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On the S_n module DH_n of Diagonal Harmonics

by

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Abstract

It can be shown there are differential operators E, F and $H = [E, F]$ for each $n \geq 1$, acting on Diagonal Harmonics, yielding that DH_n is a representation of $sl[2]$. Our main effort here is to use $sl[2]$ theory to predict a basis for the Diagonal Harmonic Alternants, DHA_n. It can be shown the irreducible representations $sl[2]$ are all of the form $P, EP, E^2P, \cdots, E^kP$, with $FP = 0$ and $E^{k+1}P = 0$. The polynomial P is called a "String Starter". From $sl[2]$ theory it follows that DHA_n is a direct sum of strings. Our main result so far is a formula for the number of string starters. A recent paper by Carlsson and Oblomkov constructs a basis for the space of Diagonal Coinvariants by Algebraic Geometrical tools. It would be interesting to see if any our result can be derived from theirs.

Introduction

We set $X_n = x_1, x_2, \ldots, x_n$ and $Y_n = y_1, y_2, \ldots, y_n$, we will be working here with polynomials $P(X_n; Y_n)$ with rational coefficients, that is $P(X_n; Y_n) \in \mathbf{Q}[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$.

The diagonal action of S_n is defined by setting for any $\sigma \in S_n$

$$\sigma\, P(X_n; Y_n) \,=\, P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n})\,. \qquad \text{I.1}$$

Another important tool in studying S_n modules that are invariant under the diagonal action is the scalar product

$$\big\langle P\,,\, Q \big\rangle \,=\, L_o P(\partial X_n; \partial Y_n) P(X_n; Y_n). \qquad \text{I.2}$$

where the differential operator $P(\partial X_n; \partial Y_n)$ is obtained by the replacements $x_i \to \partial_{x_i}$ and $y_i \to \partial_{y_i}$. It is easy to see that we have

$$\big\langle \sigma P\,,\, \sigma Q \big\rangle \,=\, \big\langle P\,,\, Q \big\rangle \qquad (\text{for all } \sigma \in S_n\ ). \qquad \text{I.3}$$

In this paper we will study the S_n module DH_n of Diagonal Harmonic polynomials. This module was originally defined as the orthogonal complement, with respect to the scalar product in I.2, of the ideal of polynomials that are invariant under the diagonal action. By a result of Hermann Weyl it follows that $P(X_n; Y_n) \in DH_n$ if and only if

$$\sum_{i=1}^{n} \partial_{x_i}^{p} \partial_{y_i}^{q}\, P(X_n; Y_n) \,=\, 0 \qquad (\text{for all } p + q \geq 1)\,. \qquad \text{I.4}$$

This simpler definition makes it obvious that DH_n is invariant under the diagonal action.

It also immediately follows from I.4 that if $P \in DH_n$ then all the bi-homgeneous components of P are in DH_n. This implies that we have the direct sum decomposition

$$DH_n \,=\, \bigoplus_{0 \leq r+s \leq \binom{n}{2}} \mathcal{H}_{r,s}\big(DH_n\big) \qquad \text{I.5}$$

where $\mathcal{H}_{r,s}(DH_n)$ is the subspace of diagonal Harmonics polynomials which are bi-homogeneous of degree r in the $x's$ and degree s in the $y's$. It then follows that the character resulting from the diagonal action of S_n on DH_n can be written in the form

$$\chi^{DH_n} \,=\, \sum_{0 \leq r+s \leq \binom{n}{2}} t^r q^s \chi^{r,s}, \qquad \big(\text{where } \chi^{r,s} \text{ is the character of } \mathcal{H}_{r,s}(DH_n)\big). \qquad \text{I.6}$$

The upper bound $\binom{n}{2}$ in I.5 follows from the operator conjecture (proved by Mark Haiman in [X]) that DH_n is the linear span of derivatives of the Vandermonde $\prod_{1 \leq i < j \leq n}(x_i - x_j)$.

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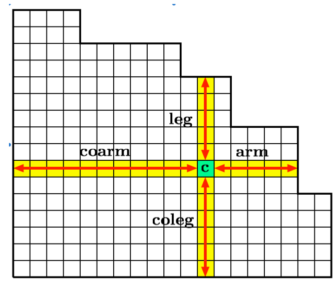
The Frobenius map "$\mathcal{F}$ " considerably simplifies the operation of computing the character of an S_n representation. Frobenius uses the dimension equality between the class functions of S_n and the space $\Lambda^{=n}$ of homogeneous symmetric functions of degree n. This given, $\mathcal{F}$ maps Class Functions onto the power basis by the formula

$$\mathcal{F} C_\mu \;=\; p_\mu / z_\mu . \tag{I.7}$$

where C_μ is the sum of all the permutations of cycle structure μ, $p_\mu = p_1 p_2 \cdots p_{l(\mu)}$, where $l(\mu)$ denotes the length of μ and

$$z_\mu \;=\; 1^{a_1} 2^{a_2} \cdots n^{a_n} a_1! a_2! \cdots a_n! \qquad\qquad (\text{when } \mu = 1^{a_1} 2^{a_2} \cdots n^{a_n} \vdash n) \tag{I.8}$$



It follows from I.7 that the character χ^λ is given by the identity

$$\chi^\lambda \;=\; \mathcal{F}^{-1} s_\lambda \tag{I.9}$$

If $\mathcal{M}_n$ is an S_n module with character $\chi^{\mathcal{M}_n}$ then the *Frobenius Characteristic* of $\mathcal{M}_n$ is the symmetric polynomial $\mathcal{F}\chi^{\mathcal{M}_n}$.

It was shown by Mark Haiman using tools of Algebraic Geometry that the Frobenius characteristic of DH_n is the symmetric rational function

$$\mathcal{F}\chi^{DH_n} \;=\; \sum_{\mu\vdash n} \frac{T_\mu \widetilde{H}_\mu(X;q,t) M B_\mu(q,t) \Pi_\mu(q,t)}{w_\mu(q,t)} . \tag{I.10}$$

To define the ingredients that appear in this formula it is convenient to use Macdonald's notation yet identify partitions with their French Ferrers diagram. Given a partition μ and a cell $c \in \mu$, as indicated in the above display, we will introduce four parameters $l_\mu(c)$, $l'_\mu(c)$, $a_\mu(c)$ and $a'_\mu(c)$ called *leg, coleg, arm* and *coarm* which give the number of lattice cells of μ strictly North, South, East and West of c. Denoting by μ' the conjugate of μ, we set

$$\begin{gathered} n(\mu) \;=\; \sum_{c\in\mu} l'_\mu(c), \qquad T_\mu = t^{n(\mu)} q^{n(\mu')}, \qquad M = (1-t)(1-q), \qquad B_\mu(q,t) = \sum_{c\in\mu} t^{l'_\mu(c)} q^{a'_\mu(c)}, \\ \Pi_\mu(q,t) = \prod_{c\in\mu; c\neq(0,0)} (1 - l'_\mu(c) q^{a'_\mu(c)}), \qquad w_\mu(q,t) = \prod_{c\in\mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}). \end{gathered} \tag{I.11}$$

This accounts for every thing that occurs in I.10 except for the modified Macdonald Basis element $\widetilde{H}_\mu(X;q,t)$. This symmetric polynomial was conjectured in [X] and proved by Mark Haiman in [X] to be the Frobenius Characteristic of the linear span of derivatives of the alternant that corresponds to the partition μ.

Our first goal is to construct operators that preserve DH_n. We will prove this property for all the differential operators

$$a) \quad F_{r,s} \;=\; \sum_{i=1}^{n} \underline{x}_i \partial_{x_i}^r \partial_{y_i}^s \,, \qquad\qquad b) \quad E_{r,s} \;=\; \sum_{i=1}^{n} y_i \partial_{x_i}^r \partial_{y_i}^s \,, \qquad\qquad (\text{for } r+s \geq 1 \text{ }) \,. \tag{I.12}$$

We will also prove the relations

$$\begin{aligned} a) \quad & \big[F_{p,q} \,,\, F_{r,s}\big] \;=\; (p-r) F_{p+r-1,q+s} \,, \\ b) \quad & \big[F_{p,q} \,,\, E_{r,s}\big] \;=\; q F_{p+r,q+s-1} \;-\; r E_{p+r-1,q+s} \,, \\ c) \quad & \big[E_{p,q} \,,\, E_{r,s}\big] \;=\; (q-s) E_{p+r,q+s-1} \,. \end{aligned} \tag{I.13}$$

Furthermore by setting

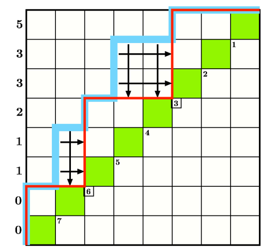
$$a) \quad F \;=\; F_{0,1} \,, \qquad\qquad b) \quad E \;=\; E_{1,0} \,, \qquad\qquad c) \quad H \;=\; [E,F] \;=\; \sum_{i=1}^{n} \big(\underline{y}_i \partial_{y_i} - \underline{x}_i \partial_{x_i}\big) \,. \tag{I.14}$$

we derive that DH_n is a direct sum of irreducible representations of $sl[2]$.

Finally, using the results of the q,t-Catalan paper we derive that the Hilbert polynomial

$$c_n(q,t) \;=\; \mathcal{F}\,\chi^{DHA_n}\Big|_{s[1^n]} \tag{I.15}$$

of the Diagonal Alternants can be obtained by a purely combinatorial construction. To describe this construction we need further definitions. By a Dyck path in the $n \times n$ lattice square $\mathcal{L}_n$ we mean a path which proceeds by n unit North steps and n unit East steps and goes from $(0,0)$ to (n,n) always remaining weakly above the main diagonal of $\mathcal{L}_n$. This is the straight line that joins $(0,0)$ to (n,n). In the illustration on the right we colored light green all the cells bisected by the main diagonal of $\mathcal{L}_8$. The Dyck path is depicted there in light blue. It is clear that we only need to give the abscissas of the of the North steps of the Dyck path D. Those are the integers that are on the left of the rows of $\mathcal{L}_8$. Thus $D = [0,0,1,1,2,3,3,5]$. Each $Dyck$ path has two statistics which we call $area(D)$ and $bounce(D)$. The area statistic is quite simple, its formula is $area(D) = [0,1,\ldots,n-1] - [0,d_1,\ldots,d_{n-1}]$. This is the number of cells between the Dyck path and the lattice diagonal (the green cells). In our case $area(D) = 13$. The bounce statistic is the sum of the places where the bounce path hits the main diagonal of $\mathcal{L}_n$. In our display we depicted it in red using a thinner line. In the general case it starts straight North until it touches the West end of an East step. Then it goes straight East until it touches the diagonal. Then goes straight North until it touches the West end of an East step... alternating straight North and straight East until it reaches (n,n). We place in the possible diagonal touching points the labels $1,2,\ldots,n-1$ as indicated in our display. The *bounce* statistic is the sum of the labels where the bouncing path touches the diagonal. In our example, $bounce(D) = 3 + 6$. We must emphasize that the bounce path does not change direction by touching the East end of an East step. In our display that happens in the 5^{th} and 8^{th} rows.



In particular we obtain

$$c_n(q,t) \;=\; \sum_{D \in \mathcal{D}_n} t^{bounce(D)} q^{area(D)} \tag{I.16}$$

This identity was conjectured by Jim Haglund in [X]. The problem to construct a further statistic that combined with area gives I.16 was formulated in [X]. For $n = 3$ we get

$$\mathcal{F}DH_3\Big|_{s[1^3]} \;=\; \sum_{D \in \mathcal{D}_3} t^{bounce(D)} q^{area(D)} \;=\; t^3 \;+\; t^2 q \;+\; tq \;+\; tq^2 \;+\; q^3. \tag{I.17}$$

This identity reveals that the alternating character $\chi^{[1,1,1]}$ occurs in bi-degrees $(3,0),(2,1),(1,2),(0,3)$ and $(1,1)$. In this particular case these are Frobenius images of the $sl[2]$ strings generated by the following two alternants

$$a)\quad \Delta_{1,1,1} \;=\; \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}, \qquad\qquad b)\quad \Delta_{2,1} \;=\; \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}. \tag{I.18}$$

It stands to reason that we should be able to construct a basis for the Diagonal Harmonics Alternants using the operators in I.12, $sl[2]$ theory and the combinatorics of the q,t-Catalan.

The computer data we have obtained so far and the Mark Haiman's proof of the Operator Conjecture make it very likely that the following is true.

Conjecture I.1

For every $n \geq 1$ and every bi-degree (a,b) for $a \geq b$ it is always possible to construct m sequences $r_1 \leq r_2 \leq \cdots \leq r_b$ such that the Diagonal Harmonic Alternants $E_{r_1,0}E_{r_2,0}\cdots E_{r_b,0}\Delta_{1^n}$ are distinct, where m is the number of $sl[2]$ strings that start at (a,b). This also requires that $\binom{n}{2} - (r_1 + r_2 + \cdots + r_b) = a$

1. The differential operators.

We start with an auxiliary fact concerning the interaction between multiplication operators and differential operators.

Propositiom 1.1

For any variable y and integer exponent $q \geq 1$ we have

$$\partial_y^q \underline{y} = q\,\partial_y^{q-1} + \underline{y}\,\partial_y^q. \tag{1.1}$$

Proof

Suppose that $P(y)$ is a polynomial in y. Then for $q=1$ we get

$$\partial_y\, y\, P(y) = P(y) + y\,\partial_y P(y). \tag{1.2}$$

Thus 1.1 is true for $q=1$. Proceeding by induction on q, suppose that 1.1 is true up to $q-1$. Then we have ,

$$\begin{aligned}\partial_y^q\, y\, P(y) &= \partial_y\,\partial_y^{q-1} y\, P(y) = \partial_y\,(q-1)\partial_y^{q-2}P(y) + \partial_y\, y\,\partial_y^{q-1}P(y) = \\ &= (q-1)\partial_y^{q-1}P(y) + \partial_y^{q-1}P(y) + y\,\partial_y^q P(y) = q\,\partial_y^{q-1}P(y) + y\,\partial_y^q P(y).\end{aligned} \tag{1.3}$$

this proves 1.1.

As an example we will show that

Theorem 1.1

The $sl[2]$ operators

$$a)\quad F = \sum_{i=1}^n \underline{x}_i \partial_{y_i}\,, \qquad\qquad b)\quad E = \sum_{i=1}^n \underline{y}_i \partial_{x_i}\,, \tag{1.4}$$

preserve $DH_n[X_n;Y_n]$.

Proof

To this end we will first compute the bracket

$$\big[\Pi^n_{p,q}\,,\, E\big] = \sum_{i=1}^n\sum_{j=1}^n \big[\partial^p_{x_i}\partial^q_{y_i}\,,\, \underline{y}_j\partial_{x_j}\big]\,, \qquad \Big(\text{where } \Pi^n_{p,q} = \sum_{i=1}^n \partial^p_{x_i}\partial^q_{y_i}\;\Big). \tag{1.5}$$

Since for $j \neq i$ the differential and multiplication operators commute, we only need to work with

$$\big[\Pi^n_{p,q}\,,\, E\big]\Big|_{j=i} = \sum_{i=1}^n \big[\partial^p_{x_i}\partial^q_{y_i}\,,\, \underline{y}_i\partial_{x_i}\big]\,,. \tag{1.6}$$

Using 1.1 for $q \geq 1$ we obtain

$$\partial^p_{x_i}\partial^q_{y_i}\underline{y}_i\partial_{x_i} = \partial^p_{x_i}\big(q\,\partial^{q-1}_{y_i} + \underline{y}_i\,\partial^q_{y_i}\big)\partial_{x_i} = q\,\partial^{p+1}_{x_i}\partial^{q-1}_{y_i} + \underline{y}_i\partial^{p+1}_{x_i}\partial^q_{y_i}. \tag{1.7}$$

We also have

$$\underline{y}_i\partial_{x_i}\partial^p_{x_i}\partial^q_{y_i} = \underline{y}_i\,\partial^{p+1}_{x_i}\partial^q_{y_i} \tag{1.8}$$

so 1.6 becomes

$$\big[\Pi^n_{p,q}\,,\, E\big] = q\sum_{i=1}^n \partial^{p+1}_{x_i}\partial^{q-1}_{y_i} + \sum_{i=1}^n \underline{y}_i\partial^{p+1}_{x_i}\partial^q_{y_i} - \sum_{i=1}^n \underline{y}_i\,\partial^{p+1}_{x_i}\partial^q_{y_i} = q\sum_{i=1}^n \partial^{p+1}_{x_i}\partial^{q-1}_{y_i}\,, \tag{1.9}$$

or equivalently

$$\Pi^n_{p,q}\,E = E\,\Pi^n_{p,q} + q\,\Pi^n_{p+1,q-1}. \tag{1.10}$$

Thus applying $\Pi^n_{p,q}\,E$ to a diagonal harmonic polynomial $P(X_n;Y_n)$, gives

$$\Pi^n_{p,q}\,EP(X_n;Y_n) = E\,\Pi^n_{p,q}P(X_n;Y_n) + q\,\Pi^n_{p+1,q-1}P(X_n;Y_n) = 0. \tag{1.11}$$

The case $q=0$ is trivial. Proving that the operator E preserves diagonal harmonics. Working with F we will reach the same result using analogous steps.

We can use the same idea on the operators

$$a) \quad F_{r,s} \;=\; \sum_{i=1}^n \underline{x}_i \partial_{x_i}^r \partial_{y_i}^s\,, \qquad\qquad b) \quad E_{r,s} \;=\; \sum_{i=1}^n y_i \partial_{x_i}^r \partial_{y_i}^s\,. \qquad 1.12$$

As before we can reduce the calculation to $j = i$ and work with

$$\Pi^n_{p,q} F_{r,s}\Big|_{j=i} \;=\; \sum_{i=1}^n \partial_{x_i}^p \underline{x}_i \partial_{x_i}^r \partial_{y_i}^{q+s} \;=\; \sum_{i=1}^n \big(p\,\partial_{x_i}^{p+r-1}\partial_{y_i}^{q+s} \,+\, \underline{x}_i \partial_{x_i}^{p+r}\partial_{y_i}^{q+s}\big). \qquad 1.13$$

Likewise we have

$$F_{r,s}\Pi^n_{p,q}\Big|_{j=i} \;=\; \sum_{i=1}^n \underline{x}_i \partial_{x_i}^{p+r}\partial_{y_i}^{q+s}\,. \qquad 1.14$$

Thus

$$\Pi^n_{p,q}\, F_{r,s} \;=\; F_{r,s}\Pi^n_{p,q} \;+\; p\,\Pi^n_{p+r-1,q+s} \qquad 1.15$$

Now if $P(X_n;Y_n)$ is Diagonal Harmonic then

$$\Pi^n_{p,q}\, F_{r,s} P(X_n;Y_n) \;=\; F_{r,s}\Pi^n_{p,q} P(X_n;Y_n) \;+\; p\,\Pi^n_{p+r-1,q+s} P(X_n;Y_n) \;=\; 0\,. \qquad 1.16$$

Proving that $F_{r,s}$ preserves Diagonal Harmonics. An analogous argument yields the same result for $E_{r,s}$.

Theorem 1.2

The following identities hold true for all $p + q \geq 1$ *and* $r + s \geq 1$.

$$\begin{aligned} a) \quad & \big[F_{p,q}\;,\;F_{r,s}\big] \;=\; (p-r)F_{p+r-1,q+s}\,, \\ b) \quad & \big[F_{p,q}\;,\;E_{r,s}\big] \;=\; qF_{p+r,q+s-1} \;-\; rE_{p+r-1,q+s}\,, \\ c) \quad & \big[E_{p,q}\;,\;E_{r,s}\big] \;=\; (q-s)E_{p+r,q+s-1}\,. \end{aligned} \qquad 1.17$$

Proof

Reducing again to the case $j = i$ we can write

$$\big[F_{p,q}\;,\;F_{r,s}\big] \;=\; \sum_{i=1}^n \big[\underline{x}_i \partial_{x_i}^p \partial_{y_i}^q \;,\; \underline{x}_i \partial_{x_i}^r \partial_{y_i}^s\big] \;=\; \sum_{i=1}^n \underline{x}_i (\partial_{x_i}^p \underline{x}_i) \partial_{x_i}^r \partial_{y_i}^{q+s} \;-\; \sum_{i=1}^n \underline{x}_i (\partial_{x_i}^r \underline{x}_i) \partial_{x_i}^p \partial_{y_i}^{q+s}\,. \qquad 1.18$$

Since we have

$$a) \quad \partial_{x_i}^p \underline{x}_i = p\,\partial_{x_i}^{p-1} + \underline{x}_i \partial_{x_i}^p \qquad\qquad b) \quad \partial_{x_i}^r \underline{x}_i = r\,\partial_{x_i}^{r-1} + \underline{x}_i \partial_{x_i}^r \qquad 1.19$$

The identity in 1.18 becomes, using 1.19

$$\big[F_{p,q}\;,\;F_{r,s}\big] \;=\; \sum_{i=1}^n \underline{x}_i (p\,\partial_{x_i}^{p-1} + \underline{x}_i \partial_{x_i}^p) \partial_{x_i}^r \partial_{y_i}^{q+s} \;-\; \sum_{i=1}^n \underline{x}_i (r\,\partial_{x_i}^{r-1} + \underline{x}_i \partial_{x_i}^r) \partial_{x_i}^p \partial_{y_i}^{q+s}\,. \qquad 1.20$$

Now this can be rearranged to

$$\big[F_{p,q}\;,\;F_{r,s}\big] \;=\; \sum_{i=1}^n \underline{x}_i (p\,\partial_{x_i}^{p-1}) \partial_{x_i}^r \partial_{y_i}^{q+s} \;-\; \sum_{i=1}^n \underline{x}_i (r\,\partial_{x_i}^{r-1}) \partial_{x_i}^p \partial_{y_i}^{q+s} \;+\; \sum_{i=1}^n (\underline{x}_i^2 - \underline{x}_i^2) \partial_{x_i}^{p+r} \partial_{y_i}^{q+s}\,.$$

From which we derive that

$$\big[F_{p,q}\;,\;F_{r,s}\big] \;=\; (p-r) \sum_{i=1}^n \underline{x}_i \partial_{x_i}^{p+r-1} \partial_{y_i}^{q+s} \;=\; (p-r)F_{p+r-1,q+s}\,. \qquad 1.21$$

This proves a) of 1.17.

Next we work on b) of 1.17. Reducing to $j=i$ we can write

$$\big[F_{p,q}\ ,\ E_{r,s}\big] \;=\; \sum_{i=1}^n \big[\underline{x}_i\partial_{x_i}^p\partial_{y_i}^q, \underline{y}_i\partial_{x_i}^r\partial_{y_i}^s\big] \;=\; \sum_{i=1}^n \underline{x}_i(\partial_{y_i}^q\underline{y}_i)\partial_{x_i}^{p+r}\partial_{y_i}^s \;-\; \sum_{i=1}^n \underline{y}_i(\partial_{x_i}^r\underline{x}_i)\partial_{x_i}^p\partial_{y_i}^{q+s} \tag{1.22}$$

Using I.19 we get

$$\begin{aligned}\big[F_{p,q}\ ,\ E_{r,s}\big] \;&=\; \sum_{i=1}^n \underline{x}_i(q\,\partial_{y_i}^{q-1}+\underline{y}_i\partial_{y_i}^q)\partial_{x_i}^{p+r}\partial_{y_i}^s \;-\; \sum_{i=1}^n \underline{y}_i(r\,\partial_{x_i}^{r-1}+\underline{x}_i\partial_{x_i}^r)\partial_{x_i}^p\partial_{y_i}^{q+s}\\ &=\; q\sum_{i=1}^n \underline{x}_i\,\partial_{x_i}^{p+r}\partial_{y_i}^{q+s-1} \;-\; r\sum_{i=1}^n \underline{y}_i\partial_{x_i}^{p+r-1}\partial_{y_i}^{q+s} \;+\; \sum_{i=1}^n(\underline{x}_i\underline{y}_i-\underline{y}_i\underline{x}_i)\partial_{x_i}^{p+r}\partial_{y_i}^{q+s}\\ &=\; q\,F_{p+r,q+s-1} \;-\; r\,E_{p+r-1,q+s}\,.\end{aligned} \tag{1.23}$$

This proves 1.17 b). The identity in 1.17 c) is proved the same way we proved a).

Remark 1.1

Using b) of 1.17 we can prove the result in c) of I.14. In fact, since $E=E_{1,0}$ and $F=F_{0,1}$, setting $p=0, q=1, r=1, s=0$ in

$$\big[F_{p,q}\ ,\ E_{r,s}\big] \;=\; qF_{p+r,q+s-1} \;-\; rE_{p+r-1,q+s}\,,$$

we obtain

$$\big[F_{0,1}\ ,\ E_{1,0}\big] \;=\; F_{1,0} \;-\; E_{0,1}\,,$$

Using 1.12 this gives that the $sl[2]$ operator in I.14

$$H \;=\; \big[E\ ,\ F\big] \;=\; \sum_{i=1}^n y_i\partial_{y_i} \;-\; \sum_{i=1}^n \underline{x}_i\partial_{x_i}\,, \tag{I.24}$$

on Diagonal Harmonics is none other than the Euler operator in the $y's$ minus the Euler operator in the $x's$.

Remark 1.2

We will make multiple uses of the operators $E_{r,0}$, we must point out that that these operators commute regardless of the values of r. This is one of the consequences of the identities in 1.17. In fact, if $q=s$ in c) that will immediately force the commutativity of $E_{p,q}$ and $E_{r,s}$. The same result can be obtained for the other differential operators pairs in 1.17.

2. Analysing the computer data for $\mathbf{n = 5, 6}$.

Although our differential operators are constructing Diagonal Harmonic alternants we will be guided by the Frobenius characteristic of the alternants in DH_n

$$c_n(q,t) \;=\; \sum_{D\in\mathcal{D}_n} t^{bounce(D)} q^{area(D)}. \qquad 2.1$$

We will do that for $n = 5$ and $n = 6$.

t^{10}	1	0	0	0	0	0	0	0	0	0	0
t^9	0	1	0	0	0	0	0	0	0	0	0
t^8	0	1	1	0	0	0	0	0	0	0	0
t^7	0	1	1	1	0	0	0	0	0	0	0
t^6	0	1	2	1	1	0	0	0	0	0	0
t^5	0	0	1	2	1	1	0	0	0	0	0
t^4	0	0	1	2	2	1	1	0	0	0	0
t^3	0	0	0	1	2	2	1	1	0	0	0
t^2	0	0	0	0	1	1	2	1	1	0	0
t^1	0	0	0	0	0	0	1	1	1	1	0
1	0	0	0	0	0	0	0	0	0	0	1
	1	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}

In the display on the right we have depicted a visual image of this polynomial for $n = 5$. For instance the term of $c_5(q,t)$ that is in the fifth row and fifth column corresponds to the Dyck path $D \in \mathcal{D}_5$ for which $t^{bounce(D)}q^{area(D)} = 2\,t^4q^4$. We can easily locate the $sl[2]$ strings in this display. In DH_n they are of the form $P(X_n;Y_n) \to EP(X_n;Y_n) \to \cdots \to E^kP(X_n;Y_n)$, where $P(X_n;Y_n)$ is killed by F and is a homogeneous polynomial of bi-degree (a,b). The operator E diminishes degree in $x's$ by one and increases degree in $y's$ by one, thus the polynomial $E^iP(X_n;Y_n)$ is homogeneous of bi-degree $(a-i, b+i)$. The final element is of bi-degree (b,a) and is killed by E. The circled entries correspond to alternants indexed by partitions. The one corresponding to t^{10} is the Vandermonde Δ_{1^5}. The one corresponding to t^6q^1 is Δ_{21^3}. $\Delta_{221}, \Delta_{311}$ and Δ_{32} form an $sl[2]$ string. Our list ends with Δ_{41} and the Vandermonde Δ_5. This display shows that the alternants in DH_5 factor into a direct sum of 7 of $sl[2]$ strings. The one of highest total degree is started by Δ_{1^5}. Let us call it *String* 1. *String* 2 starts with bi-degree $(8,1)$ and ends in bi-degree $(1,8)$. String 3 starts with bi-degree $(7,1)$ and ends in bi-degree $(1,7)$. String 4 starts with bi-degree $(6,2)$ and ends in bi-degree $(2,6)$. That accounts for the $2's$ in the image of its path. String 5 starts with bi-degree $(6,1)$ and ends in bi-degree $(1,6)$. These two ends are Δ_{21^3} and Δ_{41}. String 6 starts with bi-degree $(6,2)$ and ends in bi-degree $(2,6)$. That accounts for all the $2's$ in the image of its path. String 7 starts with bi-degree $(4,2)$ and ends in bi-degree $(2,4)$.

Our tentative starters are: Δ_{1^5}, $E_{2,0}\Delta_{1^5}$, $E_{3,0}\Delta_{1^5}$, $E_{2,0}E_{2,0}\Delta_{1^5}$, $E_{4,0}\Delta_{1^5}$, $E_{2,0}E_{2,0}E_{2,0}\Delta_{1^5}$ and $E_{4,0}E_{2,0}\Delta_{1^5}$. We need to computer check the strings they start are distinct. We should mention that the tool we use to that effect, is making shure that the "dlex" leading monomial of each alternant $P(X_n, Y_n)$ that generates a string has coefficient 1. Thus to check that the strings are distinct we need only verify the leading monomials are distinct.

But before we get involved in this project we need to make a few observations that will simplify our task. Firstly, Remark 1.2 tells us that the operators $E_{r,0}$ commute whatever is the value of r. From Conjecture I.1 we know that to obtain an alternant of bi-degree (a,b) in DH_n using the polynomial $E_{r_1,0}E_{r_2,0}\cdots E_{r_b,0}\Delta_{1^n}$ we must require the equality

$$r_1 + r_2 + \cdots + r_b \;=\; \binom{n}{2} - a \qquad 2.2$$

Due to the commutativity fact, there is no loss in assuming that our sequences are weakly decreasing $(r_1 \geq r_2 \geq \cdots \geq r_b)$ and satisfy 2.2. This is the number of b parts partitions of $\binom{n}{2} - a$.

Notice that our display at the row indexed by t^5 and column indexed by q^2 reveals that there is only one alternant in DH_5 with bi-degree $(5,2)$. Yet the 2 parts partitions of 5 are 32 and 41. Thus when we reduce the two polynomials $E_{3,0}E_{2,0}\Delta_{1^5}$ and $E_{4,0}E_{1,0}\Delta_{1^5}$ to have leading monomial with coefficient 1 the resulting alternants must be identical!

This fact shows that we can only use partitions as an upper bound to the number of strings that start at a given bi-degree. However, in our computer experimentation with DHA_6 we discovered that pairs that were discarded in DHA_5 had also to be discarded as factors in the construction of string starters in DHA_6. This suggests that the construction of a basis for DHA_n might demand a recursion on n.

These findings suggest that it might be possible to obtain a recursive construction of a basis. The following result gives us a tool for not using factors that have been discarded for $n-1$ in the construction of string starters for n. For simplicity we will state it in the simplest useful case.

Theorem 2.1

Suppose that $C_{n-1}(q,t)\big|_{t^c q^{b-1}} = k-1$ and $C_n(q,t)\big|_{t^a q^b} = k$, then the polynomial

$$P(X_n;Y_n) \,=\, E_{r_1,0}E_{r_2,0}\cdots E_{r_b,0}\Delta_{1^n} \tag{2.3}$$

cannot be used as a starter in bi-degree (a,b) if

$$a \,=\, n-1+c-r_b \tag{2.4}$$

for any discarded pairs of solutions of

$$r_1+r_2+\cdots+r_{b-1} \,=\, \tbinom{n-1}{2} \,-\, c\,. \tag{2.5}$$

Proof

Since our conjecture requires that

$$r_1+r_2+\cdots+r_b \,=\, 1+2+\cdots+n-1 \,-\, a\,. \tag{2.6}$$

this can be rewritten in an inductive way by relating for n what we already obtained for $n-1$

$$r_1+r_2+\cdots+r_{b-1} \,-\, (1+2+\cdots+n-2 \,-\, c) \,=\, n-1 \,-\, a \,+\, c \,-\, r_b\,. \tag{2.7}$$

We will soon use this result in working for $n=6$.

For $n=5$ the only things we need to assure are that the 3^{rd} and 4^{rth} strings are different and the same is true for the 5^{th} and 6^{th}. This happens to be true in both cases but for different reasons. In the first case the two leading monomials are identical. However the difference $EE_{3,0}\Delta_{1^5} - E_{2,0}E_{2,0}\Delta_{1^5}$ does not vanish since its leading monomial is $-x_1^3x_3^2x_4y_1y_2$. In the second case the leading monomial of $EEE_{4,0}\Delta_{1^5}$ is $x_1x_3^2x_4y_1^2y_2$ and the leading monomial of $E_{2,0}E_{2,0}E_{2,0}\Delta_{1^5}$ is $x_2x_3^2x_4y_1^2y_2$. Since in the dlex order we have $x_1x_3^2x_4y_1^2y_2 > x_2x_3^2x_4y_1^2y_2$ the strings are distinct. Thus we have proved our conjecture for $n=5$, Our task is now to explore $n=6$

t^{15}	1	0	0	0	0	0	0	0	0	0	0	0	**1**				
t^{14}	0	1	0	0	0	0	0	0	0	0	0	0					
t^{13}	0	1	1	0	0	0	0	0	0	0	0	0		**2**			
t^{12}	0	1	1	1	0	0	0	0	0	0	0	0		**3**			
t^{11}	0	1	2	1	1	0	0	0	0	0	0	0		**5**	**4**		
t^{10}	0	1	2	2	1	1	0	0	0	0	0	0		**8**	**6**		
t^{9}	0	0	2	3	2	1	1	0	0	0	0	0			**9**	**7**	
t^{8}	0	0	1	3	3	2	1	1	0	0	0	0			**12**	**10**	
t^{7}	0	0	1	2	4	3	2	1	1	0	0	0			**15**	**13**	**11**
t^{6}	0	0	0	2	3	4	3	2	1	1	0	0				**16**	**14**
t^{5}	0	0	0	0	2	3	4	3	2	1	1	0					
t^{4}	0	0	0	0	1	2	3	4	3	2	1	1				**17**	
	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	1	q	q^2	q^3	q^4

On the right we have a visual display of a portion of the Frobenius characteristic of the alternants in DH_6. This is the polynomial $c_6(q,t)$, The number of strings start at bi-degree (a,b) is simply the difference

$$c_6(q,t)\big|_{t^{a+1}q^{b-1}} \,-\, c_6(q,t)\big|_{t^aq^b} \tag{2.6}$$

This given the tentative string starters are Δ_{1^6}, $E_{2,0}\Delta_{1^6}$, $E_{3,0}\Delta_{1^6}$, $E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{4,0}\Delta_{1^6}$, $E_{3,0}E_{2,0}\Delta_{1^6}$, $E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{5,0}\Delta_{1^6}$, $E_{3,0}E_{3,0}\Delta_{1^6}$, $E_{3,0}E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{2,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{4,0}E_{3,0}\Delta_{1^6}$, $E_{3,0}E_{3,0}E_{2,0}\Delta_{1^6}$, $E_{3,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{4,0}E_{4,0}\Delta_{1^6}$, $E_{5,0}E_{2,0}E_{2,0}\Delta_{1^6}$, $E_{5,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$. Our next task is to find out if the $sl[2]$ strings they generate are all distinct. The right most part of the above display has red arrows. If the tip of an arrow points to a cell in row t^a and column q^b, then in the left part of the display that cell has k then we need to make sure that k distinct strings start at that point. For instance the first arrow points to the cell $t^{11}q^2$. On the left that cell has a 2. The 3^{rd} string starter is $E_{3,0}\Delta_{1^6}$ in cell $t^{11}q^2$ there is $EE_{3,0}\Delta_{1^6}$ and $E_{2,0}E_{2,0}\Delta_{1^6}$. Normalizing these two polynomials to have leading monomial with coefficients 1 gives both the monomial $x_1x_2^4x_3^3x_4^2x_5y_1^2$ so the leading monomials cancel when we compute the difference of the resulting polynomials. This difference is not zero since its leading monomial is $x_1^4x_2x_3^3x_4^2x_5y_1y_2$. The 3 diagonal harmonic alternants of Bi-degree $(9,3)$ start 3 distinct strings since their normalized leading monomials are different. Now the 8^{th} string starter in our list starts a string that meets 3 more string starters, that accounts for the sequence of $1,2,3,4$ that starts in the cell $t^{10}q$. we will begin by normalizing the two polynomials $EE_{5,0}\Delta_{1^6}$ and $E_{3,0}E_{3,0}\Delta_{1^6}$. In this case it turned out that these alternants have leading monomials $x_1^3x_3^3x_4^2x_5y_1y_2$ and $x_1^2x_2x_3^3x_4^2x_5y_1y_2$. But their difference has the same leading monomial as the first. However it is different from the first in that it has 3600 terms while the first has only 2160.

In bi-degree $(8,3)$ we should have 3 alternants $EEE_{5,0}\Delta_{1^6}$, $EE_{3,0}E_{3,0}\Delta_{1^6}$ and $E_{3,0}E_{2,0}E_{2,0}\Delta_{1^6}$. to show that they are distinct we need to compute their normalized leading monomials. The computer data gives that the respective leading monomials are $x_1x_2x_3^3x_4^2x_5y_1^2y_2$, $x_1^2x_3^3x_4^2x_5y_1^2y_2$ and $x_1x_2x_3^3x_4^2x_5y_1^2y_2$. Since the first and the last have the same leading monomial all we need do check is that the difference does not vanish. However the difference has leading monomial $x_2^2x_3^3x_4^2x_5y_1^2y_2$ and therefore it does not vanish.

In bi-degree $(7,4)$ we have one more starter $E_{2,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$ and we have established that the 3 strings that started in bi-degree $(8,3)$ are distinct. Therefore the alternants $EEEE_{5,0}\Delta_{1^6}$, $EEE_{3,0}E_{3,0}\Delta_{1^6}$ and $EE_{3,0}E_{2,0}E_{2,0}\Delta_{1^6}$ will also be distinct. Thus all that remains to do in bi-degree $(7,4)$ is show that these 3 alternants are different from $E_{2,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$. This turns out to be trivially true since the leading monomials in the previous 3 end with the factor $y_1^3y_2$ and the last factor in the new one is $y_1^2y_2^2$.

Next we need to prove that the string started in bi-degree $(8,2)$ is different than the string that starts in $(7,3)$. In other words, since the latter is our 13^{th} string starter we want the normalized form of the polynomials $E_{1,0}E_{4,0}E_{3,0}\Delta_{1^6}$ and $E_{3,0}E_{2,0}E_{3,0}\Delta_{1^6}$ to be different. Unfortunately they are not. The reason is due to Theorem 2.1. Since we chose one of our starters to be $E_{4,0}E_{3,0}\Delta_{1^6}$ choosing the next starter to be $E_{3,0}E_{3,0}E_{2,0}\Delta_{1^6}$ was a mistake! In fact, the successor of $E_{4,0}E_{3,0}\Delta_{1^6}$ is the alternant $E_{1,0}E_{4,0}E_{3,0}\Delta_{1^6}$ and the alternant $E_{3,0}E_{3,0}E_{2,0}\Delta_{1^6}$ would contain the pair $3,2$. That forces the next starter to be $E_{4,0}E_{2,0}E_{2,0}\Delta_{1^6}$ or $E_{5,0}E_{2,0}E_{1,0}\Delta_{1^6}$. Both are OK. But the first is simpler to prove that it is OK. Let us tentatively choose the normalized $EE_{4,0}E_{2,0}E_{2,0}\Delta_{1^6}$ and $EEE_{4,0}E_{3,0}\Delta_{1^6}$ and since we know they are distinct we have only to assure that they are both different for the normalized 14^{th} starter, which is $E_{3,0}E_{2,0}E_{2,0}E_{2,0}\Delta_{1^6}$. We have now proved the validity of the $n=6$ starters as modified since the remaining starters have not presented any difficulty.

We can obtain an algorithm for obtaining the number of starters for any n, that MAPLE permits, from the following calculation. Since every string is of the form

$$P(X_n;Y_n)\rightarrow EP(X_n;Y_n)\rightarrow\cdots\rightarrow E^kP(X_n;Y_n). \tag{2.7}$$

If $P(X_n;Y_n)$ is an alternant homogeneous of bi-degree (u,v) then the alternant $E^iP(X_n;Y_n)$ is homogeneous of bi-degree $(u-i,v+i)$, with $E^kP(X_n;Y_n)$ homegeous of bi-degree (v,u). The contribution of this string to the Hilbert series of DHA_n is the polynomial $\sum_{i=0}^{u-v}t^{u-i}q^{v+i}$. Note that

$$t^uq^v \;=\; (qt)^{\frac{u+v}{2}}\big(\tfrac{q}{t}\big)^{\frac{v-u}{2}}, \tag{2.8}$$

so we can write

$$\begin{aligned} t^uq^v\Big(1+\big(\tfrac{q}{t}\big)+\cdots+\big(\tfrac{q}{t}\big)^{u-v}\Big) \;&=\; (qt)^{\frac{u+v}{2}}\big(\tfrac{t}{q}\big)^{\frac{u-v}{2}}\Big(1+\big(\tfrac{q}{t}\big)+\cdots+\big(\tfrac{q}{t}\big)^{u-v}\Big)\\ &=\; (qt)^{\frac{u+v}{2}}\big(\tfrac{t}{q}\big)^{\frac{u-v}{2}}\frac{1-\big(\tfrac{q}{t}\big)^{u-v+1}}{1-\big(\tfrac{q}{t}\big)} \;=\; (qt)^{\frac{u+v}{2}}\frac{\big(\tfrac{t}{q}\big)^{\frac{u-v}{2}}-\big(\tfrac{q}{t}\big)^{\frac{u-v}{2}+1}}{1-\big(\tfrac{q}{t}\big)}\\ &=\; (qt)^{\frac{u+v}{2}}\frac{\big(\tfrac{t}{q}\big)^{\frac{u-v}{2}}\big(\tfrac{t}{q}\big)^{\frac12}-\big(\tfrac{q}{t}\big)^{\frac{u-v}{2}+1}\big(\tfrac{t}{q}\big)^{\frac12}}{\big(\tfrac{t}{q}\big)^{\frac12}-\big(\tfrac{q}{t}\big)\big(\tfrac{t}{q}\big)^{\frac12}} \;=\; (qt)^{\frac{u+v}{2}}\frac{\big(\tfrac{t}{q}\big)^{\frac{u-v+1}{2}}-\big(\tfrac{q}{t}\big)^{\frac{u-v+1}{2}}}{\big(\tfrac{t}{q}\big)^{\frac12}-\big(\tfrac{q}{t}\big)^{\frac12}} \end{aligned} \tag{2.9}$$

Thus the Hilbert series of DHA_n is the polynomial

$$h(q,t) \;=\; \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}} b_{u,v}\;(qt)^{\frac{u+v}{2}}\frac{\big(\tfrac{t}{q}\big)^{\frac{u-v+1}{2}}-\big(\tfrac{q}{t}\big)^{\frac{u-v+1}{2}}}{\big(\tfrac{t}{q}\big)^{\frac12}-\big(\tfrac{q}{t}\big)^{\frac12}} \tag{2.10}$$

where $b_{u,v}$ is the number of strings that start in bi-degree (u,v). Making the specialization $t\rightarrow q^{-1}$ gives

$$h(q,q^{-1}) \;=\; \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}} b_{u,v}\;\frac{q^{u-v+1}-q^{-(u-v+1)}}{q-q^{-1}}$$

or better

$$(q-q^{-1})h(q,q^{-1}) \;=\; \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}} b_{u,v}\;\big(q^{u-v+1}-q^{-(u-v+1)}\big) \tag{2.11}$$

We can rewrite the identity in 2.11 in the form

$$(q - q^{-1})h(q,q^{-1}) = \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}} \chi\big(u-v+1=r\big)\big(q^r - q^{-r}\big)\sum_{u-v+1=r} b_{u,v} \qquad 2.12$$

If we let $c_r = \sum_{u-v+1=r} b_{u,v}$ then 2.12 becomes

$$\begin{aligned}(q - q^{-1})h(q,q^{-1}) &= \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}} \chi\big(u-v+1=r\big)\,\big(q^r - q^{-r}\big)\,c_r \\ &= \sum_{r=1}^{\binom{n}{2}}\big(q^r - q^{-r}\big)\,c_r \sum_{u=0}^{\binom{n}{2}}\sum_{v=0}^{\binom{n}{2}}\chi\big(r=1-u+v\big) = \sum_{r=1}^{\binom{n}{2}}\big(q^r - q^{-r}\big)\,c_r\,.\end{aligned} \qquad 2.13$$

Haiman has proved in [X] that $h(q,t)$ is exactly the Garsia-Haiman q,t-Catalan. In [X] the specialization $t{\rightarrow}q^{-1}$ is derived from Macdonald identities to be related to the none other than the q-analogue of the number of Dyck paths in the $n\times n$ lattice rectangle, more precisely we have:

$$q^{\binom{n}{2}}h(q,q^{-1}) = \frac{1}{[n+1]_q}\begin{bmatrix}2\,n\\ n\end{bmatrix}_q. \qquad 2.14$$

Thus the left hand side of 2.13 is

$$(q - q^{-1})q^{-\binom{n}{2}}\frac{q-1}{q^{n+1}-1}\frac{(q^{n+1}-1)\cdots(q^{2\,n}-1)}{(q-1)\cdots(q^n-1)} = q^{-\binom{n}{2}-1}\frac{(q^{n+2}-1)\cdots(q^{2\,n}-1)}{(q^3-1)\cdots(q^n-1)} \qquad 2.15$$

and 2.13 becomes

$$q^{-\binom{n}{2}-1}\frac{(q^{n+2}-1)\cdots(q^{2\,n}-1)}{(q^3-1)\cdots(q^n-1)} = \sum_{r=1}^{\binom{n}{2}} c_r\,q^r - \sum_{r=1}^{\binom{n}{2}} c_r\,q^{-r} \qquad 2.16$$

Since the number of starters is $\sum_{r=1}^{\binom{n}{2}} c_r$ all we need is to sum the positive coefficients of the powers of q. Of course this is really only an algorithm, but we can also obtain a formula from 2.16.

To do this we first apply an odd power of the Euler operator $q\partial_q$ to both side of 2.16 :

$$\big(q\partial_q\big)^{2k+1}\left(q^{-\binom{n}{2}-1}\frac{(q^{n+2}-1)\cdots(q^{2\,n}-1)}{(q^3-1)\cdots(q^n-1)}\right) = \sum_{r=1}^{\binom{n}{2}} r^{2k+1}\,c_r\,(q^r+q^{-r}) \qquad 2.17$$

The determinant of the matrix $\big\|z_r^{2k}\big\|_{r,k=1}^{n(n-1)/2+1}$ factorized into a product of $(z_r^2 - z_s^2)$ and each is different from zero as long as $z_r \neq z_s$. this proves that the matrix $\big\|r^{2k+1}\big\|_{r,k=1}^{n(n-1)/2+1}$ has non zero determinant. Denoting by $\big\|d_{s,r}\big\|_{s,r=1}^{n(n-1)/2+1}$ the inverse, it follows that

$$\sum_{r=1}^{\binom{n}{2}} d_{s,r}\big(q\partial_q\big)^{2k+1}\left(q^{-\binom{n}{2}-1}\frac{(q^{n+2}-1)\cdots(q^{2\,n}-1)}{(q^3-1)\cdots(q^n-1)}\right) = \sum_{r=1}^{\binom{n}{2}} c_r\,(q^r+q^{-r}) \qquad 2.18$$

Thus

$$\tfrac{1}{2}\sum_{r=1}^{\binom{n}{2}} d_{s,r}\big(q\partial_q\big)^{2k+1}\left(q^{-\binom{n}{2}-1}\frac{(q^{n+2}-1)\cdots(q^{2\,n}-1)}{(q^3-1)\cdots(q^n-1)}\right)\Big|_{q=1} = \sum_{r=1}^{\binom{n}{2}} c_r \qquad 2.19$$

as desired.