF)}q^{dinv(PF)}\chi(p(PF)=p)\ . \tag{2.50}$$

Now in [18] Angela Hicks gives a bijective proof for each of the following two recursions .

Proposition 2.8 (Angela Hicks)

For all compositions $p=(p_1,p_2,\ldots,p_k)$ *and for all* $p_1>1$ *and* $1\le r\le n$ *we have*

$$S^{(p_1,p_2,p_3,\ldots,p_k)}_{r,n}(q,t) \;=\; t^{p_1-1}q^{k-1}\sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{l(\alpha)})\models p_1-1} S^{(p_2,p_3,\ldots,p_k,\alpha_1,\alpha_2,\ldots,\alpha_{l(\alpha)})}_{r-1,n-1}(q,t)\ . \tag{2.51}$$

Moreover, for $p_1=1$ *we have*

$$S^{(1,p_2,p_3,\ldots,p_k)}_{r,n}(q,t) \;=\; q^{k-1}S^{(p_2,p_3,\ldots,p_k)}_{r-1,n-1}(q,t) + \chi(r<n)S^{(p_2,p_3,\ldots,p_k)}_{r,n-1}(q,t)\ . \tag{2.52}$$

Thus it follows from these two propositions that the polynomials $S^p_{r,n}(q,t)$ and the polynomials

$$\Pi^p_{r,n}(q,t) \;=\; \Big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}\,1\,,\,e_rh_{n-r}\Big\rangle$$

satisfy the same recursions. This given, to show the equality

$$S^p_{r,n} \;=\; \Pi^p_{r,n} \qquad\qquad (\text{for all } 0 \le r \le n \text{ and } p \models n) \tag{2.53}$$

we only need to verify this equality for all the base cases.

To obtain the monomial expansion of a polynomial $\Pi^p_{r,n}$, we can visualize the successive applications of our recursions as the construction of a tree with root labeled $\Pi^p_{r,n}$. Starting from the root, the offspring of a node labeled by a polynomial $\Pi^{p'}_{r',n'}$ are nodes labeled by the terms on the right hand side of its recursion, with the branches emanating from that node labelled by the corresponding monomials $t^{p_1-1}q^{k-1}$. For our recursion the leaves of the resulting tree are the nodes whose label evaluates to 1 or 0. Those are our base cases and we only need to identify them. To this end note that at each step of the recursion either r or $n-r$ decreases by one, thus before $n-2$ steps either r or $n-r$ are reduced to 0, We claim that a node indexed by the polynomial

$$\Pi^p_{0,n}(q,t) \;=\; \big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}1 \,,\, h_n \big\rangle \tag{2.54}$$

is a leaf. In fact, passing to the $*$-scalar product, and using the relations in 2.26, 2.54 may be rewritten in the form

$$\Pi^p_{0,n}(q,t) \;=\; \big\langle 1 \,,\, \mathbf{C}^*_{p_k}\cdots \mathbf{C}^*_{p_2}\mathbf{C}^*_{p_1}\nabla e^*_n \big\rangle_* \;=\; \big\langle 1 \,,\, \mathbf{C}^*_{p_k}\cdots \mathbf{C}^*_{p_2}\mathbf{C}^*_{p_1} h^*_n \big\rangle_* \,. \tag{2.55}$$

Now we have seen in the proof of Proposition 2.6 that

$$\mathbf{C}^*_a h^*_n \;=\; \chi(a=1) h^*_{n-1} \tag{2.63}$$

thus successive uses of this relation yield that $\Pi^p_{0,n}(q,t)$ is 1 or 0 according as all the components of $p = (p_1, p_2, \ldots, p_k)$ are equal to 1 or not. This proves our assertion.

Let us next examine the polynomial

$$\Pi^p_{n,n}(q,t) \;=\; \big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k}1 \,,\, e_n \big\rangle \tag{2.57}$$

and note that for $p_1 > 1$ the recursion in 2.48 shows that every node labelled by $\Pi^p_{n,n}(q,t)$ has children labeled by polynomials $\Pi^{p'}_{n-1,n-1}(q,t)$ for various $p' \models n-1$. Since at each iteration parts are replaced by lesser parts, all the paths in the subtree emanating from a node labelled by $\Pi^{p'}_{n-1,n-1}(q,t)$ will eventually lead to a node labelled by $\Pi^{p'}_{n',n'}(q,t)$ with $p' = (1, p'_2, \ldots, p'_s)$. Since the identity in 2.47 gives that

$$\Pi^{(1,p'_2,\ldots,p'_s)}_{n',n'}(q,t) \;=\; q^{s-1}\Pi^{(p'_2,\ldots,p'_s)}_{n'-1,n'-1}(q,t)$$

we see that all the paths descending from a node labelled by $\Pi^p_{n,n}$ will eventually lead to a node labelled by $\Pi^{(1)}_{1,1}$ which trivially evaluates to 1 and therefore it is a leaf.

The same reasoning applied to the tree corresponding to the recursion satisfied by the polynomial $S^p_{r,n}(q,t)$ yields that every path of this tree will eventually lead to a node labelled by $S^p_{0,n}$ or a node labeled by $S^1_{1,1}$. Now we can trivially see from 2.50 that $S^1_{1,1} = 1$. On the other hand for $r = 0$ the right hand side of 2.50 reduces to a sum over parking functions PF with $\sigma(PF) = 12\cdots n$. However, by the column increasing condition, there are no parking functions with this diagonal permutation and composition p if any component of p is greater than 1 and the sum reduces to 0 in this case. But if all the components of p are equal to 1. Thus 2.50 reduces to a single summand obtained by placing $12\cdots n$ on the main diagonal and from right to left. Since such arrangement creates no dinv we see that $S^p_{0,n}(q,t)$ evaluates to 1 in this case. This verifies the equality in 2.53 for all the base cases and completes the proof of Theorem I.2.

We terminate by adding a few words to justify the last statement at the end of Theorem I.2. The easiest way to do this is to use the fact that, for any Dyck path U, the quasi-symmetric polynomial

$$F_U(X;q,t) \;=\; \sum_{U(PF)=U} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}(X)$$

is also symmetric. Thus it follows from Gessel's theorem [12] that for $1\le r\le n-1$ we have

$$\sum_{U(PF)=U} t^{area(PF)} q^{dinv(PF)} \chi(ides(PF)=\{1,2,\cdots,r\}) \;=\; \big\langle F_U \, , \, Z_{\{1,2,\cdots,r\}}\big\rangle.$$

Since we may write

$$e_r h_{n-r} \;=\; Z_{\{1,2,\cdots,r-1\}} + Z_{\{1,2,\cdots,r\}} \;=\; Z_{\{n-r+1,2,\cdots,n-1\}} + Z_{\{n-r,2,\cdots,n-1\}}$$

we necessarily have the equality

$$\sum_{U(PF)=U} t^{area(PF)} q^{dinv(PF)} \chi(ides(PF)=\{1,2,\cdots,r-1\} \;or\; \{1,2,\cdots,r\}) \;=$$
$$= \sum_{U(PF)=U} t^{area(PF)} q^{dinv(PF)} \chi(ides(PF)=\{n-r+1,2,\cdots,n-1\} \;or\; \{n-r,2,\cdots,n-1\}) \;.$$

This given, the equality asserted the end of Theorem I.2 is simply obtained by summing these equalities for all Dyck paths of diagonal composition p.

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