APPENDIX

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The preceding paper by S. Milne deals with a number of issues in the theory of symmetric polynomials which are of great present interest. In this writing we show that certain identities established in the paper are connected to several recent developments. In particular, we show a connection between some specializations of Milne's polynomials $F_\lambda(Z,\gamma;q,t)$ with corresponding specializations of the polynomials $P(X;q,t)$ recently introduced by MacDonald in []. We shall also show that Milne's polynomials may be given a *Vertex* operator setting which extends that given by N. Jing to the Hall-Littlewood polynomials in a recent Yale doctoral thesis []. In particular this setting brings to evidence in which manner the Milne polynomials $F_\lambda(Z,0;q,t)$ differ from the MacDonald polynomials $P(X;q,t)$. We should note that the considerations that follow also contain a new proof of the basic *duality* result of [] for the polynomials $P(X;q,t)$ and a more transparent way to deal with the ubiquitous *raising* operators of D. H. Littlewood.

1. The Gramm-Schmidt process in reverse lex and the duality of the MacDonald polynomials.

One of the exciting features of the two parameter family of polynomials $P(X;q,t)$ introduced by MacDonald in [] is that, by appropriate choice of q and t they specialize to well known classical bases of symmetric polynomials. In particular letting $q\to t$ yields the classical Schur functions $S_\lambda(X)$, letting $q\to 0$ yields the Hall-Littlewood basis and setting $q=t^\alpha$ then letting $t\to 1$ yields the Jack polynomials. Each of these bases has a natural setting, we shall show here that Milne's polynomials $F_\lambda(Z,0;q,0)$ yield a setting for the polynomials obtained by letting $t\to 0$ in $P(X;q,t)$. To make this more precise we need to recall two basic results of []. We start with some notation.

Recall that for $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k)$

$$p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_k}$$

denotes the complete power symmetric function indexed by λ. Moreover, in λ-ring notation, we have

$$p_\lambda\Big(\frac{1-q}{1-t}\Big) = \prod_{i=1}^{k}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\ . \qquad 1.1$$

As in [], we shall let $\langle\ ,\ \rangle_{q,t}$ denote the scalar product defined by setting

$$\langle p_\lambda\ ,\ p_\mu\rangle_{q,t} \;=\; \begin{cases} p_\lambda\left(\frac{1-q}{1-t}\right)\ z_\lambda & \text{if } \lambda \;=\; \mu \\ 0 & \text{otherwise} \end{cases} \quad . \qquad 1.2$$

Where, as customary, for $\lambda \;=\; 1^{m_1}1^{m_2}1^{m_3}\cdots$ we set

$$z_\lambda \;=\; 1^{m_1}1^{m_2}1^{m_3}\cdots\ m_1!\ m_2!\ m_3!\cdots$$

The starting point in MacDonald development is the following

Theorem 1.1 (see [])

For each $n>0$ there exists a unique family of homogeneous symmetric polynomials $\{P_\lambda(X;q,t)\}_{\lambda\vdash n}$ satisfying the following conditions:

$$\begin{aligned} &i) \qquad P_\lambda(X;q,t) \;=\; \textstyle\sum_{\substack{\mu\vdash n\\ \mu\leq\lambda}} S_\mu(X)\,\eta_{\mu\lambda} \qquad (\eta_{\mu\mu} \;=\; 1) \\ &ii) \qquad \langle P_\lambda\ ,\ P_\mu\rangle_{q,t} \;=\; 0 \qquad \textit{for } \lambda\neq\mu \end{aligned} \qquad 1.3$$

Here the sign $\leq$ denotes the dominance partial order of partitions.

It should be noted that one of the consequences of this existence theorem is that the basis $\{P_\lambda(X;q,t)\}_{\lambda\vdash n}$ may be obtained by applying the Gramm-Schmidt algorithm (with scalar product $\langle\ ,\ \rangle_{q,t}$) to the Schur basis $\{S_\lambda\}_{\lambda\vdash n}$ arranged according to **any** total order of partitions that is compatible with dominance, such as the lexicographic order, for instance. We recall that we say that λ precedes μ in the lex order and write $\lambda<_L\mu$ if and only λ precedes μ when both are viewed as words in the alphabet of natural numbers. Note that we use the *English* convention and let partitions be vectors with weakly decreasing components. We shall also use here the order $<_{L'}$ obtained by setting $\lambda<_{L'}\mu$ if and only if $\lambda'>_L\mu'$. We can easily see that $<_{L'}$ is also compatible with dominance.

Before we can state the next basic result of [] we need further notation. Given an alphabet $X \;=\; x_1+x_2+\ldots+x_n$ we shall set

$$\Omega(X) \;=\; \prod_{i=1}^{n}\frac{1}{1-x_i} \;=\; \prod_{x\in X}\frac{1}{1-x}$$

In particular, for the products XY and XYZ of two or three different alphabets we have

$$\Omega(XY) \;=\; \prod_{x\in X}\prod_{y\in Y}\frac{1}{1-xy} \quad \textit{and} \quad \Omega(XYZ) \;=\; \prod_{x\in X}\prod_{y\in Y}\prod_{z\in Z}\frac{1}{1-xyz}\ .$$

Moreover, in λ-ring notation, for the difference $X-Y$ of two alphabets we set

$$\Omega(X-Y) \;=\; \prod_{x\in X}\frac{1}{1-x}\prod_{y\in Y}(1-y) \qquad 1.2\ .$$

Applying to the alphabet XYZ the customary expansion

$$\Omega(X) \;=\; \sum_{\rho}\frac{p_\rho(X)}{z_\rho}$$

we may write

$$\Omega(XYZ) \;=\; \sum_{\rho}\frac{p_\rho(X)p_\rho(Y)p_\rho(Z)}{z_\rho}\ . \qquad 1.3$$

Interpreting $1/1-q$ as the alphabet $1+q+q^2+q^3+\cdots$ we may then write

$$\Omega(XY\tfrac{1-t}{1-q}) \;=\; \sum_{\rho}\frac{p_\rho(X)\,p_\rho(Y)}{z_\rho}\,p_\rho(\tfrac{1-t}{1-q}) \qquad 1.4$$

with $p_\rho(1-t/1-q)$ given by 1.2 for $\lambda = \rho$. We shall use MacDonald's notation and denote this kernel by $\Omega_{q,t}(XY)$. It easy to show by a standard argument that two bases $\{a_\lambda\}$ and $\{b_\lambda\}$ are dual with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$ if and only if

$$\sum_{\lambda} a_\lambda(X)\,b_\lambda(Y) \;=\; \Omega_{q,t}(XY)\ .$$

We recall that in [] the basis dual to $P_\lambda(X;q,t)$, with respect to $\langle\ ,\ \rangle_{q,t}$, is denoted by $Q_\lambda(X;q,t)$ and since the former is already an orthogonal basis we must have

$$Q_\lambda(X;q,t) \;=\; d_\lambda(q,t)\;P_\lambda(X;q,t)$$

where $d_\lambda(q,t)$ is a rational function of its arguments.

Let now $\theta_{q,t}$ be a linear operator on symmetric polynomials defined by setting

$$\theta_{q,t}\;p_\lambda(X) \;=\; p_\lambda(X\tfrac{1-q}{1-t}) \;=\; p_\lambda(X)\,p_\lambda(\tfrac{1-q}{1-t})\ ,$$

and let ω be the classic involution which permutes the Schur functions according to the formula

$$\omega S_\lambda(X) \;=\; S_{\lambda'}(X)\ .$$

Thus the operator $\omega_{q,t}$ of [] may be written as

$$\omega_{q,t} \;=\; \omega\;\theta_{q,t}$$

With this notation the *duality* result of MacDonald may be stated as follows

Theorem 1.2 []

$$\omega\, \theta_{q,t}\;\; P_\lambda(X;q,t) \;=\; Q_{\lambda'}(X;t,q) \qquad 1.5(a)$$

$$\omega\, \theta_{t,q}\;\; Q_\lambda(X;t,q) \;=\; P_{\lambda'}(X;q,t) \qquad 1.5(b)$$

This given, the connection between Milne's polynomials $H_\lambda(Z,\gamma;q)$ and the MacDonald polynomials $P_\lambda(X;q,t)$ is given by

Theorem 1.3

$$H_\lambda(X,0;q) \;=\; \theta_{0,q}\, Q_\lambda(X;0,q) \;=\; \omega\, P_{\lambda'}(X;q,0)\ . \qquad 1.6$$

Moreover, the polynomials $\{H_\lambda(Z,0;q)\}_{\lambda\vdash n}$ can be obtained by applying the Gramm-Schmidt algorithm (with scalar product $\langle\ ,\ \rangle_{0,q}$) to the Schur function basis in the reverse lex order.

The proof of this result is an immediate consequence of the following

Proof of Theorem 1.2

Our point of departure is the identity

$$\Omega_{q,t}(XY) \;=\; \sum_\lambda \sum_\mu \;S_\lambda(X) S_\mu(Y)\;\; \Gamma_{\lambda,\mu}\ . \qquad 1.7$$

with

$$\Gamma_{\lambda,\mu} \;=\; \langle S_\lambda\ ,\ S_\mu\rangle_{t,q}\ \ ,$$

To show 1.7, we substitute in the product rule 1.4, the expansions

$$p_\rho(X) \;=\; \sum_\lambda\; S_\lambda(X)\frac{\chi_\rho^\lambda}{z_\rho} \qquad\qquad p_\rho(Y) \;=\; \sum_\mu\; S_\mu(Y)\frac{\chi_\rho^\mu}{z_\rho}$$

of the power symmetric functions in terms of the Schur functions. Here χ_ρ^λ and χ_ρ^μ denote the values of the corresponding irreducible characters at the conjugacy class indexed by ρ. This gives

$$\Omega_{q,t}(XY) \;=\; \sum_\lambda \sum_\mu \;S_\lambda(X) S_\mu(Y) \sum_\rho \frac{\chi_\rho^\lambda \chi_\rho^\mu}{z_\rho}\;\; p_\rho(\tfrac{1-t}{1-q})\ \ ,$$

from which we derive that

$$\Gamma_{\lambda\mu} \;=\; \sum_\rho \frac{\chi_\rho^\lambda \chi_\rho^\mu}{z_\rho}\;\; p_\rho(\tfrac{1-t}{1-q})\ \ . \qquad 1.8$$

On the other hand we have

$$\langle S_\lambda \ , \ S_\mu \rangle_{t,q} \ = \ \sum_\rho \chi_\rho^\lambda \chi_\rho^\mu \ (\frac{1}{z_\rho})^2 \ \langle p_\rho \ , \ p_\rho \rangle_{t,q} \ = \ \sum_\rho \ \chi_\rho^\lambda \ \chi_\rho^\mu \ \frac{1}{z_\rho} \ \ p_\rho\big(\tfrac{1-t}{1-q}\big) \ \ .$$

Comparing with 1.8 we see that 1.7 holds true as asserted.

This given we let $\{\pi_\lambda(X)\} \ = \ \{\pi_\lambda(X;q,t)\}$ be the orthogonal basis obtained by applying Gramm-Schmidt (with scalar product $\langle \ , \ \rangle_{t,q}$) to the Schur basis $\{S_\lambda(X)\}$ in reverse lex. More precisely, we have

$$\pi_\lambda(X) \ = \ \sum_{\lambda_1 \geq_L \lambda} S_{\lambda_1}(X) \ \xi_{\lambda_1,\lambda} \qquad\qquad (\ \mathit{with} \quad \xi_{\lambda,\lambda} \ = \ 1 \ ) \ , \qquad 1.9(a)$$

and

$$\langle \pi_\lambda \ , \ \pi_\mu \rangle_{t,q} \ = \ \sum_{\substack{\lambda_1 \geq_L \lambda \\ \mu_1 \geq_L \mu}} \xi_{\lambda_1,\lambda} \ \xi_{\mu_1,\mu} \ \langle S_{\lambda_1} \ , \ S_{\mu_1} \rangle_{t,q} \ = \ 0 \quad \mathit{if} \ \ \lambda \neq \mu \ \ . \qquad 1.9(b)$$

In other words, letting d be the diagonal matrix with entries $d_\lambda \ = \ \langle \pi_\lambda, \pi_\lambda \rangle_{t,q}$ and setting $\xi = \xi(q,t) = \|\xi_{\lambda,\mu}\|$, (both matrices written according to the total order $<_L$) we see that 1.9 (b), in matrix form, may be rewritten as

$$d \ = \ \xi^T \ \Gamma \ \xi \qquad 1.10$$

where $\Gamma = \|\Gamma_{\lambda,\mu}\|$ and "T" denotes transposition.

Set now

$$\eta \ = \ \|\eta_{\lambda,\mu}\| \ = \ \big(\xi^T\big)^{-1} \qquad *$$

so that we have

$$\Gamma \ = \ \eta \ d \ \eta^T \ \ .$$

We can thus write 1.7 in the matrix form

$$\Omega_{q,t}(XY) \ = \ \langle S(X) \rangle \ \eta \ d \ \big(\langle S(Y)\eta \rangle\big)^T \ \ .$$

where $\langle S(X) \rangle$ and $\langle S(Y) \rangle$ denote the Schur bases in the X and Y variables written as row vectors and again in the total order $<_L$ of their index.

* Although ξ is an infinite matrix, it may be viewed as a block diagonal matrix, each block being a unitriangular matrix indexed by the partitions of a fixed integer n. Thus there is no problem in taking inverses here.

This means that the basis

$$\langle P(X)\rangle \;=\; \langle S(X)\rangle\ \eta \qquad 1.11$$

is orthogonal with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$. Moreover, from 1.9 (a) we derive that η is upper unitriangular (since ξ is *lower*). That is we have

$$P_\lambda(X) \;=\; \sum_{\mu\le_L\lambda} S_\mu(X)\ \eta_{\mu\lambda} \qquad (\ with \quad \eta_{\lambda\lambda}=1\ )$$

and

$$\Omega(\tfrac{1-t}{1-q}XY) \;=\; \langle P(X)\rangle\ d\ \langle P(Y)\rangle^T\ . \qquad 1.12$$

Since $<_L$ is compatible with dominance, Theorem 1.1 implies that

$$P_\lambda(X) \;=\; P_\lambda(X;q,t)\ .$$

In the same vein we can also show that

$$\pi_\lambda(X) \;=\; \omega P_{\lambda'}(X;t,q)\ . \qquad 1.13$$

To see this, note that we can write

$$\omega\pi_{\lambda'}(X) \;=\; \sum_{\mu\ge_L\lambda'} S_{\mu'}(X)\ \xi_{\mu\lambda'} \;=\; \sum_{\mu'\le_{L'}\lambda} S_{\mu'}(X)\ \xi_{\mu\lambda'} \;=\; \sum_{\mu\le_{L'}\lambda} S_\mu(X)\ \xi_{\mu'\lambda'}\ .$$

Since $\xi_{\lambda'\lambda'}=1$ and

$$\langle\omega\pi_{\lambda'},\omega\pi_{\mu'}\rangle_{t,q} \;=\; \langle\pi_{\lambda'},\pi_{\mu'}\rangle_{t,q} \;=\; 0 \qquad (\ if \quad \lambda\ne\mu\ )$$

with $\le_{L'}$ compatible with dominance, we must necessarily conclude that 1.13 holds true as well. But, from 1.11 and the definition of η we derive that

$$\langle\pi(X)\rangle\ \langle P(X)\rangle^T \;=\; \langle S(X)\rangle\ \xi\eta^T\langle S(Y)\rangle^T \;=\; \langle S(X)\rangle\langle S(Y)\rangle^T \;=\; \prod_{i,j}\frac{1}{1-x_iy_j}\ , \qquad 1.14$$

as well as

$$\langle\omega\pi(X)\rangle\ \langle P(X)\rangle^T \;=\; \langle\omega S(X)\rangle\xi\eta^T\langle S(Y)\rangle^T \;=\; \langle\omega S(X)\rangle\langle S(Y)\rangle^T \;=\; \prod_{i,j}(1+x_iy_j)\ . \qquad 1.15$$

Now 1.13 gives that $\omega\pi_\lambda=P_{\lambda'}(X;t,q)$ and we may thus rewrite 1.15 as

$$\sum_\lambda P_{\lambda'}(X;t,q)\ P_\lambda(Y;q,t) \;=\; \prod_{i,j}(1+x_iy_j)$$

which is one of the basic identities in [].

Now using 1.13 we can rewrite 1.14 as

$$\sum_{\lambda} \omega P_{\lambda'}(X;t,q)\ P_{\lambda}(Y;q,t) \ = \ \Omega(XY)$$

applying $\theta_{t,q}$ to both sides of this equation gives

$$\sum_{\lambda} \omega_{t,q} P_{\lambda'}(X;t,q)\ P_{\lambda}(Y;q,t) \ = \ \Omega_{q,t}(XY) \ ,$$

which necessarily implies the duality relation

$$Q_{\lambda}(X;q,t) \ = \ \omega_{t,q} P_{\lambda'}(X;t,q) \ .$$

This is clearly equivalent to 1.5 (a) upon interchanging q and t. Equation 1.5 (b) follows from 1.5 (a) because the operators $\theta_{q,t}$ and $\theta_{t,q}$ are inverses of each other and ω commutes with the $\theta' s$.

Proof of Theorem 1.3

Formula 4.30 a) of Milne's paper, with $Z = X$ and $\gamma = 0$, may be written as

$$H_{\lambda}(X,0;q) \ = \ \sum_{\mu \leq \lambda} S_{\mu}(X)\ K_{\lambda,\mu}(q) \ . \qquad 1.16$$

On the other hand, since the specialization $q \to 0$ in $Q_{\lambda}(X;q,t)$ yields the Hall-Littlewood polynomial $Q_{\lambda}(X;t)$, formula 6.3 (page 126) of [] may be rewritten as

$$Q_{\lambda}(X;0,t) \ = \ \sum_{\mu \leq \lambda} S_{\mu}(X(1-t))\ K_{\lambda,\mu}(t) \ .$$

Applying $\theta_{0,t}$ to both sides of this equation gives

$$\theta_{0,t}\ Q_{\lambda}(X;0,t) \ = \ \sum_{\mu \leq \lambda} S_{\mu}(X)\ K_{\lambda,\mu}(t) \ .$$

Comparing this with 1.16 we see that we must have

$$H_{\lambda}(X,0;q) \ = \ \theta_{0,q}\ Q_{\lambda}(X;0,q) \ .$$

Thus the first equality in 1.6 holds true as asserted. The second equality then follows from 1.5 (b).

Remark

The special case $q = 0$ of this proof of Theorem 1.2, yields an interesting byproduct. Indeed, from 1.6, 1.13 and 1.9 (a) we get that

$$H_\lambda(X,0;q) \;=\; \pi_\lambda(X;0,q) \;=\; \sum_{\mu\geq\lambda} S_\mu(X)\,\xi_{\mu,\lambda}(0,q) \;.$$

Comparing with 1.16 we deduce that

$$\xi_{\mu,\lambda}(0,q) \;=\; K_{\lambda,\mu}(q) \;.$$

But this means that

> *The entries in the q-Kotska matrix can be obtained by applying to the Schur basis in reverse lex order, the Gramm-Schmidt algorithm with scalar product* $\langle\ ,\ \rangle_{q,0}$

This fact makes it even more remarkable that its entries are polynomials with positive integer coefficients.

2. On the associativities of Raising operators

There are some misconceptions about *Raising operators* that we wish to point out here. Raising operators acting on shapes most probably appeared the first time in the work of Littlewood. Some attempts at giving a precise definition can be found in [],[] and []. Although the presentations in [],[] and [], resulted from a need to make some sense out of some of A. Young's identities, the Raising operators, as discussed in those publications, are not the same as those introduced by A. Young in QSA IV ([] page). To begin with, Young's operators act on *tableaux* and not on *shapes*. In previous work [] we have given a rigorous setting to Young's manipulations in the framework of tableau idempotents. From [], the reader should get at least an idea of what Young, and later on G. d. B. Robinson [] had in mind. Here, we would like consider in some detail some of the associativities that are most commonly used in recent expositions involving Littlewood's Raising operators.

The symbol $R_{i,j}$ (for $i < j$) is defined in [] (page 8) as acting on a vector $a \;=\; (a_1, a_2, \ldots, a_n)$ according to the formula

$$R_{i,j}\,a \;=\; (a_1,\ldots,a_i+1,\ldots,a_j-1,\ldots,a_n) \qquad 2.1$$

a Raising operator is then any product

$$R(m) \quad = \quad \prod_{i<j} R_{i,j}^{m_{i,j}} \quad .$$

Its action on a vector is obtained by successive applications of 2.1. To be precise it will be good to express it entirely in vector form. Let

$$e_1, e_2, \ldots, e_n$$

be the unit coordinate vectors, with e_s having 1 in the s^{th} position and zeros in all other positions. We can easily see then that

$$R(m)\, a \quad = \quad a \;+\; \sum_{i<j} m_{i,j}\, (e_i - e_j) \quad . \qquad 2.2$$

Later on in [] (page 26) R is made to act on the complete homogeneous symmetric function h_λ by setting

$$R(m)\, h_\lambda \quad = \quad h_{R(m)\lambda} \qquad 2.3$$

and finally the formula expressing the Schur function S_λ in terms of the homogeneous basis is written as

$$S_\lambda \quad = \quad \prod_{i<j} (1 - R_{i,j})\, h_\lambda \quad . \qquad 2.4$$

There is no problem with all of this until we run into formulas which suggest that

$$\prod_{i<j} \frac{1}{(1 - R_{i,j})} \prod_{i<j} (1 - R_{i,j}) \quad = \quad 1 \qquad 2.5$$

which seems to be implied by combining 2.4 with ([] :I §6 Ex. 3).

$$h_\lambda \quad = \quad \prod_{i<j} \frac{1}{(1 - R_{i,j})}\, S_\lambda \quad . \qquad 2.6$$

In Young's work, the group algebra analogue of 2.6, comes *naturally* out a close study of idempotents of Young subgroups of S_n. Then, quite brutally, the analogue of 2.4 is derived from it, by *inverting* $\prod_{i<j} (1 - R_{i,j})$ without much justification. Even his proof of 2.6 is quite sketchy at best, and only in the case when λ is a 2-part partition. A completion of Young's argument can be found in []. There is reason to believe that, in this branch of representation theory, Raising operator formulas do come naturally, and from first principles. Thus it is important that we develop a mechanism for using them that is free of misinterpretations.

Note, that there is no problem with 2.5 as long as we interpret Raising operators as acting on vectors. However, we may be misled into thinking that 2.3 defines them as linear operators acting on symmetric polynomials. This interpretation immediately leads to problems. This can be quickly illustrated. Since by definition $R(m)h_\lambda$ is to be taken equal to zero *if any component of* $R(m)\lambda$ *is negative*, then from 2.3 we get that

$$R_{1,2}^2\, h_{1,1,1} \;=\; 0$$

on the other hand, from 2.3 we also get that

$$R_{1,2}^2\, R_{2,3}\, h_{1,1,1} \;=\; h_3 \;.$$

The question then arises, is it *associativity* or *commutativity* that does not hold here? We could fix things up as far as this example is concerned by postulating that in applying a monomial $\prod_{i<j} R_{i,j}^{m_{i,j}}$ to a homogeneous function indexed by a composition we should apply the factors $R_{i,j}^{m_{i,j}}$ in order of *decreasing* j*'s*. However, this modification, doesn't help in justifying such a step as

$$\prod_{i<j} \frac{1-R_{i,j}}{1-tR_{i,j}} \;=\; \prod_{i<j}(1-R_{i,j}) \;\prod_{i<j}\frac{1}{1-tR_{i,j}} \;.$$

which seems to be implied by the derivation of formula (6.3) in [] (page 126). The plot thickens when we realize that, in the cases commonly occuring in present literature, these seemingly injustifiable associativities, do indeed hold true. An instance in point is the derivation of formula (4.30a) in Milne's paper.

In the following pages we show how a formalism may be introduced that enjoyes all the flavor and convenience of Raising operators without sharing all of their puzzling features. Indeed, we can even keep the same notation if we let Raising operators only act on vectors. To work with symmetric functions, we only need to replace 2.3 by something more precise. We illustrate our approach by deriving some classical formulas for the Hall-Littlewood polynomials and rederiving 1.16 (formula 4.30a in Milne's paper). But before we can do this we need some notation.

Let us start by rewriting 2.4. Note that since the generating function of the ordinary homogeneous functions $\{h_m(X)\}_{m\geq 0}$ can be written as

$$\sum_{m\geq 0} h_m(X)\, y^m \;=\; \Omega(Xy) \;, \qquad 2.7$$

For any composition $p \;=\; (p_1,p_2,\ldots,p_n)$ we have

$$h_{p_1}(X)h_{p_2}(X)\cdots h_{p_n}(X) \;=\; \Omega(Xy_1)\Omega(Xy_2)\cdots\Omega(Xy_n)\,\big|_{y^{p_1}y^{p_2}\cdots y^{p_n}}$$

where the symbol "|" denotes the operation of taking a coefficient. We see then that, for any vector vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, the expression

$$\Omega(Xy_1)\cdots\Omega(Xy_n)\,|_{y^{p_1+\alpha_1}\cdots y^{p_n+\alpha_n}} \;=\; \Omega(Xy_1)\cdots\Omega(Xy_n)\; y^{-\alpha_1}\cdots y^{-\alpha_n}\,|_{y^{p_1}\cdots y^{p_n}}$$

yields the symmetric function

$$h_{p+\alpha}(X) \;=\; h_{p_1+\alpha_1}(X)h_{p_2+\alpha_2}(X)\cdots h_{p_n+\alpha_n}(X)$$

with the desired feature that it vanishes when some of the components of $p+\alpha$ are negative. Thus, using 2.2, we can rewrite the right hand side of 2.3 in the form

$$\Omega(Xy_1)\Omega(Xy_2)\cdots\Omega(Xy_n)\prod_{i<j}(y_j/y_i)^{m_{i,j}}\;\;|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}}$$

In this vein the "Raising operator" formula 2.4 is written as

$$S_\lambda(X) \;=\; \Omega(Xy_1)\Omega(Xy_2)\cdots\Omega(Xy_n)\prod_{1\le i<j\le n}(1-y_j/y_i)\;\;|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \qquad 2.8$$

We shall see that this formalism achieves whatever purpose 2.4 was expected to achieve, and more, without resorting to a symbolism suggesting a direct action of the Raising operators on symmetric polynomials. The subtle difference is that, in this manner of writing, Raising operator formulas **appear** now as the composition of an action of a Raising operator on vectors followed by the application of a *linear functional* (the taking of a coefficient). We emphasize the word "appear" here since, it has secretely been this way all the time, but the old notation didn't not reflect it. This led to confusions.

Let us resort to a subscript to indicate the size of an alphabet, thus $X_n = x_1 + x_2 + \cdots + x_n$, $Y_n = y_1 + y_2 + \cdots + y_n$, etc. Let Δ be the operation of taking the Vandermonde determinant of an alphabet and set

$$\delta_n \;=\; (n-1, n-2, \ldots, 1, 0)\ .$$

With this notation 2.8 can be abbreviated to

$$S_\lambda(X_n) \;=\; \Omega(X_nY_n)\,\Delta(Y_n)\,|_{y^{\lambda+\delta_n}}\ . \qquad 2.9$$

For a composition $p = (p_1, p_2, \ldots, p_n)$ set

$$\Delta_p(X_n) \;=\; \sum_{\sigma\in S_n} sign(\sigma)\;\sigma\; x^{p+\delta_n} \;=\; det\,\|x_i^{p_j+n-j}\| \qquad 2.10$$

Suppose we only knew Schur functions through formula 2.8 (that is the situation that Young found himself in QSA IV). We shall show that all the other formulas involving S_λ are encoded in 2.9 and may be derived from it with the greatest of ease. Note first that since taking the coefficient of $y^{\lambda+\delta_n}$ in an alternating function of Y_n is equivalent to taking the coefficient of $\Delta_\lambda(Y_n)$ we immediately derive the expansion

$$\Omega(X_nY_n) \;=\; \sum_\lambda S_\lambda(X_n)\,\frac{\Delta_\lambda(Y_n)}{\Delta(Y_n)}\;. \qquad 2.11$$

So once we show the bialternant formula

$$S_\lambda(X_n) \;=\; \frac{\Delta_\lambda(X_n)}{\Delta(X_n)} \;=\; \sum_{\sigma\in S_n} \sigma\,\frac{x^{\lambda+\delta_n}}{\Delta(X_n)}\;. \qquad 2.12$$

the Cauchy formula

$$\Omega(X_nY_n) \;=\; \sum_\lambda S_\lambda(X_n)\,S_\lambda(Y_n)\;. \qquad 2.13$$

immediately follows.

In turn, 2.12 is easily derived from the recursion

$$S_{\lambda_1,\lambda_2,\ldots,\lambda_n}(X_n) \;=\; \sum_{s=1}^n \frac{x_s^{n-1+\lambda_1}}{\prod_{i\neq s}(x_s-x_i)}\,S_{\lambda_2,\ldots,\lambda_n}(X_n-x_s) \qquad 2.14$$

To see this just rewrite the second expression on the right hand side of 2.12 by decomposing $S_n = S_{[1,\ldots,n]}$ into left cosets of $S_{[2,\ldots,n]}$.

In summary, we have *recursion* $\to$ *bialternant* $\to$ *Cauchy*. Now, to complete the picture we need to go from *raising* to *recursion*. The missing link turns out to be the *partial fraction* expansion

$$\Omega(X_ny) \;=\; \prod_{i=1}^n \frac{1}{1-x_iy} \;=\; \sum_{s=1}^n \frac{x_s^{n-1}}{\prod_{i\neq s}(x_s-x_i)}\,\frac{1}{1-x_sy}\;. \qquad 2.15$$

To get 2.14 out of 2.8 we use 2.15 to extract the coefficient of $y_1^{\lambda_1}$ from the product. More precisely, we write

$$S_\lambda(X_n) = \Omega(X_ny_1)\prod_{1<j\leq k}(1-y_j/y_1)\,\Big|_{y_1^{\lambda_1}}\;\prod_{j=2}^n\Omega(X_ny_j)\prod_{2\leq i<j\leq k}(1-y_j/y_i)\;\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}}\;, \qquad 2.16$$

Now, if we set for a moment

$$\prod_{1<j\leq k}(1-y_j/y_1) \;=\; \sum_{m\geq 0} c_m\,y_1^{-m} \qquad 2.17$$

we get

$$\Omega(Xy_1) \prod_{1<j\leq k} (1-y_j/y_1)\,\big|_{y_1^{\lambda_1}} \;=\; \sum_{m\geq 0} c_m\, y_1^{-m}\, \Omega(Xy_1)\,\big|_{y_1^{\lambda_1}} \;=\; \sum_{m\geq 0} c_m\, \Omega(Xy_1)\,\big|_{y_1^{\lambda_1+m}} \;.$$

Using 2.15 (with $y = y_1$) this becomes

$$\sum_{m\geq 0} c_m \sum_{s=1}^{n} \frac{x_s^{n-1+\lambda_1+m}}{\prod_{i\neq s}(x_s-x_i)} \;=\; \sum_{s=1}^{n} \frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)} \sum_{m\geq 0} c_m x_s^m \;.$$

Replacing $1/y_1$ by x_s in 2.17 yields the identity

$$\Omega(Xy_1) \prod_{1<j\leq k} (1-y_j/y_1)\,\big|_{y_1^{\lambda_1}} \;=\; \sum_{s=1}^{n} \frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)} \prod_{2\leq i\leq n} (1-x_s y_j) \;.$$

Substituting in 2.16 and canceling out the factor involving x_s from each of the remaining kernels $\Omega(X_n y_j)$, we finally obtain

$$S_\lambda(X_n) \;=\; \sum_{s=1}^{n} \frac{x_s^{\lambda_1+n-1}}{\prod_{i\neq s}(x_s-x_i)} \prod_{j=2}^{n} \Omega\big((X_n-x_s)y_j\big) \prod_{2\leq i<j\leq k} (1-y_j/y_i)\,\big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;,$$

from which 2.14 immediately follows by another application of 2.8.

Of course, formulas 2.4,2.12,2.13,2.14 are well known and have been derived in many ways and in various order of succession. In fact, they are all equivalent to 2.8. We have carried out this derivation here to illustrate the sequence of implications

$$\left.\begin{array}{r} \textit{raising}\ (2.8) \\ \textit{partial fraction}\ (2.15) \end{array}\right\} \;\rightarrow\; \textit{recursion}\ (2.14) \;\rightarrow\; \textit{bialternant}\ (2.12) \;\rightarrow\; \textit{orthogonality}\ (2.13) \;,$$

in its simplest setting. However, there are settings where some of the corresponding identities are rather surprising and difficult to predict. Yet, given a raising operator formula, the method can be applied almost mechanically. We will give further examples.

Let us first see how the method applies to the Hall-Littlewood functions $P_\lambda(X;t)$ and $Q_\lambda(X;t)$ (we use here the same notation as in []). Let us set

$$\Omega_t(X_n) \;=\; \Omega\big((1-t)X_n\big) \;=\; \prod_{i=1}^{n} \frac{1-tx_i}{1-x_i}$$

Recalling the expansion

$$\Omega(XY) \;=\; \sum_{\lambda} h_\lambda(X)\, m_\lambda(Y) \;,$$

where h_λ and m_λ are respectively the complete homogeneous and the monomial symmetric functions, and making the replacement $X \to X\,(1-t)$ we deduce that

$$\Omega_t(XY) \;=\; \sum_\lambda h_\lambda\big(X(1-t)\big)\;\; m_\lambda(Y)\;\;. \tag{2.18}$$

Setting, as in [], for an integer m and a partition λ

$$q_m(X;q,t) \;=\; h_m\big(X(1-t)\big)\;\; and\;\; q_\lambda(X;t) \;=\; h_\lambda\big(X(1-t)\big)\;\;, \tag{2.19}$$

the *Raising operator* formula in this case is (III, 2.15 of [])

$$Q_\lambda(X_n;t) \;=\; \prod_{1\le i<j\le n} \frac{1-R_{i,j}}{1-tR_{i,j}}\;\; q_\lambda(X_n;t)\;\;. \tag{2.20}$$

Now, since the generating function of the sequence $\{q_m(X_n;t)\}_{m\ge 0}$ is

$$\Omega_t(X_ny) \;=\; \sum_{m\ge 0} q_m(X_n;t)\; y^m \;=\; \prod_{i=1}^n \frac{1-tx_iy}{1-x_iy}\;\;,$$

the corresponding *partial fraction* expansion becomes

$$\Omega_t(X_ny) \;=\; 1+(1-t)\sum_{s=1}^n \prod_{i\ne s} \frac{x_s-tx_i}{x_s-s_i}\;\frac{x_sy}{1-x_sy}\;\;. \tag{2.21}$$

Let us then apply our sequence of steps, starting from the Raising operator formula written in our notation. That is,

$$Q_\lambda(X_n;t) \;=\; \prod_{j=1}^n \Omega_t(X_ny_j) \prod_{1\le i<j\le n} \frac{1-y_j/y_i}{1-ty_j/y_i}\;\;\Big|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}}\;\;. \tag{2.22}$$

Imitating what we did in the Schur function case, we are led to evaluate first the expression

$$\Omega_t(X_ny_1) \prod_{1<j\le n} \frac{1-y_j/y_1}{1-ty_j/y_1}\;\;\Big|_{y_1^{\lambda_1}}$$

which, setting

$$\prod_{1<j\le n} \frac{1-y_j/y_1}{1-ty_j/y_1} \;=\; \sum_{m\ge 0} c_m(t)\; y_1^{-m}$$

reduces to

$$\sum_{m\ge 0} c_m(t)\;\Omega_t(Xy_1)\;\Big|_{y_1^{\lambda_1+m}}\;\;.$$

The partial fraction expansion 2.21 then gives

$$
\begin{aligned}
\Omega_t(X_n y_1) \prod_{1<j\le n} \frac{1-y_j/y_1}{1-ty_j/y_1}\ \Big|_{y_1^{\lambda_1}} &= (1-t)\sum_{m\ge 0} c_m(t) \sum_{s=1}^{n} x_s^{\lambda_1+m} \prod_{i\ne s}\frac{x_s-tx_i}{x_s-x_i}\\
&= (1-t)\sum_{s=1}^{n} x_s^{\lambda_1} \prod_{i\ne s}\frac{x_s-tx_i}{x_s-x_i} \sum_{m\ge 0} c_m(t) x_s^m\\
&= (1-t)\sum_{s=1}^{n} x_s^{\lambda_1} \prod_{i\ne s}\frac{x_s-tx_i}{x_s-x_i} \prod_{2\le i\le n} \frac{1-x_s y_j}{1-tx_s y_j}\ .
\end{aligned}
$$

Substituting this last expression in 2.22 we finally obtain

$$
Q_\lambda(X_n;t) = (1-t)\sum_{s=1}^{n} x_s^{\lambda_1} \prod_{i\ne s}\frac{x_s-tx_i}{x_s-x_i} \prod_{j=2}^{n} \Omega_t\big((X_n-x_s)y_j\big) \prod_{2\le i<j\le k} \frac{1-y_j/y_i}{1-ty_j/y_i}\ \Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}}\ ,
$$

from which, by another application of 2.22, we derive the recursion

$$
Q_{\lambda_1,\lambda_2,\ldots,\lambda_n}(X_n;t) = (1-t)\sum_{s=1}^{n} x_s^{\lambda_1} \prod_{i\ne s}\frac{x_s-tx_i}{x_s-x_i}\ Q_{\lambda_2,\ldots,\lambda_n}(X_n-x_s;t) \qquad 2.23
$$

Finally, we are led, for $\lambda=(\lambda_1,\ldots,\lambda_k,0,\ldots)$ and by k iterations of this recursion, to the *bialternant formula*

$$
\begin{aligned}
Q_\lambda(X_n;t) &= (1-t)^k \sum_{\sigma\in S_n/S_{[k+1,n]}} \sigma \prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\ x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_k^{\lambda_k}\\
&= \frac{(1-t)^k}{[n-k]_t!}\ \frac{1}{\Delta(X_n)} \sum_{\sigma\in S_n} sign(\sigma)\sigma \prod_{i<j}(x_i-tx_j)\ x^\lambda
\end{aligned} \qquad 2.24
$$

To complete the picture we only need to establish the orthogonality of the $Q_\lambda(X;t)$ with respect to the Hall-Littlewood scalar product $\langle\ ,\ \rangle_{0,t}$. This is easily obtained by an induction argument based on the recursion 2.23. We first note that the raising operator formula 2.22 implies that for any symmetric polynomial $A(X_n)$ we have

$$
\langle A, Q_\mu\rangle_{0,t} = A(Y_n)\prod_{i<j}\frac{y_i-y_j}{y_i-ty_j}\ \Big|_{y^\mu}
$$

Indeed, we have

$$
\begin{aligned}
\langle A, Q_\mu\rangle_{0,t} = \langle\ A\ ,\ \Omega_t(X_nY_n)\prod_{i<j}\frac{1-y_j/y_i}{1-ty_j/y_i}\ \Big|_{y^\mu}\ \rangle_{0,t} &= \langle A, \Omega_t(X_nY_n)\ \rangle_{0,t} \prod_{i<j}\frac{1-y_j/y_i}{1-ty_j/y_i}\ \Big|_{y^\mu}\\
&= A(Y_n)\prod_{i<j}\frac{y_i-y_j}{y_i-ty_j}\ \Big|_{y^\mu}
\end{aligned} \qquad 2.25
$$

the last equality here holds true because $\Omega_t(X_nY_n)$ is the reproducing kernel for the Hall-Littlewood scalar product. This given, using the recursion 2.23 we get

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; (1-t)\sum_{s=1}^{n}\prod_{i\neq s}\frac{y_s-ty_i}{y_s-y_i}\;\; y_s^{\lambda_1}\, Q_{\lambda*}(Y^{(s)})\quad \prod_{i<j}\frac{y_i-y_j}{y_i-ty_j}\;\;\Big|_{y^\mu}\;, \qquad 2.26$$

where, for convenience we set $\lambda* = (\lambda_2,\ldots,\lambda_n)$, $\mu* = (\mu_2,\ldots,\mu_n)$ and $Y_n^{(s)} = Y_n - y_s$. Note now that we can write

$$\prod_{i\neq s}\frac{y_s-ty_i}{y_s-y_i}\prod_{i<j}\frac{y_i-y_j}{y_i-ty_j} \;=\; \frac{ty_1-y_s}{y_1-ty_s}\cdots\frac{ty_{s-1}-y_s}{y_{s-1}-ty_s}\;\prod_{i<j}{}^{(s)}\frac{y_i-y_j}{y_i-ty_j} \qquad 2.27$$

where in the last product neither i nor j are to take the value s. Substituting 2.27 in 2.26 gives

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{s-1}\frac{ty_i-y_s}{y_i-ty_s}\; y_s^{\lambda_1}\, Q_{\lambda*}(Y^{(s)})\quad\prod_{i<j}{}^{(s)}\frac{y_i-y_j}{y_i-ty_j}\;\;\Big|_{y^\mu}\;.$$

Taking the coefficient of y_s first (in the s^{th} summand) reduces this to

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{s-1}\frac{ty_i-y_s}{y_i-ty_s}\;\Big|_{y_s^{\mu_s-\lambda_1}}\; Q_{\lambda*}(Y^{(s)})\prod_{i<j}{}^{(s)}\frac{y_i-y_j}{y_i-ty_j}\;\;\Big|_{y^\mu/y_s^{\mu_s}}\;.$$

By symmetry we can suppose that $\lambda_1 \geq \mu_1$, but then the first coefficient taking yields 0 if $\lambda_1 > \mu_s$ and t^{s-1} if $\lambda_1 = \mu_s$. That is we have

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; (1-t)\sum_{s=1}^{m} t^{s-1}\;\; Q_{\lambda*}(Y^{(s)})\prod_{i<j}{}^{(s)}\frac{y_i-y_j}{y_i-ty_j}\;\;\Big|_{y^\mu/y_s^{\mu_s}}\;.$$

where m is the multiplicity of λ_1 in μ. In other words we have shown that

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; \begin{cases} 0 & \text{if } \lambda_1\neq\mu_1, \text{ and} \\ (1-t^m)\,\langle Q_{\lambda*}, Q_{\mu*}\rangle_{0,t} & \text{if } \lambda_1=\mu_1 \end{cases}\;.$$

From this recursion we can easily deduce the classical formula

$$\langle Q_\lambda, Q_\mu\rangle_{0,t} \;=\; \begin{cases} 0 & \text{if } \lambda\neq\mu \\ \prod_{i\geq 1}(1-t)(1-t^2)\cdots(1-t^{m_i}) & \text{if } \lambda=\mu=1^{m_1}2^{m_2}\cdots \end{cases}$$

This implies that the basis $\{P_\lambda(X_n;t)\}$ dual to $\{Q_\lambda(X_n;t)\}$, with respect to the H-L scalar product is given by the formula

$$P_\lambda(X_n;t) \;=\; \frac{1}{\prod_{i\geq 1}(1-t)(1-t^2)\cdots(1-t^{m_i})}\, Q_\lambda(X_n;t)\;.$$

Combining this with formula 2.24 we can finally derive that (see [] page 104)

$$P_\lambda(X_n;t) \quad = \quad \sum_{\sigma\in S_n/S_n^\lambda} \sigma\, x^\lambda \prod_{\substack{\lambda_i>\lambda_j\\ i<j}} \frac{x_i - t x_j}{x_i - x_j}$$

where S_n^λ denotes the stabilizer of λ in S_n. In analogy with the Schur function case, we can also write this in the form

$$P_\lambda(X_n;t) \quad = \quad \frac{\Delta(X_n;t)}{\Delta(X_n)} \tag{2.28}$$

with

$$\Delta(X_n;t) \quad = \quad \sum_{\sigma\in S_n/S_n^\lambda} sign(\sigma)\sigma\, x^\lambda \prod_{\substack{\lambda_i>\lambda_j\\ i<j}} (x_i - t x_j) \prod_{\substack{\lambda_i=\lambda_j\\ i<j}} (x_i - x_j) \quad . \tag{2.29}$$

We should point out that since the system $\{P_\lambda(X_n;t)\}_{\lambda\vdash n}$ is a basis for the homogeneous symmetric functions of degree n, then the system $\{\Delta_\lambda(X_n;t)\}_{\lambda\vdash n}$ is a basis for the homogeneous alternating functions of degree $n+\binom{n}{2}$. From equation 2.25 it is not difficult to derive that for any symmetric polynomial $A(X_n)$ we have

$$A(X_n)\Delta(X_n)\,\big|_{\Delta_\lambda(X_n;t)} \quad = \quad A(X_n)\Delta(X_n) \prod_{1\le i<j\le n} \frac{1}{1-t x_j/x_i}\,\Big|_{x^{\lambda+\delta_n}} \quad . \tag{2.30}$$

This may be viewed as a formula for calculating expansions of alternating polynomials in terms of the basis $\{\Delta_\lambda(X_n;t)\}_{\lambda\vdash n}$. A direct proof of 2.30 should be of interest. The case $t=0$ is trivial of course because the polynomials $\Delta_\lambda(X_n)$, for different $\lambda' s$, have no monomials in common. This suggests that there must be some underlying combinatorial mechanism that explains the formula, also in the general case. With a direct proof of 2.30, our derivation of orthogonality from the bialternant formula would be entirely identical here to what we did in the Schur function case.

At any rate, with the establishment of orthogonality, the process in the H-L case is now complete and, in its general lines, may be summarized in the same manner as in the Schur case, namely

$$\left.\begin{array}{c} raising\ (2.20) \\ partial\ fraction\ (2.21) \end{array}\right\} \rightarrow\ recursion\ (2.23)\ \rightarrow\ bialternant\ (2.30)\ \rightarrow\ orthogonality\ (2.25)\quad .$$

It develops that the same process may be applied, at least in parts, in a number of other cases which are not so classical. We shall give two further examples: the Milne polynomials which we now briefly write as $H_\lambda(X_n;t)$ and the Schur functions of the difference of two alphabets, which in λ-ring notation, are written as $S_\lambda(X_n - Y_m)$. The latter have recently been given a beautiful bialternant formula by A. Sergeev. We shall refer to these examples as the *Milne* and *Sergeev* cases respectively.

In the Milne case the raising operator formula is

$$H_\lambda(X_n;q) \quad = \quad \prod_{i<j} \frac{1-R_{i,j}}{1-qR_{i,j}}\ h_\lambda(X_n)$$

(this is 4.9a of [] for $\gamma=0$), in our notation, we can write it as

$$H_\lambda(X_n;q) \quad = \quad \prod_{i<j} \frac{1-y_j/y_i}{1-qy_j/y_i}\ \Omega(X_nY_n)\ |_{y^\lambda} \tag{2.31}$$

Formula 4.30 a) in [] for $\gamma=0$ (and 1.16 of the present paper) is

$$H_\lambda(X_n;q) \quad = \quad \prod_{i<j} \frac{1}{1-qR_{i,j}}\ S_\lambda(X_n)\quad , \tag{2.32}$$

To derive 2.32 from 2.31 we recall that the action of a raising operator $R(m)$ on a Schur function is defined by setting

$$R(m)\ S_\lambda \quad = \quad S_{R(m)\lambda}\quad ,$$

where (see [] or []) for a general vector $p=(p_1,\ldots,p_n)$ we must let

$$S_p(X_n) \quad = \quad \frac{\Delta_p(X_n)}{\Delta(X_n)}$$

with $\Delta_p(X_n)$ given by 2.10. Now 2.31 can also be rewritten as

$$H_\lambda(X_n;q) \quad = \quad \prod_{i<j} \frac{1}{1-qy_j/y_i}\ \Omega(X_nY_n)\Delta(Y_n)\ |_{y^{\lambda+\delta_n}} \tag{2.33}$$

Using this convention, it easy to see that 2.32 and 2.33 are equivalent.

The path from 2.31 to the Milne recursion (4.7 of [] for $\gamma=0$) can also be obtained by following our process.

Since the Cauchy kernel $\Omega(X_nY_n)$ is involved, the *partial fraction* expansion must then be the same as in the Schur case, that is

$$\Omega(X_ny) \quad = \quad \prod_{i=1}^{n} \frac{1}{1-x_iy} \quad = \quad \sum_{s=1}^{n} \frac{x_s^{n-1}}{\prod_{i\neq s}(x_s-x_i)}\ \frac{1}{1-x_sy}\quad . \tag{4.34}$$

As before, we split off the first coefficient taking in 2.33 by writing

$$H_\lambda(X_n;q)=\Omega(X_ny_1)\prod_{1<j\le k}\frac{1-y_j/y_1}{1-qy_j/y_1}\,\Big|_{y_1^{\lambda_1}}\;\prod_{j=2}^{n}\Omega(X_ny_j)\prod_{2\le i<j\le k}\frac{1-y_j/y_i}{1-qy_j/y_i}\,\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}}\;, \qquad 2.35$$

Setting for a moment

$$\prod_{1<j\le k}\frac{1-y_j/y_1}{1-qy_j/y_1}\;=\;\sum_{m\ge 0}c_m(q)\;y_1^{-m} \qquad 2.36$$

we get

$$\Omega(Xy_1)\prod_{1<j\le k}\frac{1-y_j/y_1}{1-qy_j/y_1}\,\Big|_{y_1^{\lambda_1}}\;=\;\sum_{m\ge 0}c_m(q)\;\Omega(Xy_1)\,\Big|_{y_1^{\lambda_1+m}}\;.$$

Using 4.34 (with $y=y_1$) this becomes

$$\begin{aligned}\Omega(Xy_1)\prod_{1<j\le k}\frac{1-y_j/y_1}{1-qy_j/y_1}\,\Big|_{y_1^{\lambda_1}}\;&=\;\sum_{m\ge 0}c_m(q)\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1+m}}{\prod_{i\ne s}(x_s-x_i)}\\&=\;\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\ne s}(x_s-x_i)}\sum_{m\ge 0}c_m(q)x_s^m\\&=\;\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\ne s}(x_s-x_i)}\prod_{2\le i\le n}\frac{1-x_sy_j}{1-qx_sy_j}\;.\end{aligned}$$

Observe now that upon substituting this in 2.35 the effect of the factor

$$\prod_{2\le i\le n}\frac{1-x_sy_j}{1-qx_sy_j}\;.$$

will be to replace x_s by qx_s in each of the remaining kernels. We may consider this replacement as a linear operator on symmetric polynomials and denote it by T_s. With this notation the resulting formula may be written as

$$H_\lambda(X_n;q)\;=\;\sum_{s=1}^{n}\frac{x_s^{\lambda_1+n-1}}{\prod_{i\ne s}(x_s-x_i)}\;T_s\;H_{\lambda*}(X_n;q)\;\;, \qquad 2.37$$

and this is Milne's recursion.

We may be tempted to continue our process and, by iterating 2.37, arrive at what should be the present *analogue* of the bialternant formula. It develops that some care must be exerted in formulating the final result. This is best illustrated by our next example, the Sergeev case. The polynomials $S_\lambda(X_n-Y_m)$ are usually defined by means of the classical determinantal formula expressing the Schur function in terms of the homogeneous basis. In terms of raising operators the formula is

$$S_\lambda(X_n - Y_m) \;=\; \prod_{i<j} (1 - R_{i,j})\, h_\lambda(X_n - Y_m) \qquad 2.38$$

where the generating function of the sequence $h_m(X_n - Y_m)$ is

$$\sum_{m\geq 0} h_m(X_n - Y_m) \;=\; \Omega\big((X_n - Y_m)z\big) \;=\; \frac{\prod_{j=1}^{m}(1 - y_j z)}{\prod_{i=1}^{n}(1 - x_i z)}$$