The Ultimate Polynomial Problem

1. The Conjecture

We will study here a new family of Angela polynomial $P_W(x;q)$, of the form

$$P_{[w_1,w_2,\ldots,w_k]}(x;q) \;=\; \sum_{\substack{S\subseteq\{x_{-1},x_0,x_1,\ldots x_{k-1}\}\\ |S|\le k}} A_S(q)\prod_{i\in S}x_i \tag{1.1}$$

with $A_S(q)\in\mathbf{N}[q]$. The $W=[w_1,w_2,\ldots,w_k]$ need only be sequences of positive integers satisfying

$$w_1=w_2=2, \quad \& \quad w_i\le w_{i-1}+1 \tag{1.2}$$

These polynomials, may be recursively constructed by means of the linear operator $\mathbf{B}_{k,w}$, obtained by setting, for $P(x;q)$ of the form 1.1

$$\mathbf{B}_{k,w}P(x;q) \;=\; \tfrac{1}{1-q}\Big((x_k-q^w)P(x,q)\;+\;(1-x_k)P(x_{-1},\ldots,x_{k-w-1},qx_{k-w},\ldots,qx_{k-1};q)\Big) \tag{1.3}$$

where in the second term we replace each variable "x_{k-i}" for $1\le i\le w$ by "qx_{k-i}".

This given, we set

$$P_{[2]}(x;q) \;=\; qx_{-1}+x_0 \tag{1.4}$$

and having constructed $P_{[w_1,w_2,\ldots,w_k]}(x;q)$ we set

$$P_{[w_1,w_2,\ldots,w_k,w_{k+1}]}(x;q) \;=\; B_{k,w_{k+1}}P_{[w_1,w_2,\ldots,w_k]}(x;q) \tag{1.5}$$

It is easy to see from 1.3 and 1.4 that we have

$$x-\mathrm{degree}(P_W(x;q))=k \qquad \& \qquad q-\mathrm{degree}(P_W(x;q))=\sum_{i=1}^{k}(w_i-1) \tag{1.6}$$

Conjecture I (Angela Hicks)

For $W=[w_1,w_2,\ldots,w_k]$ set

$$Q_{[w_1,w_2,\ldots,w_k,w_{k+1}]}(x;q) \;=\; P_{[w_1,w_2,\ldots,w_k,w_{k+1}]}(x;q)\Big|_{x_{-1}=x_0=x_1=\cdots=x_k=x} \tag{1.7}$$

then for all W satisfying 1.2 the Q polynomials satisfy the following functional equation

$$(1-\tfrac{q}{x})Q(x;q)+x^d(1-qx)Q(\tfrac{1}{x},q) \;=\; A(q)+x^dB(q) \tag{1.8}$$

where d denotes the degree of Q, and $A(q),B(q)$ are polynomials only in the variable q.

The importance of this conjecture derives from the following result

Theorem 1.1 (Angela Hicks)

The validity of Conjecture I reduces the Compositional Shuffle Conjecture of Haglund-Morse-Zabrocki to the partitions case

More precisely, recall that we have set for any composition $p=(p_1,p_2,\ldots,p_k)$

$$\Pi_{p_1,p_2,\ldots,p_k}[X;q,t] \;=\; \sum_{PF\in\mathcal{PF}_n} t^{area(PF)}q^{dinv(PF)}Q_{ides(PF)}[X]\chi(p(PF)=p) \tag{1.9}$$

where $\mathcal{PF}_n$ denote the collection of Parking Functions in the $n \times n$ lattice square, and the sum is over the Parking Functions whose Dyck path hits the diagonal according to the composition p. This given we have

Conjecture II (Haglund-Morse-Zabrocki)

For all compositions $p = (p_1, p_2, \ldots, p_k)$ we have

$$\Pi_{p_1,p_2,\ldots,p_k}[X;q,t] \;=\; \nabla C_{p_1} C_{p_2} \cdots C_{p_k}\, 1 \tag{1.10}$$

where the C operators are defined by setting for every symmetric function $F[X]$

$$C_a F[X] \;=\; (-\tfrac{1}{q})^{a-1} F\big[X - \tfrac{1-1/q}{z}\big]\Omega[zX]\Big|_{z^a}, \tag{1.7}$$

Theorem 1. of Angela Hicks states that on the validity of Conjecture I, Conjecture II holds true for all compositions if it holds true when p is a partition.

2. An important identity

Lets us set for $P(x_{-1}, x_0, x_1, \ldots, x_{k-1}; q)$ of degree d in q

$$\downarrow P(x_{-1}, x_0, x_1, \ldots, x_{k-1}; q) \;=\; q^d x_{-1} x_0 x_1 \cdots x_{k-1} P(\tfrac{1}{x_{-1}}, \tfrac{1}{x_0}, \tfrac{1}{x_1}, \ldots \tfrac{1}{x_{k-1}}; \tfrac{1}{q})$$

We have the following basic property

Theorem 2.1(Angela Hicks)

For all W satisfying 1.2

$$\downarrow P_W(x;q) \;=\; P_W(x;q) \tag{2.1}$$

Proof

We proceed by induction on k. In the base case we have

$$\downarrow P_{[2]}(x;q) \;=\; q x_{-1} x_0 \big(\tfrac{1}{q x_{-1}} + \tfrac{1}{x_0}\big) \;=\; P_{[2]}(x;q)$$

as needed. This given, assuming 2.1 true for k we have, from 1.5 and 1.3 with $w = w_{k+1}$, and $P(x;q) = P_{[w_1,w_2,\ldots,w_k]}(x;q)$

$$\begin{aligned}
\downarrow P_{[w_1,w_2,\ldots,w_k,w_{k+1}]}(x;q) \;&=\; \frac{q^{w-1} x_k}{1-1/q}\Big((1/x_k - q^{-w}) \downarrow P(x,q) \;+\; (1-1/x_k) \downarrow P(x_{-1}, \ldots, x_{w-k-1}, q x_{k-w}, \ldots, q x_{k-1}; q)\Big) \\
&=\; \frac{1}{1-q}\Big((x_k - q^w) \downarrow P(x,q) \;+\; (1-x_k) q^w \downarrow P(x_{-1}, \ldots, x_{w-k-1}, q x_{k-w}, \ldots, q x_{k-1}; q)\Big)
\end{aligned}$$

Now the inductive hypothesis gives $\downarrow P(x,q) = P(x,q)$ and this also implies that we have

$$q^w \downarrow P(x_{-1}, \ldots, x_{w-k-1}, q x_{k-w}, \ldots, q x_{k-1}; q) \;=\; P(x_{-1}, \ldots, x_{w-k-1}, q x_{k-w}, \ldots, q x_{k-1}; q)$$

this completes the induction and the proof.

Theorem 1.2

The polynomials

$$Q_W(x;q) \;=\; P_W(x_{-1},x_0,\ldots,x_{k-1},q)\Big|_{x_{-1}=x_0=\cdots=x_{k-1}=x}$$

are none other than the original Angela polynomials without ides information. In fact, for $W(\tau)=[w_1,w_2,\ldots,w_k]$ we have

$$P_W(x_{-1},x_0,\ldots,x_{k-1},q) \;=\; \sum_{\tau(PF)=\tau} q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j} \qquad 2.2$$

Proof

Note that if τ_1 acts on w of the next elements of τ then the w descendants of the summand PF in 1.8 will yield the monomial sum

$$\big(x_k(1+q+\cdots+q^{i-1})+q^i+\cdots+q^{w-1}\big)q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j} \qquad 2.3$$

where "i" is the number of $x_{k-w},\ldots x_{k-1}$ that are in the top set of PF. This given note that

$$\prod_{\tau_j\in topset(PF)} x_{k-j}\Big|_{\substack{x_{k-j}=qx_{k-j}\\ 1\le j\le w}} \;=\; q^i \prod_{\tau_j\in topset(PF)} x_{k-j}$$

Now 2.3 can be rewritten in the form

$$\begin{aligned}
&\tfrac{1}{1-q}\big(x_k(1-q^i)+q^i(1-q^{w-i})\big)q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j} \;=\\
&\quad \tfrac{1}{1-q}\big(x_k-q^w+q^i(1-x_k)\big)q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j} \;=\\
&\quad\quad \tfrac{1}{1-q}(x_k-q^w)q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j} \;+\; \tfrac{1}{1-q}(1-x_k)q^{dinv(PF)} \prod_{\tau_j\in topset(PF)} x_{k-j}\Big|_{\substack{x_{k-j}=x_{k-j}q\\ 1\le j\le w}}
\end{aligned} \qquad 2.4$$

Using this for every summand in 2.3 the obvious inductive argument yields 2.2 in full generality.

Attempting the inductive argument

We assume by induction that the polynomial

$$Q_W(x;q) \;=\; P_W(x_{-1},x_0,\ldots,x_{k-1},q)\Big|_{x_{-1}=x_0=\cdots=x_{k-1}=x}$$

satisfies the functonal identity

$$(1-\tfrac{q}{x})Q_W(x;q) \,+\, (1-qx)x^kQ_W(\tfrac{1}{x};q) \;=\; A(q)+x^kB(q) \tag{1.10}$$

We want to show that the same is true for

$$Q_{W,w}(x;q) \;=\; P_{W,w}(x_{-1},x_0,\ldots,x_k,q)\Big|_{x_{-1}=x_0=\cdots=x_k=x}$$

where $w=w_{k+1}$ and

$$P_{W,w}(x;q) \;=\; \tfrac{1}{1-q}\Big((x_k-q^w)P_W(x,q) \,+\, (1-x_k)P_W(x_{-1},\ldots,x_{w-k-1},qx_{k-w},\ldots,qx_{k-1};q)\Big) \tag{1.11}$$

For convenience let us set

$$Q^*_W(x,q) \;=\; P_W(x_{-1},\ldots,x_{w-k-1},qx_{k-w},\ldots,qx_{k-1};q)\Big|_{x_{-1}=x_0=\cdots=x_k=x}$$

since

$$q^{\sum_{i=1}^k(w_i-1)}q^w\prod_{i=-1}^{k-1}x_iP_W(1/x_{-1},\ldots,1/x_{w-k-1},1/qx_{k-w},\ldots,1/qx_{k-1};1/q) \;=$$

$$=\; P_W(x_{-1},\ldots,x_{w-k-1},qx_{k-w},\ldots,qx_{k-1};q)$$

we derive that

$$q^{\sum_{i=1}^k(w_i-1)}q^wx^{k+1}Q^*_W(1/x,1/q) \;=\; Q^*_W(x,q)$$

Thus replacing q by $1/q$ we get

$$q^{-\sum_{i=1}^k(w_i-1)}q^{-w}x^{k+1}Q^*_W(1/x,q) \;=\; Q^*_W(x,q)$$

or better

$$Q^*_W(1/x,q) \;=\; x^{-k-1}q^{\sum_{i=1}^k(w_i-1)}q^wQ^*_W(x,q)$$

So 1.11 becomes

$$Q_{W,w}(x;q) \;=\; \tfrac{1}{1-q}\Big((x-q^w)Q_W(x,q) \,+\, (1-x)Q^*_W(x,q)\Big) \tag{1.12}$$

which also gives

$$Q_{W,w}(1/x;q) \;=\; \tfrac{1}{1-q}\Big((1/x-q^w)Q_W(1/x,q) \,+\, (1-1/x)Q^*_W(1/x,q)\Big)$$

or better

$$Q_{W,w}(1/x;q) \;=\; \tfrac{1}{1-q}x^{-1}\Big((1-xq^w)Q_W(1/x,q) \,+\, (x-1)x^{-k-2}q^{\sum_{i=1}^k(w_i-1)}q^wQ^*_W(x,q)\Big) \tag{1.13}$$

Thus

$$(1-q)\Big((1-\tfrac{q}{x})Q_{W,w}(x;q) \,+\, (1-qx)x^{k+2}Q_{W,w}(\tfrac{1}{x};q)\Big) \;=$$

$$\begin{aligned}
&=\; (1-\tfrac{q}{x})\Big((x-q^w)Q_W(x,q) \,+\, (1-x)Q^*_W(x,q)\Big)\\
&\quad+\, (1-qx)x^{k+2}x^{-1}\Big((1-xq^w)Q_W(1/x,q) \,+\, (x-1)x^{-k-2}q^{\sum_{i=1}^k(w_i-1)}q^wQ^*_W(x,q)\Big)\\
&=\; (1-\tfrac{q}{x})\Big((x-q^w)Q_W(x,q) \,+\, (1-x)Q^*_W(x,q)\Big)\\
&\quad+\, \Big((1-xq^w)x^{k+1}(1-qx)Q_W(1/x,q) \,+\, (1-qx)(x-1)x^{-1}q^{\sum_{i=1}^k(w_i-1)}q^wQ^*_W(x,q)\Big)
\end{aligned}$$

To this end we start with

$$\begin{aligned}\prod_{i=-1}^{k} x_i P_{W,w}(\tfrac{1}{x};q) &= \tfrac{\prod_{i=-1}^{k} x_i}{1-q}\Big((1/x_k-q^w)P_W(1/x,q)\ +\ (1-1/x_k)P_W(1/x_{-1},\dots,1/x_{w-k-1},q/x_{k-w},\dots,q/x_{k-1};q)\Big)\\ &= A(x;q)\ +\ B(x;q)\end{aligned}$$

Now

$$\begin{aligned}A(x;q) &= \tfrac{x_k}{1-q}(1/x_k-q^w)\prod_{i=-1}^{k-1} x_i P_W(1/x,q)\\ &= \tfrac{1}{1-q}(1-x_kq^w)q^{\sum_{i=1}^{k}(w_i-1)}P_W(x,1/q)\\ &= \tfrac{1}{1-q}(1-x_kq^w)q^{-w+1+\sum_{i=1}^{k+1}(w_i-1)}P_W(x,1/q)\\ &= \tfrac{1}{1-q}(q^{-w+1}-x_kq)q^{\sum_{i=1}^{k}(w_i-1)}P_W(x,1/q)\end{aligned}$$

and so

$$A(x;q)\Big|_{x_{-1}=x_0=\cdots=x_k=x} = q^{\sum_{i=1}^{k}(w_i-1)}\tfrac{1}{1-q}(q^{-w+1}-xq))Q_W(x;1/q)$$

Next we have

$$\begin{aligned}B(x;q) &= \tfrac{\prod_{i=-1}^{k} x_i}{1-q}(1-1/x_k)P_W(1/x_{-1},\dots,1/x_{w-k-1},q/x_{k-w},\dots,q/x_{k-1};q)\\ &= q^{\sum_{i=1}^{k}(w_i-1)+w}\tfrac{1}{1-q}(x_k-1)P_W(x_{-1},\dots,x_{w-k-1},x_{k-w}/q,\dots,x_{k-1}/q;1/q)\end{aligned}$$

But let us recall that

$$P_{W,w}(x;q) = \tfrac{1}{1-q}\Big((x_k-q^w)P_W(x,q)\ +\ (1-x_k)P_W(x_{-1},\dots,x_{w-k-1},qx_{k-w},\dots,qx_{k-1};q)\Big)$$

Diagonal Harmonics Revisited

Let $H_n(X)$ denote the space of ordinary S_n harmonics in $x_1, x_2, \ldots, x_n$ and $H_n(Y)$ denote the space of ordinary S_n harmonics in $y_1, y_2, \ldots, y_n$. Recall that the "*descent monomial* " corresponding to a permutation $\alpha \in S_n$ is by definition the monomial

$$m_\alpha(x) \;=\; \prod_{\substack{i=1 \\ \alpha_i > \alpha_{i+1}}}^{n-1} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_i}$$

For instance the descent monomials corresponding to the $\alpha \in S_3$ are

$$1,\;\; x_1 x_3,\;\; x_2\;\; x_3 x_2,\;\; x_3,\;\; x_3^2 x_2$$

It can be shown that, with $\Delta_n(x)$ the Vandemonde in $x_1, x_2, \ldots, x_n$, the collection

$$\{m_\alpha(\partial_x)\Delta_n(x)\}_{\alpha \in S_n}$$

is a basis for $H_n(X)$. Since the Diagonal Harmonic polynomials are a subspace of the harmonics of $S_n \otimes S_n$, and the later have basis

$$\{m_\alpha(\partial_x)\Delta_n(x) \times m_\beta(\partial_y)\Delta_n(y)\}_{\alpha,\beta \in S_n}$$

I wrote a program for expanding a Diagonal Harmonic polynomial in terms of this basis. To my surprise I observed that the expansions, for a general Diagonal Harmonic $P(x; y)$ have the peculiar form

$$P(x,y) \;=\; \frac{1}{C_P}\Big(\sum_{\alpha,\beta \in \mathcal{C}_P} \pm m_\alpha(\partial_x) m_\beta(\partial_y)\Delta_n(x)\Delta_n(y)\Big) \tag{$*$}$$

where C_P is a constant and $\mathcal{C}_P$ is a collection of pairs $\alpha, \beta \in S_n$ depending on P

Then I explored the expansions of the polynomials $\Delta_\mu(x; y)$. Recall that for a partition $\mu \vdash n$, if $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are the coordinates of thge cells of the Ferrers diagram of μ then

$$\Delta_\mu(x,y) \;=\; \det \| x_i^{a_j} y_i^{b_j} \|_{i,j=1}^n$$

For general μ I did not notice anything different than in (*). However for μ a hook I noticed expansions of the form

$$\Delta_\mu(x,y) \;=\; \frac{1}{C_\mu}\Big(\sum_{\sigma \in \mathcal{F}_\mu} \pm m_\sigma(\partial_x) m_{\sigma'}(\partial_y)\Delta_n(x)\Delta_n(y)\Big)$$

where σ' denotes the reverse of σ, and $\mathcal{F}_\mu \subseteq S_n$ is again a family of permutations (depending on μ). More over if $\mathcal{F}_\mu$ contains both a permutation σ and its reverse σ' then the signs of the corresponding terms are opposite. If this is true in general, it must be due to a curious combinatorial mechanism that might be worth understanding.

Note that in order for an element

$$P(x,y) \;=\; \sum_{\alpha,\beta \in S_n} c_{\alpha,\beta}\, m_\alpha(\partial_x) m_\beta(\partial_y)\Delta_n(x)\Delta_n(y)\Big)$$

to be Diagonal Harmonic it is necessary and sufficient that we have

$$\sum_{\alpha,\beta\in S_n} c_{\alpha,\beta} m_\alpha(\partial_x) m_\beta(\partial_y) \sum_{i=1}^{n} \partial_{x_i}^r \Delta_n(x) \partial_{y_i}^s \Delta_n(y)\Big) \;=\; 0 \qquad \forall \quad 0 < r+s \leq n$$

By homogeneity we must thus also have

$$\sum_{\substack{\alpha,\beta\in S_n\\ |\alpha|=a,\,|\beta|=b}} c_{\alpha,\beta} m_\alpha(\partial_x) m_\beta(\partial_y) \sum_{i=1}^{n} \partial_{x_i}^r \Delta_n(x) \partial_{y_i}^s \Delta_n(y)\Big) \;=\; 0 \qquad \forall \quad 0 < r+s \leq n \quad \text{and} \quad 0 \leq a,b \leq \binom{n}{2}$$