Rodriguez Formulas and Orthogonality for Schur, Hall-Littlewood and Macdonald (?) Polynomials

1. The Schur function case

As customary, we start by defining Schur functions by means of the Jacobi formula

$$S_\lambda(x_1,x_2,\ldots,x_n) \;=\; \frac{\Delta_\lambda(x_1,x_2,\ldots,x_n)}{\Delta(x_1,x_2,\ldots,x_n)} \;\;,$$

where for convenience we have set

$$a) \qquad \Delta_\lambda(x_1,x_2,\ldots,x_n) \;=\; det \parallel x_i^{\lambda_j+n-j} \parallel_{i,j=1}^{n} \;\;,$$

$$b) \qquad \Delta(x_1,x_2,\ldots,x_n) \;=\; \Delta_0(x_1,x_2,\ldots,x_n) \;=\; det \parallel x_i^{n-j} \parallel_{i,j=1}^{n} \;\;.$$

We shall make extensive use of plethystic notation here. This is a notational device which expresses the “substitution” of an arbitrary expression $E(t_1,t_2,t_3\ldots)$ into a symmetric polynomial P. The result will be denoted by $P[E]$ and it is simply obtained by expanding P in terms of the power basis and then “substituting” E into the power symmetric functions occcurring in this expansion. More precisely if the power sum expansion of P is

$$P \;=\; \sum_\rho c_\rho \; p_\rho \;\;,$$

then

$$P[E] \;=\; \sum_\rho c_\rho \; p_\rho \;\Big|_{p_k \rightarrow E(t_1^k,t_2^k,t_3^k,\ldots)}$$

We shall always use square brackets for plethystic substitutions and ordinary parentheses for customary ones. For instance $P(y_1,y_2,\ldots,y_n)$ denotes P evaluated at $y_1,y_2,\ldots,y_n$. Note that since we clearly have

$$P(y_1,y_2,\ldots,y_n) \;=\; \sum_\rho c_\rho \; p_\rho \;\Big|_{p_k \rightarrow y_1^k+y_2^k+\cdots+y_n^k}$$

we see that if $Y_n = y_1+y_2+\cdots+y_n$ then we may write

$$P(y_1,y_2,\ldots,y_n) \;=\; P[Y_n] \;\;.$$

Keeping this in mind, we can state the basic recursion satisfied by Schur functions.

Proposition 1.1

For any $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ we have

$$S_{\lambda_1\lambda_2\cdots\lambda_n}(x_1,x_2,\ldots,x_n) \;=\; \sum_{i=1}^{n} \frac{x_i^{\lambda_i}}{\prod_{j=1,j\neq i}^{n}(1-x_j/x_i)} \; S_{\lambda_2\cdots\lambda_n}[X_n - x_i] \qquad 1.1$$

Proof

We may write the left coset decomposition of $S_{[2,n]}$ in S_n in the form

$$S_n \;=\; \tau_1\, S_{[2,n]} \;+\; \tau_2\, S_{[2,n]} \;+\; \cdots \;+\; \tau_n\, S_{[2,n]} \;, \qquad \text{where} \;\; \tau_i = (1,i)\;.$$

Thus

$$\begin{aligned}
S_{\lambda_1\lambda_2\cdots\lambda_n}(x_1,x_2,\ldots,x_n) &= \sum_{\sigma\in S_n} \sigma\; \frac{x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n+n-n}}{\prod_{1\le i<j\le n}(x_i-x_j)} \\
&= \sum_{i=1}^n \tau_i \left(\frac{x_1^{\lambda_1+n-1}}{\prod_{j=2}^n (x_1-x_j)} \sum_{\sigma\in S_{[2,n]}} \sigma\; \frac{x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n+n-n}}{\prod_{2\le i<j\le n}(x_i-x_j)} \right) \\
&= \sum_{i=1}^n \tau_i \left(\frac{x_1^{\lambda_1+n-1}}{\prod_{j=2}^n (x_1-x_j)} \; S_{\lambda_2\cdots\lambda_n}[X_n-x_1] \right) \\
&= \sum_{i=1}^n \frac{x_i^{\lambda_i+n-1}}{\prod_{j=1,j\ne i}^n (x_i-x_j)} \; S_{\lambda_2\cdots\lambda_n}[X_n-x_i] \\
&= \sum_{i=1}^n \frac{x_i^{\lambda_i}}{\prod_{j=1,j\ne i}^n (1-x_j/x_i)} \; S_{\lambda_2\cdots\lambda_n}[X_n-x_i]
\end{aligned}$$

QED

This result may be recast in a manner which is reminiscent of a classical formula of Rodriguez in the one variable theory of orthogonal polynomials. We simply let $\mathbf{H}_m$ be the operator on symmetric polynomials in $x_1, x_2, \cdots, x_{n-1}$ defined by setting

$$\mathbf{H}_m\, P \;=\; \sum_{i=1}^n \Big(\prod_{j=1}^{(i)} \frac{1}{1-x_j/x_i} \Big)\, x_i^m\; P[X_n - x_i]\;. \qquad 1.2$$

It should be note that $\mathbf{H}_m$ transforms a homogeneous symmetric polynomial of degree k in $y_1, y_2, \ldots, y_{n-1}$ into one of degree $m+k$ in $x_1, x_2, \ldots, x_n$.

This given, we obtain, as an immediate corollary of 1.1, the so-called "Rodriguez" formula for Schur functions:

Theorem 1.1

For $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ we have

$$S_{\lambda_1\lambda_2\cdots\lambda_n}(x_1,x_2,\ldots,x_n) \;=\; \mathbf{H}_{\lambda_1}\mathbf{H}_{\lambda_2}\cdots\mathbf{H}_{\lambda_n}\;\mathbf{1} \qquad 1.3$$

Proof

Formula 1.1 simply says that

$$S_{\lambda_1\lambda_2\cdots\lambda_n}(x_1,x_2,\ldots,x_n) \;=\; \mathbf{H}_{\lambda_1}\; S_{\lambda_2\cdots\lambda_n}\;.$$

Iterating this relation gives 1.3, provided the relation is true at the start. That is we need only verify that

$$\mathbf{H}_{\lambda_n}\,\mathbf{1} \;=\; S_{\lambda_n}[x_1, x_2, \ldots, x_n] \ .$$

Since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, we see that we must have $\lambda_n = 0$ or $\lambda_n = 1$. In either case this identity is an immediate consequence of the partial fraction expansion

$$\prod_{j=1}^{n} \frac{1}{1 - z\,x_i} \;=\; \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j\neq i}^{n} (1 - x_j/x_i)} \; \frac{1}{1 - z\,x_i} \qquad 1.4$$

In fact, setting $z = 0$ here gives that

$$\sum_{i=1}^{n} \frac{1}{\prod_{j=1, j\neq i}^{n} (1 - x_j/x_i)} \;=\; 1 \ .$$

Thus we obtain that $\mathbf{H}_0\,\mathbf{1} = 1$ as desired. On the other hand equating coefficients of z in 1.4 yields that

$$\sum_{i=1}^{n} \frac{x_i}{\prod_{j=1, j\neq i}^{n} (1 - x_j/x_i)} \;=\; x_1 + x_2 + \cdots + x_n \ .$$

Thus we see that $\mathbf{H}_1\,\mathbf{1} = S_1(x_1, x_2, \ldots, x_n)$. This compltetes our proof.

We should mention at this point that the bialternant formula yields an algorithm for computing Schur function expansions. In fact we have the following useful result:

Lemma 1.1

If P is any symmetric polynomial, then the coefficients $c_\lambda(P)$ in the expansion

$$P(x_1, x_2, \ldots, x_n) \;=\; \sum_{\lambda} c_\lambda(P)\;\; S_\lambda(x_1, x_2, \ldots, x_n)$$

are given by the formula

$$c_{\lambda_1\lambda_2\cdots\lambda_n}(P) \;=\; P(x_1, x_2, \ldots, x_n) \prod_{1<\leq i<j\leq n} (x_i - x_j)\ \Big|_{x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n+n-n}}$$

Proof

Since the product $\Delta(x_1, x_2, \ldots, x_n)\; P(x_1, x_2, \ldots, x_n)$ is an alternant it may be expanded as a linear combination of the polynomials $\Delta_\lambda(x_1, x_2, \ldots, x_n)$. From the bialternant formula we see that the coefficient of Δ_λ in this expansion must be given precisely by $c_\lambda(P)$. Thus our formula follows from the fact that the Δ_λ is the only one of the Δ_μ which contains the monomial

$$x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n+n-n}$$

.

Our next move is to show that the Rodriguez formula yields the orthogonality of the Schur functions with respect to the Hall scalar product. To this end recall that this scalar product is defined by setting for the power basis

$$\langle p_\lambda \, , \, p_\mu \rangle \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu \, , \\ z_\mu & \text{if } \lambda = \mu \, , \end{cases}$$

where for a partition $\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n}$

$$z_\mu \;=\; 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n! \ . \qquad 1.5$$

The following result will be useful in the sequel:

Proposition 1.2

Let $\langle \, , \, \rangle_f$ be a scalar product defined by setting for the power basis

$$\langle p_\lambda \, , \, p_\mu \rangle_f \;=\; \begin{cases} 0 & \textit{if } \lambda \neq \mu \, , \\ f_\mu & \textit{if } \lambda = \mu \, , \end{cases} \qquad 1.6$$

where f_μ is any family of non zero scalars indexed by partitions. Set

$$\Omega_f[X_n, Y_n] \;=\; \sum_\lambda \frac{p_\lambda[X_n] p_\lambda[Y_n]}{f_\lambda} \ . \qquad 1.7$$

Then the two symmetric function bases $\{A_\lambda\}_\lambda$, $\{B_\lambda\}_\lambda$ are dual with respect to the scalar product in 1.6 if and only if

$$\sum_\lambda A_\lambda[X_n] \;\; B_\lambda[Y_n] \;=\; \Omega_f[X_n, Y_n] \ . \qquad 1.8$$

Proof

Note that from 1.6 and 1.7 we derive that

$$\langle p_\lambda[X_n] \, , \, \Omega_f[X_n, Y_n] \rangle_f \;=\; p_\lambda[Y_n]$$

This implies that for every symmetric polynomial P we have

$$\langle P[X_n] \, , \, \Omega_f[X_n, Y_n] \rangle_f \;=\; P[Y_n]$$

In particular for every λ we must have

$$\langle A_\lambda[X_n] \, , \, \Omega_f[X_n, Y_n] \rangle_f \;=\; A_\lambda[Y_n] \ . \qquad 1.9$$

Thus if $\{A_\lambda\}_\lambda$ and $\{B_\lambda\}_\lambda$ are $\langle \, , \, \rangle_f$-dual this gives that

$$\Omega_f[X_n, Y_n] \;=\; \sum_\mu A_\mu[X_n] \;\; B_\mu[Y_n] \ .$$

Conversely, if this holds true, then 1.9 gives that

$$\sum_{\mu} \left\langle A_\lambda[X_n]\ ,\ A_\mu[X_n]\right\rangle_f\ B_\mu[Y_n] \ =\ A_\lambda[Y_n]\ .$$

from which we derive that

$$\left\langle A_\lambda[X_n]\ ,\ A_\mu[X_n]\right\rangle_f \ =\ \begin{cases} 0 & \text{if } \lambda \neq \nu \\ 1 & \text{if } \lambda = \nu \end{cases}$$

Thus the condition in 1.8 is indeed necessary and sufficent for the asserted duality of the two bases.

Our next result is interesting in itself and may be used as a templet for constructing identities that will play the same role in similar calculations.

Proposition 1.3

For any polynomial P (in one variable) we have

$$P\,(\,1/z\,)\ \prod_{j=1}^{n} \frac{1}{1-z\ y_i}\ \Big|_{z^m} \ =\ \sum_{i=1}^{n} \frac{y_i^m}{\prod_{j=1,j\neq i}^{n}(1-y_j/y_i)}\ P(y_i) \qquad 1.10$$

Proof

The partial fraction decomposition in 1.4 (rewritten with $y_1, y_2, \ldots, y_n$)

$$\prod_{j=1}^{n} \frac{1}{1-z\ y_i} \ =\ \sum_{i=1}^{n} \frac{1}{\prod_{j=1,j\neq i}^{n}(1-y_j/y_i)}\ \frac{1}{1-z\ y_i}$$

gives

$$P\,(\,1/z\,)\ \prod_{j=1}^{n} \frac{1}{1-z\ y_i}\ \Big|_{z^m} \ =\ \sum_{i=1}^{n} \frac{1}{\prod_{j=1,j\neq i}^{n}(1-y_j/y_i)} \left(\ P\,(\,1/z)\quad \frac{1}{1-z\ y_i}\ \Big|_{z^m}\right) \qquad 1.11$$

Now if $P(x) = x^p$ we see that

$$P\,(\,1/z)\quad \frac{1}{1-z\ y_i}\ \Big|_{z^m} \ =\ \frac{1}{z^p}\ \ \frac{1}{1-z\ y_i}\ \Big|_{z^m} \ =\ y_i^{m+p} \ =\ y_i^m\ \ P(y_i)\ .$$

Thus by linearity we must have for all polynomials P

$$P\,(\,1/z)\quad \frac{1}{1-z\ y_i}\ \Big|_{z^m} \ =\ y_i^m\ \ P(y_i)\ .$$

Substituting this in 1.11 gives 1.10 precisely as stated.

It will be convenient here and after to set, for any expression E:

$$\Omega[E] \ =\ exp\Big[\sum_{k\geq 1} \frac{p_k[E]}{k}\ \Big]\ . \qquad 1.12$$

We should also note that by repetitive uses of the exponential formula

$$\frac{1}{1-x} \;=\; exp\Big[\sum_{k\geq 1} \frac{x^k}{k}\Big] \qquad 1.13$$

we can easily derive that we have

$$\Omega[E] \;=\; exp\Big[\sum_{k} \frac{p_k[E]}{k}\Big] \;=\; \sum_{\lambda} \frac{p_\lambda[E]}{z_\lambda} \;. \qquad 1.14$$

where the sum on the right is over all partitions and z_λ is as given in 1.5.

Another basic identity we shall often use may be stated as follows:

Proposition 1.4

Let $E^+ = \sum_{m^+\in\mathcal{M}^+} m^+$ and $E^- = \sum_{m^-\in\mathcal{M}^-} m^-$ with $\mathcal{M}^+$ and $\mathcal{M}^-$ two multisets of monomials, then

$$\Omega[E^+ - E^-] \;=\; \sum_{\lambda} \frac{p_\lambda[E^+ - E^-]}{z_\lambda} \;=\; \prod_{m^+\in\mathcal{M}^+} \frac{1}{1-m^+} \prod_{m^-\in\mathcal{M}^-} (1-m^-) \;. \qquad 1.15$$

Proof

From our definition of plethystic substitution it immediately follows that

$$p_k[E^+ - E^-] \;=\; p_k[E^+] \;-\; p_k[E^-] \;.$$

This gives

$$\Omega[E^+ - E^-] \;=\; exp\Big[\sum_{k\geq 1} \frac{p_k[E^+]}{k}\Big] \Big/ exp\Big[\sum_{k\geq 1} \frac{p_k[E^-]}{k}\Big] \qquad 1.16$$

Now since

$$p_k[E^+] \;=\; \sum_{m^+\in\mathcal{M}^+} (m^+)^k \;,$$

repetitive uses of 1.13 yields that

$$exp\Big[\sum_{k} \frac{p_k[E^+]}{k}\Big] = exp\Big[\sum_{k} \frac{\sum_{m^+\in\mathcal{M}^+} (m^+)^k}{k}\Big] = \prod_{m^+\in\mathcal{M}^+} exp\Big[\sum_{k} \frac{(m^+)^k}{k}\Big] = \prod_{m^+\in\mathcal{M}^+} \frac{1}{1-m^+} \;.$$

Similarly we deduce that

$$exp\Big[\sum_{k} \frac{p_k[E^-]}{k}\Big] \;=\; \prod_{m^-\in\mathcal{M}^-} \frac{1}{1-m^-} \;.$$

and 1.15 follows from 1.16.

In view of Proposition 1.2 to show the Hall inner product orthogonality of the Schur functions we need only verify that

$$\sum_{\lambda} S_\lambda[X_n] S_\lambda[Y_n] \;=\; \sum_{\lambda} \frac{p_\lambda[X_n] p_\lambda[Y_n]}{z_\lambda} \;.$$

Now 1.15 for $E^- = 0$ and

$$E^+ \;=\; X_nY_n \;=\; \sum_{i=1}\sum_{j=1} x_iy_j$$

gives

$$\sum_\lambda \frac{p_\lambda[X_n]p_\lambda[Y_n]}{z_\lambda} \;=\; \prod_{i=1}^n\prod_{j=1}^n \frac{1}{1-x_iy_j} \;.$$

This given, we see that the Hall orthogonality of the Schur function turns out to be one of the Corollaries of the celebrated Cauchy formula:

Theorem 1.2

$$\Omega[X_nY_n] \;=\; \prod_{i=1}^n\prod_{j=1}^n \frac{1}{1-x_iy_j} \;=\; \sum_\lambda S_\lambda[X_n]\,S_\lambda[Y_n] \;. \tag{1.17}$$

Proof

Since $\Omega[X_nY_n]$ is symmetric in $y_1, y_2, \ldots, y_n$ and the Schur functions are a basis we may write

$$\Omega[X_nY_n] \;=\; \sum_\lambda c_\lambda[X_n]\,S_\lambda[Y_n]$$

with the coefficient $c_\lambda[X_n]$ necessarily a symmetric function of $x_1, x_2, \ldots, x_n$. To see what it is we make use of Lemma 1.1 and get that for $\lambda = \lambda_1\lambda_2\cdots\lambda_n$

$$c_\lambda[X_n] \;=\; \Delta[Y_n]\,\Omega[X_nY_n]\,\Big|_{y_1^{\lambda_1+n-1}y_2^{\lambda_2+n-2}\cdots y_n^{\lambda_n+n-n}} \;.$$

More explicitely we have

$$c_\lambda[X_n] \;=\; \prod_{1\le i<j\le n}(1-y_j/y_i)\prod_{i=1}^n\prod_{j=1}^n \frac{1}{1-x_iy_j}\,\Big|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;.$$

Factoring out the terms that contain y_1 we may write

$$c_\lambda[X_n] \;=\; \prod_{j=2}^n(1-y_j/y_1)\prod_{i=1}^n\frac{1}{1-x_iy_1}\,\Big|_{y_1^{\lambda_1}}\;\prod_{2\le i<j\le n}(1-y_j/y_i)\prod_{i=1}^n\prod_{j=2}^n\frac{1}{1-x_iy_j}\,\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;. \tag{1.18}$$

Using Proposition 1.3 we derive that

$$\prod_{j=2}^n(1-y_j/y_1)\prod_{i=1}^n\frac{1}{1-x_iy_1}\,\Big|_{y_1^{\lambda_1}} \;=\; \sum_{s=1}^n \frac{x_s^{\lambda_1}}{\prod_{r=1,r\neq s}^n(1-x_r/x_s)}\prod_{j=2}^n(1-y_jx_s) \;.$$

Substituting this into 1.18 we get

$$c_\lambda[X_n] \;=\; \sum_{s=1}^n \frac{x_s^{\lambda_1}}{\prod_{r=1,r\neq s}^n(1-x_r/x_s)}\;\prod_{2\le i<j\le n}(1-y_j/y_i)\;\prod_{i=1,i\neq s}^n\;\prod_{j=2}^n\frac{1}{1-x_iy_j}\,\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;.$$

Note that this may also be rewritten as

$$c_{\lambda_1\lambda_2\cdots\lambda_n}[X_n] \;=\; \sum_{s=1}^{n} \frac{x_s^{\lambda_1}}{\prod_{r=1,r\neq s}^{n}(1-x_r/x_s)}\; c_{\lambda_2\cdots\lambda_n}[X_n - x_s]$$

In other words we have shown (see the proof of Theorem 1.1), that $c_\lambda[X_n]$ satisfies the same recursion as S_λ. Since $c_0[X_n] = S_0[X_n] = 1$ we must necessarily have

$$c_\lambda[X_n] \;=\; S_\lambda[X_n]$$

as desired. This completes the proof of the Cauchy identity.

We should mention that our operators $\mathbf{H}_m$ may be imbedded into a single operator which may be viewed as their generating function. We simply set for any symmetric polynomial P

$$\mathbf{H}(z)\,P \;=\; \Omega[zX_n]\;P[X_n - \tfrac{1}{z}] \;. \tag{1.19}$$

Then it immediately follows from Proposition 1.3 that

$$\mathbf{H}_m\,P \;=\; \Omega[zX_n]\;P[X_n - \tfrac{1}{z}]\,\Big|_{z^m} \;. \tag{1.20}$$

To see this we simply note that from our definition of plethystic substitution it immediately follows that if P is any symmetric polynomial, then we have symmetric polynomials $P^{(k)}$ yielding the expansion

$$P[X_n - x] \;=\; \sum_{k\geq 0} x^k\,P^{(k)}[X_n] \tag{1.21}$$

Thus, for a fixed alphabet X_n, the expression $P[X_n - \frac{1}{z}]$ may be interpreted as a polynomial in one variable evaluated at $1/z$. This given, we can apply formula 1.10 with the $y's$ replaced by $x's$ and derive that 1.20 is another way of writing 1.2. We should note that 1.20 together with our observations concerning the nature of $P[X_n - \frac{1}{z}]$ constitutes a proof that $\mathbf{H}_m$ sends symmetric polynomials into symmetric polynomials.

To complete this case we need to see what becomes of the Rodriguez formula when instead of of indexing the successive operators by the parts of a partition we index them by the parts of a composition. It turns out that the answer is as good as can be possibly expected.

Proposition 1.5

For any integers a, b we have

$$\mathbf{H}_a\;\mathbf{H}_b \;=\; -\;\mathbf{H}_{b-1}\,\mathbf{H}_{a+1} \;. \tag{1.22}$$

Proof

Note that applying our definition 1.20 for $z = z_1$ and $z = z_2$ we derive that for any symmetric polynomial P

$$\mathbf{H}(z_1)\;\mathbf{H}(z_2)\;P[X_n] \;=\; \Omega\big[z_1X_n\big]\Omega\big[z_2(X_n - \tfrac{1}{z_1})\big]\;P\big[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2}\big] \;,$$

and since

$$\Omega\big[z_2(-\tfrac{1}{z_1})\big] \;=\; \Omega\big[-\tfrac{z_2}{z_1}\big] \;=\; (1 - z_2/z_1) \;\;,$$

we finally derive that

$$\mathbf{H}(z_1)\;\mathbf{H}(z_2)\;\; P[X_n] \;\;=\;\; \Omega\big[(z_1+z_2)X_n\big]\,(1-z_2/z_1)\,P\big[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2}\big] \;\;.$$

Interchanging z_1 and z_2 we also get that

$$\mathbf{H}(z_2)\;\mathbf{H}(z_2)\;\; P[X_n] \;\;=\;\; \Omega\big[(z_1+z_2)X_n\big]\,(1-z_1/z_2)\,P\big[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2}\big] \;\;.$$

Consequently we must have

$$z_1\,\mathbf{H}(z_1)\,\mathbf{H}(z_2) \;\;+\;\; z_2\,\mathbf{H}(z_2)\,\mathbf{H}(z_1) \;\;=\;\; 0 \;\;.$$

In particular for any pair of integers a, b we get

$$\mathbf{H}_a\,\mathbf{H}_b \;\;=\;\; z_1\,\mathbf{H}(z_1)\,\mathbf{H}(z_2)\,\Big|_{z_1^{a+1}}\Big|_{z_2^b} \;\;=\;\; -\;z_2\,\mathbf{H}(z_2)\,\mathbf{H}(z_1)\,\Big|_{z_1^{a+1}}\Big|_{z_2^b} \;\;=\;\; -\;\mathbf{H}_{b-1}\,\mathbf{H}_{a+1} \;\;.$$

QED

An immediate corollary of this identity is the following useful fact:

Theorem 1.3

If we set, for any composition $p = (p_1, p_2, \cdots, p_n)$:

$$S_{p_1p_2\cdots p_n}[X_n] \;\;=\;\; \frac{\det \|x_i^{p_j+n-j}\|_{i=1}^n}{\det \|x_i^{n-j}\|_{i=1}^n} \;\;, \qquad 1.23$$

then for $p_i < p_{i+1}$ we have

$$S_{p_1\cdots p_i p_{i+1}\cdots p_n}[X_n] \;\;=\;\; \begin{cases} -S_{p_1\cdots(p_{i+1}-1)(p_i+1)\cdots p_n}[X_n] & \text{if } p_i + 1 \neq p_{i+1} \\ 0 & \text{otherwise} \;\;. \end{cases} \qquad 1.24$$

Proof

Note that in our proof of the Rodriguez formula we made no use of the fact that the parts of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are weakly decreasing. Thus our argument there gives that for any composition $p = (p_1, p_2, \ldots, p_n)$ we have just as well

$$\frac{\det \|x_i^{p_j+n-j}\|_{i=1}^n}{\det \|x_i^{n-j}\|_{i=1}^n} \;\;=\;\; \mathbf{H}_{p_1}\mathbf{H}_{p_2}\cdots\mathbf{H}_{p_n}\,\mathbf{1} \;\;.$$

Note then that when $p_i + 1 = p_{i+1}$ the identity in 1.22 simply yields that $\mathbf{H}_{p_i}\mathbf{H}_{p_{i+1}} = 0$. Otherwise 1.22 gives that

$$\mathbf{H}_{p_i}\mathbf{H}_{p_{i+1}} \;\;=\;\; -\,\mathbf{H}_{p_{i+1}-1}\mathbf{H}_{p_i+1} \;\;.$$

Thus 1.24 holds true precisely as asserted.

We should note that in the course of proof of Theorem 1.2 we established that the Schur function S_λ may also be computed according to the formula

$$S_{\lambda_1\lambda_2\cdots\lambda_n}[X_n] \;=\; \prod_{1\le i<j\le n}(1-y_j/y_i)\prod_{i=1}^n\prod_{j=1}^n\frac{1}{1-x_iy_j}\Big|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;. \qquad 1.25$$

Now it was already note by Alfred Young (in QSA IV) that this expresion can be easily converted into the very useful Jacobi-Trudi identity:

Theorem 1.4

For any partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ we have

$$S_{\lambda_1\lambda_2\cdots\lambda_n}[X_n] \;=\; det\,\big\|h_{\lambda_i+j-i}[X_n]\,\big\|_{i,j=1}^n \qquad 1.26$$

with the convention that $h_k[X_n]=0$ for negative k.

Proof

Recall that the homogeneous symmetric function $h_k[X_n]$ is simply the coefficient of y^k in $\Omega[yX_n]$. That is we have

$$\prod_{i=1}^n\frac{1}{1-y\,x_i} \;=\; \sum_{p\ge 0}y^p\,h_p[X_n]\;.$$

Thus we may write

$$\prod_{i=1}^n\prod_{j=1}^n\frac{1}{1-x_iy_j} \;=\; \sum_{p_1,p_2,\ldots,p_n\ge 0}y_1^{p_1}y_2^{p_2}\cdots y_n^{p_n}\;h_{p_1}[X_n]h_{p_2}[X_n]\cdots h_{p_n}[X_n]\;. \qquad 1.27$$

Note next that we may write

$$\begin{aligned}\prod_{1\le i<j\le n}(1-y_j/y_i) &= \prod_{1\le i<j\le n}(y_i-y_j)\prod_{i=1}^n\frac{1}{y_i^{n-i}}\\ &= \sum_{\sigma\in S_n}sign(\sigma)\,\frac{y_1^{n-\sigma_1}y_2^{n-\sigma_2}\cdots y_n^{n-\sigma_n}}{y_1^{n-1}y_2^{n-2}\cdots y_n^{n-n}}\\ &= \sum_{\sigma\in S_n}sign(\sigma)\,\frac{1}{y_1^{\sigma_1-1}y_2^{\sigma_2-2}\cdots y_n^{\sigma_n-n}}\;.\end{aligned}$$

Substituting this in 1.25 we derive that

$$\begin{aligned}S_{\lambda_1\lambda_2\cdots\lambda_n}[X_n] &= \sum_{\sigma\in S_n}sign(\sigma)\,\frac{1}{y_1^{\sigma_1-1}y_2^{\sigma_2-2}\cdots y_n^{\sigma_n-n}}\prod_{i=1}^n\prod_{j=1}^n\frac{1}{1-x_iy_j}\Big|_{y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_n^{\lambda_n}}\\ &= \sum_{\sigma\in S_n}sign(\sigma)\,\prod_{i=1}^n\prod_{j=1}^n\frac{1}{1-x_iy_j}\Big|_{y_1^{\lambda_1+\sigma_1-1}y_2^{\lambda_2+\sigma_2-2}\cdots y_n^{\lambda_n+\sigma_n-n}} \qquad 1.25\\ &= \sum_{\sigma\in S_n}sign(\sigma)\;h_{\lambda_1+\sigma_1-1}h_{\lambda_2+\sigma_2-2}\cdots h_{\lambda_n+\sigma_n-n}\;.\end{aligned}$$

Where the last equality follows from 1.27 and the convention that $h_k[X_n] = 0$ for k negative. This completes our proof since, with this convention, this last sum is just the expansion of the determinant in the right hand side of 1.26

2. The Hall-Littlewood case

Following Macdonald we start by setting, for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$,

$$R_{\lambda_1\lambda_2\cdots\lambda_n}(x_1, x_2, \ldots, x_n) \;=\; \frac{1}{\Delta(x)} \sum_{\sigma \in S_n} sign(\sigma)\, \sigma \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \prod_{1 \leq i < j \leq n} (x_i - t x_j) \right) \; . \qquad 2.1$$

Our first result here is a complete analogue of Proposition 1.1. Namely, we have

Proposition 2.1

For any $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

$$R_{\lambda_1\lambda_2\cdots\lambda_n}(x_1, x_2, \ldots, x_n; t) \;=\; \sum_{i=1}^{n} \Big(\prod_{j=1, j \neq i}^{n} \frac{x_i - t x_j}{x_i - x_j} \Big)\, x_i^{\lambda_1}\; R_{\lambda_2\cdots\lambda_n}[X_n - x_i] \; . \qquad 2.2$$

Proof

Since we can also write

$$R_{\lambda_1\lambda_2\cdots\lambda_n}(x_1, x_2, \ldots, x_n; t) \;=\; \sum_{\sigma \in S_n} \sigma \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{x_i - t x_j}{x_i - x_j} \right) \; ,$$

the same left coset decomposition $S_n = \tau_1 S_{[2,n]} + \tau_2 S_{[2,n]} + \cdots + \tau_n S_{[2,n]}$ we used in section 1, gives

$$\begin{aligned} R_{\lambda_1\lambda_2\cdots\lambda_n}(x_1, x_2, \ldots, x_n; t) \;&=\; \sum_{i=1}^{n} \tau_i \left(x_1^{\lambda_1} \prod_{j=2}^{n} \frac{x_1 - t x_j}{x_1 - x_j} \sum_{\sigma \in S_{[2,n]}} \sigma \Big(x_2^{\lambda_2} \cdots x_n^{\lambda_n} \prod_{2 \leq i < j \leq n} \frac{x_i - t x_j}{x_i - x_j} \Big) \right) \\ &=\; \sum_{i=1}^{n} x_i^{\lambda_1} \Big(\prod_{j=1, \neq i}^{n} \frac{x_i - t x_j}{x_i - x_j} \Big)\; \tau_i\;\; R_{\lambda_2\cdots\lambda_n}[X_n - x_1; t] \\ &=\; \sum_{i=1}^{n} x_i^{\lambda_1} \Big(\prod_{j=1, \neq i}^{n} \frac{x_i - t x_j}{x_i - x_j} \Big)\; R_{\lambda_2\cdots\lambda_n}[X_n - x_i; t] \; . \end{aligned}$$

QED

Now it develops that the polynomials $\{R_\lambda(x_1, x_2, \ldots, x_n; t)\}_\lambda$ are an orthogonal basis with respect to the "Hall-Littlewood" scalar product $\big\langle \;,\; \big\rangle_t$ which is defined by setting for the power basis

$$\big\langle p_\lambda \,,\, p_\mu \big\rangle_t \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu \,, \\ z_\mu / p_\mu[1-t] & \text{if } \lambda = \mu \,. \end{cases} \qquad 2.3$$

Note that since

$$\prod_{i=1}^{n} \prod_{j=1}^{n} \frac{1 - t x_i y_j}{1 - x_i y_j} \;=\; \Omega[(1-t) X_n Y_n] \;=\; \sum_{\rho} \frac{p_\rho[X_n] p_\rho[X_n]}{z_\rho / p_\rho[1-t]}$$

from Proposition 1.2 we immediately derive the following basic facts

Proposition 2.2

Two bases $\{A_\lambda\}_\lambda$, $\{B_\lambda\}_\lambda$ are dual with respect to the scalar product $\langle\ ,\ \rangle_t$ if and only if

$$\Omega[(1-t)X_nY_n] \;=\; \sum_\lambda A_\lambda[X_n]B[Y_n]$$

Moreover, for any symmetric polynomial $P[X_n]$ we have

$$\Big\langle\, P[X_n]\,,\ \Omega[(1-t)X_nY_n]\Big\rangle_t \;=\; P[Y_n] \tag{2.5}$$

Proof

We need only note that 2.5 is but the particular case $f_\mu = z_\mu/p_\mu[1-t]$ of one of the identities derived in the proof of Proposition 1.2.

We call $\Omega[(1-t)X_nY_n]$ the Hall-Littlewood Kernel and shall refer to 2.5 as the reproducing property. We shall also define the Hall-Littlewood polynomial $Q_\lambda(x_1,x_2,\ldots,x_n;t)$ by the so called "Raising Operator formula". Namely we set

$$Q_\lambda(x_1,x_2,\ldots,x_n;t) \;=\; \prod_{1\le i<j\le n}\frac{1-y_j/y_i}{1-ty_j/y_i}\,\Omega[(1-t)X_nY_n]\,\big|_{y_1^{\lambda_1}\cdots y_n^{\lambda_n}}\ . \tag{2.6}$$

Now it develops that this polynomial differs only by a factor from the polynomial $R_\lambda(x_1,x_2,\ldots,x_n;t)$ we defined in 2.1. To see how this comes about we need the following Hall-Littlewood analogue of Proposition 1.3.

Proposition 2.3

For any formal series $F(x)=\sum_{k\ge 0}\ c_k\ x^k$ and for all integers $m\ge 1$ we have

$$F(1/z)\,\Omega[(1-t)zX_n]\,\Big|_{z^m} \;=\; (1-t)\sum_{i=1}^n\prod_{j=1,j\ne i}^n\frac{x_i-tx_j}{x_i-x_j}\ x_i^m\,F(x_i) \tag{2.7}$$

Proof

This is an immediate consequence of the partial fraction decomposition

$$\Omega[(1-t)zX_n] \;=\; t^n \;+\; (1-t)\sum_{i=1}^n\prod_{j=1,j\ne i}^n\frac{x_i-tx_j}{x_i-x_j}\,\frac{1}{1-zx_i}\ . \tag{2.8}$$

In fact, using this in the left hand side of 2.7 gives

$$F(1/z)\,\Omega[(1-t)zX_n]\,\Big|_{z^m} \;=\; (1-t)\sum_{i=1}^n\prod_{j=1,j\ne i}^n\frac{x_i-tx_j}{x_i-x_j}\,\frac{1}{1-zx_i}\ F(1/z)\,\Big|_{z^m} \tag{2.9}$$

and since for $F(x)=x^l$ we have

$$\frac{1}{1-zx_i}\ F(1/z)\,\Big|_{z^m} \;=\; \frac{1}{1-zx_i}\ \frac{1}{z^l}\,\Big|_{z^m} \;=\; x_i^{m+l} \;=\; x_i^m\,F(x_i)$$

the desired result in 2.7 follows by linearity from 2.9.

Proposition 2.4

$$Q_\lambda(x_1,x_2,\ldots,x_n;t) \;=\; (1-t)^n\, R_\lambda(x_1,x_2,\ldots,x_n;t) \qquad 2.10$$

Proof

Factoring out the dependence on y_1 in the right hand side of 2.6 we can write

$$\begin{aligned} Q_\lambda[X_n;t] = \prod_{j=2}^{n} \frac{1-y_j/y_1}{1-ty_j/y_1}\Omega[(1-t)y_1X_n]\Big|_{y_1^{\lambda_1}} \times \\ \times \prod_{2\le i<j\le n} \frac{1-y_j/y_i}{1-ty_j/y_i}\,\Omega[(1-t)X_n(Y_n-y_1)]\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;. \end{aligned} \qquad 2.11$$

Now, an application of Proposition 2.3 gives

$$\begin{aligned} \prod_{j=2}^{n} \frac{1-y_j/y_1}{1-ty_j/y_1}\Omega[(1-t)y_1X_n]\Big|_{y_1^{\lambda_1}} &= (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1,j\ne i}^{n}\frac{x_i-tx_j}{x_i-x_j}\Big)\; x_i^m \prod_{j=2}^{n}\frac{1-x_iy_j}{1-tx_iy_j} \\ &= (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1,j\ne i}^{n}\frac{x_i-tx_j}{x_i-x_j}\Big)\; x_i^m\,\Omega[-(1-t)x_iY_n] \;. \end{aligned}$$

Substituting this back into 2.11 gives

$$\begin{aligned} Q_\lambda[X_n;t] = (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1,j\ne i}^{n}\frac{x_i-tx_j}{x_i-x_j}\Big)\; x_i^m \;\times \\ \times \prod_{2\le i<j\le n} \frac{1-y_j/y_i}{1-ty_j/y_i}\,\Omega[(1-t)(X_n-x_i)(Y_n-y_1)]\Big|_{y_2^{\lambda_2}\cdots y_n^{\lambda_n}} \;, \end{aligned}$$

and this may be rewritten as

$$Q_{\lambda_1\lambda_2\cdots\lambda_n}[X_n] \;=\; (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1,j\ne i}^{n}\frac{x_i-tx_j}{x_i-x_j}\Big)\, x_i^{\lambda_1}\, Q_{\lambda_2\cdots\lambda_n}[X_n-x_i] \;. \qquad 2.12$$

This shows that $Q_\lambda/(1-t)^n$ satisfies the same recursion as R_λ thus they must be the same if they are the same at the start. However for $n=1$ the definition in 2.6 gives

$$Q_{\lambda_1}(x_1;t) \;=\; \Omega[(1-t)x_1y_1]\Big|_{y_1^{\lambda_1}} \;=\; \tfrac{1-tx_1y_1}{1-x_1y_1}\Big|_{y_1^{\lambda_1}} \;=\; (1-t)\,x_1^{\lambda_1} \;.$$

On the other hand 2.1 for $n=1$ reduces to

$$R_{\lambda_1}(x_1;t) \;=\; x_1^{\lambda_1} \;=\; Q_{\lambda_1}(x_1;t)/(1-t) \;.$$

This completes our proof.

It develops that the Raising Operator formula 2.6 implies a Hall-Littlewood analogue of Lemma 1.1:

Lemma 2.1

For any symmetric polynomial $R[X_n]$ and any partition $\mu=(\mu_1\geq\mu_2\geq\cdots\geq\mu_k)$ we have

$$\langle\, R\,,\, Q_\mu\,\rangle_t \;=\; R[Z_n]\prod_{1\leq i<j\leq n}\frac{1-z_j/z_i}{1-tz_j/z_i}\,\Big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \qquad 2.13$$

Proof

From the definition 2.6 we get (using the linearity of the scalar product)

$$\langle\, R[X_n]\,,\, Q_\mu[X_n;t]\,\rangle_t \;=\; \langle\, R[X_n]\,,\, \Omega[(1-t)X_nZ_n]\,\rangle_t \prod_{1\leq i<j\leq n}\frac{1-z_j/z_i}{1-tz_j/z_i}\,\Bigg|_{z_1^{\mu_1}\cdots z_n^{\mu_n}}\;. \qquad 2.14$$

On the other hand the reproducing property in 2.5 yields that

$$\langle\, R[X_n]\,,\, \Omega[(1-t)X_nZ_n]\,\rangle_t \;=\; R[Z_n]$$

and 2.13 follows by substituting this in 2.14

We are now in a position to prove the basic orthogonality result.

Theorem 2.1

For any two partitions λ and μ we have

$$\langle\, Q_\lambda\,,\, Q_\mu\,\rangle_t \;=\; \begin{cases} 0 & \textit{if } \lambda\neq\mu \\ \prod_{i=1}^n (t)_{m_i} & \textit{if } \lambda=\mu=1^{m_1}2^{m_2}\cdots n^{m_n} \end{cases}\;, \qquad 2.15$$

where for any integer m we set $(t)_m=(1-t)(1-t^2)\cdots(1-t^m)$.

Proof

It is not difficult to see that only monomials of degree $\lambda_1+\lambda_2+\cdots+\lambda_n$ can come out of the the right hand side of 2.6. That means that $Q_\lambda[X_n;t]$ will necessarily be a homogeneous symmetric polynomial of that degree. Thus the orthogonality of Q_λ and Q_μ when λ and μ are partitions of different integers is trivial. We may assume that λ and μ are both partitions of n. We shall set

$$\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)\quad,\quad \mu=(\mu_1\geq\mu_2\geq\cdots\geq\mu_n\geq 0)\;. \qquad 2.16$$

It is also convenient to set

$$\lambda^*=(\lambda_2,\lambda_3,\ldots,\lambda_n)\;,\;\mu^*=(\mu_2,\mu_3,\ldots,\mu_n)\;.$$

Clearly by the symmetry of the scalar product we can also assume that

$$\lambda_1\geq\mu_1 \qquad 2.17$$

Under these conventions we shall show that

$$\langle\, Q_\lambda\,,\, Q_\mu\,\rangle_t \;=\; \begin{cases} 0 & \text{if } \lambda_1>\mu_1 \\ (1-t^m)\langle\, Q_{\lambda^*}\,,\, Q_{\mu^*}\,\rangle_t & \text{if } \lambda_1=\mu_1 \end{cases} \qquad 2.18$$

where m denotes the multiplicity of λ_1 in μ. This given, 2.15 will immediately follow by induction on n, (the theorem is trivially true for $n=0$).

Now, it develops that 2.18 is a simple consequence of the identities 2.12 and 2.13. In fact, using 2.13 we can write

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; Q_\lambda[Z_n] \prod_{1\le i<j\le n} \frac{1-z_j/z_i}{1-tz_j/z_i}\Big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \;, \qquad 2.19$$

on the other hand 2.12 with the $x_i's$ replaced by the $z_i's$ gives

$$\langle\, Q_\lambda, Q_\mu \,\rangle_t = (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\frac{z_s-tz_i}{z_s-z_i}\, z_s^{\lambda_1}\, Q_{\lambda^*}[Z_n-z_s;t] \prod_{1\le i<j\le n}\frac{1-z_j/z_i}{1-tz_j/z_i}\Big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \;. \qquad 2.20$$

Note that we can write

$$\begin{aligned}\prod_{i=1}^{n}{}^{(s)}\frac{z_s-tz_i}{z_s-z_i}\prod_{1\le i<j\le n}\frac{z_i-z_j}{z_i-tz_j} \;&=\; \frac{z_s-tz_1}{z_s-z_1}\cdots\frac{z_s-tz_{s-1}}{z_s-z_{s-1}}\\ &\quad\times\;\frac{z_1-z_s}{z_1-tz_s}\cdots\frac{z_{s-1}-z_s}{z_{s-1}-tz_s}\prod_{1\le i<j\le n}{}^{(s)}\frac{z_i-z_j}{z_i-tz_j}\end{aligned}$$

where the superscript (s) in the last product is to indicate that neither i nor j are to take the value s. Carrying out the cancellations we get

$$\frac{tz_1-z_s}{z_1-tz_s}\cdots\frac{tz_{s-1}-z_s}{z_{s-1}-tz_s}\prod_{1\le i<j\le n}{}^{(s)}\frac{z_i-z_j}{z_i-tz_j}$$

and substituting in 2.20 gives

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{s-1}\frac{tz_i-z_s}{z_i-tz_s}\, z_s^{\lambda_1}\, Q_{\lambda^*}[Z_n-z_s;t]\prod_{1\le i<j\le n}{}^{(s)}\frac{z_i-z_j}{z_i-tz_j}\Big|_{z^\mu} \;.$$

Taking first the coefficient of $z_s^{\mu_s}$ in the s^{th} summand reduces this to

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{s-1}\frac{tz_i-z_s}{z_i-tz_s}\Big|_{z_s^{\mu_s-\lambda_1}}\, Q_{\lambda^*}[Z_n-z_s;t]\prod_{1\le i<j\le n}{}^{(s)}\frac{z_i-z_j}{z_i-tz_j}\Big|_{z^\mu/z_s^{\mu_s}} \;.$$

This clearly evaluates to zero if $\lambda_1>\mu_1$. On the other hand, when $\lambda_1=\mu_1$ we see that the first factor in the s^{th} summand evaluates to t^{s-1} if $\mu_s=\lambda_1$ and yields zero if $\mu_s<\lambda_1$. This implies that

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t)\sum_{s=1}^{m} t^{s-1}\, Q_{\lambda^*}[Z_n-z_s;t]\prod_{1\le i<j\le n}{}^{(s)}\frac{z_i-z_j}{z_i-tz_j}\Big|_{z^\mu/z_s^{\mu_s}} \;.$$

Making a further use of 2.14, we get that when $\lambda_1=\mu_1$

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t)\Big(\sum_{s=1}^{m} t^{s-1}\Big)\;\langle\, Q_{\lambda^*} \,,\, Q_{\mu^*} \,\rangle_t \;.$$

This establishes 2.18 as desired.

This theorem has the following immediate corollary:

Theorem 2.2

For $\mu = 1^{m_1}2^{m_2}\cdots n^{m_n}$ *set*

$$d_\mu(t) \;=\; \prod_{i=1}^{n} (t)_{m_i} \tag{2.21}$$

and let

$$P_\mu[X_n;t] \;=\; \frac{1}{d_\mu(t)}\, Q_\mu[X_n;t] \ . \tag{2.22}$$

Then

$$\Omega[(1-t)X_nY_n] \;=\; \sum_\mu Q_\mu[X_n;t]P_\mu[Y_n;t] \tag{2.23}$$

Proof

Formula 2.15 simply says that the family $\{P_\mu[X_n;t]\}_\mu$ defined by 2.21 and 2.22 is dual to $\{Q_\mu[X_n;t]\}_\mu$ with respect to the scalar product $\langle\ ,\ \rangle_t$. Thus the identity in 2.23 is an immediate consequence of Proposition 2.2.

Note that the family of polynomials $\{S_\lambda[(1-t)X_n]\}_\lambda$ is also a basis for the symmetric functions in n variables. In fact, from the Frobenius formula

$$S_\lambda[X_n] \;=\; \sum_\rho \chi^\lambda_\rho\, p_\rho[X_n]/z_\rho \ , \tag{2.24}$$

we derive the expansion

$$S_\lambda[(1-t)X_n] \;=\; \sum_\rho \chi^\lambda_\rho\, p_\rho[X_n]p_\rho[1-t]/z_\rho \ , \tag{2.25}$$

and the assertion follows from the invertibility of the matrix $\chi = \|\chi^\lambda_\rho\|$.

Remark 2.1

More generally we can easily see that if $\{A_\lambda[X_n]\}_\lambda$ is a symmetric function basis so must be the family $\{A_\lambda[(1-t)X_n]\}_\lambda$. This is due to the simple reason that the matrix expressing $\{A_\lambda[(1-t)X_n]\}_\lambda$ in terms of the power basis differs from that expressing $\{A_\lambda[X_n]\}_\lambda$ only by pre-multiplication by the invertible diagonal matrix with elements $p_\rho[1-t]$.

We must therefore have an expansion of the form

$$Q_\mu[X_n;t] \;=\; \sum_\lambda S_\lambda[(1-t)X_n]\; K_{\lambda\mu}(t) \tag{2.26}$$

In view of the definitions 2.6 and 2.24 all we can say about the coefficients $K_{\lambda\mu}(t)$ is that they are rational functions of t. However, it is well known that they are indeed polynomials with non negative integer coefficients. This result was originally a consequence of deep developments in algebraic geometry (see []). It was also proved by Lascoux-Schützenberger ([],[] and []) who showed that the coefficients of $K_{\lambda\mu}(t)$ count certain column strict tableau of shape λ and content μ. Just the same, their proof, though elementary, is considerably intricate and only accessible to a handful of experts in the combinatorics of the Robinson-Schensted correspondence. An elementary and reasonably accessible representation theoretical proof was recently given in []. Here we shall show that the integral polynomiality of the $K_{\lambda\mu}(t)$ may be easily derived from the machinery we have put together. But before we can do so we need to establish some further identities.

To this end it is convenient to set

$$H_\mu[X_n;t] \;=\; Q_\mu[X_x/(1-t);t] \tag{2.27}$$

In view of the defining formula 2.6 we see that this is equivalent to setting

$$H_\mu[X_n;t] \;=\; \prod_{1\le i<j\le n} \frac{1-z_j/z_i}{1-tz_j/z_i}\,\Omega[X_nZ_n]\Big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \;. \tag{2}$$

This yields us the following remarkable fact

Theorem 2.3

The coefficients $K_{\lambda\mu}(t)$ are polynomials which are different from zero if and only if $\lambda \ge \mu$. Moreover we have

$$H_\mu[X_n;t] \;=\; \sum_{\lambda\ge\mu} S_\lambda[X_n]\,K_{\lambda\mu}(t) \tag{2.29}$$

with

$$a)\quad K_{\lambda\mu}(t)\,|_{t=1} = K_{\lambda\mu} \qquad\qquad b)\quad K_{\mu\mu}(t)\equiv 1 \tag{2.30}$$

the latter denoting the classical Kostka coefficients yielding the expansion

$$h_\mu[X_n] \;=\; \sum_{\lambda\ge\mu} S_\lambda[X_n]\,K_{\lambda\mu} \tag{2.31}$$

of the homogeneous basis in terms of the Schur basis

Proof

Note that our definition 2.28 can also be rewritten in the form

$$H_\mu[X_n;t] \;=\; \left(\prod_{1\le i<j\le n}\frac{1}{1-tz_j/z_i}\right)\frac{\Delta[Z_n]}{\prod_{i=1}^n z_i^{n-i}}\,\Omega[X_nZ_n]\Big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \;. \tag{2.32}$$

Now we may expand the first product into a formal Laurent series in the variables $z_1, z_2, \ldots, z_n$ and obtain

$$\prod_{1\le i<j\le n}\frac{1}{1-tz_j/z_i} \;=\; \sum_p \frac{c_p(t)}{z^p} \tag{2.32}$$

where the sum is carried out over all exponent vectors p which may be written in the form

$$p \;=\; \sum_{1\le i<j\le n} p_{i,j}(e^{(i)}-e^{(j)}) \;\;, \tag{2.33}$$

with $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$ the unit coordinate vectors of n-dimensional space and the $p_{i,j}$ arbitrary non-negative integers. Moreover, it is easy to see that

$$c_p(t) \;=\; t^{\sum_{1\le i<j\le n}\; p_{ij}} \tag{2.34}$$

Substituting this into 2.32 gives

$$H_\mu[X_n;t] \;=\; \sum_p c_p(t)\,\Delta[Z_n]\,\Omega[X_nZ_n]\Big|_{z^{\mu+\delta+p}} \;\;. \tag{2.35}$$

where for convenience we have set

$$\delta \;=\; (n-1,n-2,\ldots,1,0) \;\;.$$

Now from the Cauchy Formula we derive that

$$\Delta[Z_n]\,\Omega[X_nZ_n] \;=\; \sum_\lambda S_\lambda[X_n]\,\Delta_\lambda[Z_n] \;\;.$$

Using this, 2.35 becomes

$$H_\mu[X_n;t] \;=\; \sum_\lambda S_\lambda[X_n] \sum_p c_p(t)\,\Delta_\lambda[Z_n]\Big|_{z^{\mu+\delta+p}} \;\;.$$

From which we derive that

$$K_{\lambda\mu}(t) \;=\; \sum_p c_p(t)\,\Delta_\lambda[Z_n]\Big|_{z^{\mu+\delta+p}} \;\;. \tag{2.36}$$

To make this formula a bit more explicit, we need to introduce a notation. Given a vector

$$q=(q_1,q_2,\cdots,q_n)$$

with distinct components, let $\sigma(q)=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ be the permutation that rearranges the components of q in decreasing order and set

$$q^*=(q^*_1,q^*_2,\cdots,q^*_n)=(q_{\sigma_1},q_{\sigma_2},\ldots,q_{\sigma_n})$$

Finally, for any q set

$$\epsilon(q) \;=\; \begin{cases} sign(\sigma(q)) & \text{if } q \text{ has distinct components and} \\ 0 & \text{otherwise} \end{cases}$$

This given we can easily see that 2.36 can also be written in the form

$$K_{\lambda\mu}(t) \;=\; \sum_p c_p(t)\,\epsilon(\mu+\delta+p)\,\chi\Big(\,(\mu+\delta+p)^*=\lambda+\delta\,\Big)\;.\;. \qquad 2.37$$

Now it is not difficult to see that the condition $(\mu+\delta+p)^*=\lambda+\delta$ together with 2.34 forces this to be a finite sum of signed monomials. Thus $K_{\lambda\mu}(t)$ is a polynomial in t with integer coefficients as asserted.

Note further that, since each of the vectors $e^{(i)}-e^{(j)}$ has non-negative partial sums, 2.33 gives that

$$p_1+p_2+\cdots+p_k\geq 0 \qquad (\;\forall\;\; k=1,2,\ldots,n\;)\;.$$

If $(\mu+\delta+p)*=\lambda+\delta$ and τ is the inverse of $\sigma(\mu+\delta+p)$ we shall have

$$\lambda_{\tau_i}+n-\tau_i \;=\; \mu_i+n-i+p_i$$

Thus for all $k=1,2,\ldots,n$:

$$\sum_{i=1}^{k}(\lambda_{\tau_i}+n-\tau_i)\;\geq\;\sum_{i=1}^{k}(\mu_i+n-i)$$

and since

$$\sum_{i=1}^{k}(\lambda_i+n-i)\;\geq\;\sum_{i=1}^{k}(\lambda_{\tau_i}+n-\tau_i)\;.$$

We deduce that

$$\sum_{i=1}^{k}\lambda_i\;\geq\;\sum_{i=1}^{k}\mu_i \qquad (\;\forall\;\; k=1,2,\ldots,n\;)\;.$$

This shows the first assertion of the theorem and 2.29 is then given by 2.26 and 2.27. Moreover, we see that the right hand side of defining formula 2.28, for $t=1$, reduces to

$$\Omega[X_nZ_n]\,\big|_{z_1^{\mu_1}\cdots z_n^{\mu_n}} \;=\; h_\mu[X_n]$$

and thus 2.30 a) is an immediate consequence of 2.31. This also shows that $K_{\lambda\mu}(t)$ cannot be vanishing identically when $\lambda\geq\mu$. Finally we see from 2.36 for $\lambda=\mu$ that the only way to get $(\mu+\delta+p)*=\mu_\delta$ is for $p=0$ in which case $\sigma(\mu+\delta+p)$ is the identity permutation. This yields 2.30 b) and our proof is now complete.

It will be good to keep in mind also the following identities.

Theorem 2.4

The bases $\{H_\mu[X_n;t]\}_\mu$ and $\{P_\mu[X_n;t]\}_\mu$ are dual with respect to the customary Hall scalar product, that is

$$\Omega[X_nY_n] \;=\; \sum_\mu H_\mu[X_n;t]P_\mu[Y_n;t] \qquad 2.38$$

in particular we must have

$$S_\lambda[X_n] \;=\; \sum_{\mu\le\lambda} K_{\lambda\mu}(t)\; P_\mu[X_n;t] \;\; . \qquad 2.39$$

Proof

The first of these identities is simply obtained by replacing X_n by $X_n/(1-t)$ in 2.23 and using the definition 2.27. The second is obtained by eliminating $H_\mu[X_n;t]$ from 2.38 by means of 2.29 obtaining

$$\Omega[X_nY_n] \;=\; \sum_{\lambda} S_\lambda[X_n]\,\Big(\sum_{\mu\le\lambda} K_{\lambda\mu}(t)\; P_\mu[Y_n;t]\Big)$$

then equating coefficients of $S_\lambda[X_n]$ on the right hand sides of this identity and the classical Cauchy formula

$$\Omega[X_nY_n] \;=\; \sum_{\lambda\ge\mu} S_\lambda[X_n]S_\lambda[Y_n] \;\; .$$

Formulas 2.29 and 2.34 yield us two important special cases.

Theorem 2.5

$$\begin{aligned} a)\quad H_n[X_n;t] \;&=\; h_n[X_n] \\ b)\quad H_{1^n}[X_n;t] \;&=\; (1-t)(1-t^2)\cdots(1-t^n)e_n[X_n/(1-t)] \end{aligned} \qquad 2.40$$

Proof

The first of these identities is an immediate consequence of 2.29. Indeed, since there is no partition that stricly dominates the ”one row” partition (n), 2.29 reduces to

$$H_n[X_n;t] \;=\; S_n[X_n]K_{nn}(t)$$

and 2.36 a) follows then from 2.30 b) and the equality of $S_n[X_n]$ and $h_n[X_n]$. For the second identity, we use 2.39 for $\lambda=1^n$. Here, since there is no partition that is strictly below 1^n in dominance, we end up with

$$S_{1^n}[X_n] \;=\; K_{1^n1^n}(t)P_{1^n}[X_n;t] \;\; .$$

Using 2.21, 2.22 and 2.30 b) we then get that

$$Q_{1^n}[X_n;t] \;=\; \prod_{i=1}^{n}(1-t^i)\; P_{1^n}[X_n;t] \;=\; \prod_{i=1}^{n}(1-t^i)\; S_{1^n}[X_n] \;\; .$$

and 2.36 b) then follows from 2.27 and the fact that $S_{1^n}[X_n]$ is another way of denoting the elementary symmetric function $e_n[X_n]$.

To complete this section we need to derive the Rodriguez formulas for Q_μ and H_μ. We start with Q_μ. Our clue for the definition of the vertex operator $\mathbf{Q}_m$ givinq Q_μ should come from the recursion in 2.12. In fact, we do set

$$\mathbf{Q}_m\; P[X_n] \;=\; (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1,j\ne i}^{n}\frac{x_i-tx_j}{x_i-x_j}\Big)\, x_i^m\; P[X_n-x_i] \;\; . \qquad 2.37$$

This may also be rewritten in an equivalent plethystic form by means of Proposition 2.3. To be precise we get.

Proposition 2.5

$$\mathbf{Q}_m\, P[X_n] \;=\; \Omega[(1-t)zX_n]\, P[X_n - \tfrac{1}{z}]\,\big|_{z^m}$$

Proof

Using the identity in 2.7 with $F(x) = P[X_n - x]$ gives that

$$\Omega[(1-t)zX_n]\, P[X_n - \tfrac{1}{z}]\,\big|_{z^m} \;=\; (1-t)\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n}\frac{x_i - tx_j}{x_i - x_j}\Big)\, x_i^m\, P[X_n - x_i]\ .$$

QED

It will be convenient to use the generating function of the operators $\mathbf{Q}_m$ and set

$$\mathbf{Q}(z)\, P \;=\; \Omega[(1-t)zX_n]\, P[X_n - \tfrac{1}{z}]\ . \qquad 2.38$$

Note that two succesive applications of this operator may be written in the form

$$\begin{aligned}\mathbf{Q}(z_1)\mathbf{Q}(z_2)\, P &= \Omega[(1-t)z_1X_n]\, \Omega[(1-t)\, z_2(X_n - \tfrac{1}{z_1})]\, P[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2}] \\ &= \Omega[(t-1)z_2/z_1]\, \Omega[(1-t)(z_1+z_2)X_n]\, P[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2}]\ .\end{aligned} \qquad 2.39$$

Proceeding by induction we can easily see that n successive iterations of $\mathbf{Q}(z)$ may be written as

$$\mathbf{Q}(z_1)\mathbf{Q}(z_2)\cdots\mathbf{Q}(z_n)\, P = \Big(\prod_{1\le i<j\le n}\frac{1 - z_j/z_i}{1 - tz_j/z_i}\Big)\Omega[(1-t)Z_nX_n]\, P[X_n - \tfrac{1}{z_1} - \tfrac{1}{z_2} - \cdots - \tfrac{1}{z_n}]\ . \qquad 2.40$$

This identity contains the Rodiguez formula for the Hall Litlewood polynomials.

Theorem 2.4

$$Q_\lambda[X_n, t] \;=\; \mathbf{Q}_{\lambda_1}\mathbf{Q}_{\lambda_2}\cdots\mathbf{Q}_{\lambda_n}\ \mathbf{1}\ . \qquad 2.41$$

Proof

Just set $P = \mathbf{1}$ in 2.40, equate the coefficents of $z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n}$ of both sides and use the definition in 2.6 with the $y\,'s$ replaced by the $z\,'s$. We should also note that 2.41 can also be obtained by iterating the recursion in 2.12 by interpreting it as an application of the operator $\mathbf{Q}_{\lambda_1}$.

As we did in the case of Schur functions, formula 2.41 may be used to define Hall-Litlewood polynomials $Q_p[X_n, t]$ indexed by compositions $p = (p_1, p_2, \ldots, p_n)$. The following identities provide an algorithm for computing the expansion of the resulting symmetric polynomial in terms of the $Q_\mu[X_n, t]$:

Theorem 2.5

$$\mathbf{Q}_{a-1}\mathbf{Q}_b \;+\; \mathbf{Q}_{b-1}\mathbf{Q}_a \;=\; t\,\mathbf{Q}_a\mathbf{Q}_{b-1} \;+\; t\,\mathbf{Q}_b\mathbf{Q}_{a-1} \qquad 2.42$$

Proof

Note that from 2.39 we derive that

$$\mathbf{Q}(z_1)\mathbf{Q}(z_2)\,P \;=\; \frac{1-z_2/z_1}{1-tz_2/z_1}\,\Omega[(1-t)(z_1+z_2)X_n]\;P[X_n-\tfrac{1}{z_1}-\tfrac{1}{z_2}]$$

Thus

$$(z_1-tz_2)\;\mathbf{Q}(z_1)\mathbf{Q}(z_2)\,P \;=\; (z_1-z_2)\;\Omega[(1-t)(z_1+z_2)X_n]\;P[X_n-\tfrac{1}{z_1}-\tfrac{1}{z_2}]\;\;.$$

Interchanging the roles of z_1 and z_2 we get

$$(z_2-tz_1)\;\mathbf{Q}(z_2)\mathbf{Q}(z_1)\,P \;=\; (z_2-z_1)\;\Omega[(1-t)(z_1+z_2)X_n]\;P[X_n-\tfrac{1}{z_1}-\tfrac{1}{z_2}]\;\;.$$

Adding these two equations yields that

$$(z_1-tz_2)\;\mathbf{Q}(z_1)\mathbf{Q}(z_2) \;+\; (z_2-tz_1)\;\mathbf{Q}(z_2)\mathbf{Q}(z_1) \;=\; 0\;\;,$$

and 2.42 immediately follows by taking the coefficients of $z_1^a z_2^b$.

We shall see next that our polynomials $H_\mu[X_n,t]$ allow an entirely analogous development. Upon combining the definition in 2.27 with the formula for the operators $\mathbf{Q}_m$ given by Proposition 2.4 it is not difficult to arrive at the conclusion that the vertex operator $\mathbf{H}_m$ yielding $H_\mu[X_n,t]$ should be defined by setting

$$\mathbf{H}_m\;P[X_n] \;=\; \Omega[zX_n]\;P[X_n-\tfrac{1-t}{z}]\Big|_{z^m} \qquad 2.43$$

Note that if we let

$$\mathbf{H}(z)\;P[X_n] \;=\; \Omega[zX_n]\;P[X_n-\tfrac{1-t}{z}]\;\;,$$

Then in complete analogy with 2.39 and 2.40 we have

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\;P[X_n] \;=\; \Omega[(t-1)z_2/z_1]\;\Omega[(z_1+z_2)X_n]\;P[X_n-\tfrac{1-t}{z_1}-\tfrac{1-t}{z_2}] \qquad 2.44$$

and

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\cdots\mathbf{H}(z_n)\;P[X_n] = \Big(\prod_{1\le i<j\le n}\frac{1-z_j/z_i}{1-tz_j/z_i}\Big)\;\Omega[Z_nX_n]\;P[X_n-\tfrac{1-t}{z_1}-\tfrac{1-t}{z_2}-\cdots-\tfrac{1-t}{z_n}]\;\;. \qquad 2.45$$

This gives the Rodiguez formula for $H_\mu[X_n,t]$.

Theorem 2.6

$$H_{\mu_1\mu_2\cdots\mu_n}[X_n,t] \;=\; \mathbf{H}_{\mu_1}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_n}\;\mathbf{1} \qquad 2.46$$

Proof

Setting $P=\mathbf{1}$ in 2.45 and equating coefficients of $z_1^{\mu_1} z_2^{\mu_2}\cdots z_n^{\mu_n}$ gives

$$\mathbf{H}_{\mu_1}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_n}\ \mathbf{1}=\Big(\prod_{1\leq i<j\leq n}\frac{1-z_j/z_i}{1-tz_j/z_i}\Big)\ \Omega[Z_nX_n]\ \Big|_{z_1^{\mu_1}z_2^{\mu_2}\cdots z_n^{\mu_n}}\ .$$

Thus 2.46 is just another way of writing formula 2.28.

It is interesting to note that the action of the operator $\mathbf{H}_m$ adds a special twist to our formulas. Let T_x^a denote the operator which replaces the variable x by ax in a polynomial. For instance we have

$$T^q_{x_i}\ P(x_1,\ldots,x_i,\ldots x_n)\ \ =\ \ P(x_1,\ldots,qx_i,\ldots x_n)\ . \qquad 2.47$$

Of course this may also be written in the plethystic form

$$T^t_{x_i}\ P[X_n]\ \ =\ \ P[X_n-(1-t)x_i] \qquad 2.48$$

This given, we have

Proposition 2.6

$$\mathbf{H}_m\ \ =\ \ \sum_{i=1}^{n}\frac{x_i^m}{\prod_{j=1,j\neq i}^{n}(1-x_j/x_i)}\ \ T^t_{x_i} \qquad 2.49$$

Proof

Using 1.10 with the one-variable polynomial $P(x)$ specialized to $P[X_n-(1-t)x]$ and the $y\,'s$ replaced by the $x\,'s$ gives

$$P[X_n-\tfrac{1-t}{z}]\ \Omega[zX_n]\ \Big|_{z^m}\ \ =\ \ \sum_{i=1}^{n}\frac{x_i^m}{\prod_{j=1,j\neq i}^{n}(1-x_j/x_i)}\ \ P[X_n-(1-t)x_i]\ .$$

Thus 2.49 follows from 2.48.

Our final result here is a recipe that shows us how to commute products of the operators $\mathbf{H}_m$.

Theorem 2.7

$$\mathbf{H}_{a-1}\mathbf{H}_b\ \ +\ \ \mathbf{H}_{b-1}\mathbf{H}_a\ \ =\ \ t\,\mathbf{H}_a\mathbf{H}_{b-1}\ \ +\ \ t\,\mathbf{H}_b\mathbf{H}_{a-1} \qquad 2.50$$

Proof

From 2.44 we derive, as we did in the proof of 2.42,

$$(z_1-tz_2)\ \mathbf{H}(z_1)\mathbf{H}(z_2)\ \ +\ \ (z_2-tz_1)\ \mathbf{H}(z_2)\mathbf{H}(z_1)\ \ =\ \ 0\ ,$$

and 2.50 follows by equating coefficients of $z_1^a z_2^b$.

3. The Macdonald case.

Writing 2.35 in matrix form we get that the Schur basis $\langle S\rangle = \{S_\lambda\}_{\lambda\vdash n}$ and the HL basis $\langle P\rangle = \{P_\lambda\}_{\lambda\vdash n}$ are related by the equation

$$\langle S\rangle \;=\; \langle P\rangle\, K^T(t) \tag{3.1}$$

where $K(t) = \|K_{\lambda\mu}(t)\|_{\lambda,\mu\vdash n}$ denotes the Kostka-Foulkes matrix and $K^T(t)$ denotes its transpose. Since $K_{\lambda\mu}(t) \neq 0$ only when $\lambda \geq \mu$ (in dominance) we see (taking also account of 2.30 b) that $K^T(t)$ is an upper unitriangular matrix. Inverting 3.1 we also get that

$$\langle P\rangle \;=\; \langle S\rangle\, K^T(t)^{-1}\ . \tag{3.2}$$

In particular we see that the $\langle P\rangle$-basis is also related to the $\langle S\rangle$-basis by an upper unitriangular matrix. Combining this with the orthogonality result in 2.15. We see that we have the following basic result.

Theorem 3.0

For each $n \geq 1$ the Hall-Littlewood basis is uniquely characterized by the following two conditions

$$\begin{cases} a) \quad P_\lambda[X_n] = S_\lambda[X_n] + \sum_{\mu<\lambda} S_\mu[X_n]\,\xi_{\mu\lambda} & \\ b) \quad \langle P_\lambda\ ,\ P_\mu\rangle_t \;=\; 0 & \text{for } \lambda\neq\mu \end{cases} \tag{3.3}$$

Proof

We only need to show that $a)$ and $b)$ uniquely determine P_λ. However, this is immediate since from the triangularity condition in 3.3 a) we derive that the elements of the basis $\langle P\rangle = \{P_\lambda\}_{\lambda\vdash n}$ may be obtained by applying the Gramm-Schmidt orthogonalization algorithm to the elements of $\langle S\rangle = \{S_\lambda\}_{\lambda\vdash n}$ arranged in any total order that extends the dominance partial order. In fact, 3.3 a) implies the surprising fact that we obtain the same basis regardless of which total extension we use.

The definition of the Macdonald polynomials is entirely analogous, the only thing we need to do is replace the Hall-Littlewood scalar product $\langle\ ,\ \rangle_t$ by the Macdonald scalar product $\langle\ ,\ \rangle_{q,t}$ defined by setting for the power basis:

$$\langle p_\lambda\ ,\ p_\mu\rangle_{q,t} \;=\; \begin{cases} 0 & \text{if } \lambda\neq\mu\ , \\ z_\mu\, p_\mu[\frac{1-q}{1-t}] & \text{if } \lambda=\mu\ . \end{cases} \tag{3.4}$$

The point of departure of the theory of Macdonald polynomials is the following fundamental fact:

Theorem 3.1

For each $n \geq 1$ there exists a unique family of symmetric polynomials $\{P_\lambda[X_n;q,t)\}_{\lambda\vdash n}$ satisfying the following two conditions

$$\begin{cases} a) \quad P_\lambda[X_n] = S_\lambda[X_n] + \sum_{\mu\leq\lambda} S_\mu[X_n]\,\xi_{\mu\lambda}(q,t) & \\ b) \quad \langle P_\lambda\ ,\ P_\mu\rangle_{q,t} \;=\; 0 & \text{for } \lambda\neq\mu \end{cases} \tag{3.3}$$

The proof of this result consists only on showing the existence part since as we have seen in the Hall-Littlewood case these two conditions uniquely determine the polynomials $P_\lambda[X_n;q,t)$.

Before we proceed with our presentation of the Rodriguez formulas of Vinet-LaPointe-Noumi-Kirillov we need to go over a brief review of the Macdonald existence proof.

We begin by recalling that Macdonald defines a sequence of operators D_k acting on polynomials in $x_1, x_2, \ldots, x_n$ by setting

$$D(u) \;=\; \sum_{k=0}^{n} u^k\, D_k \tag{3.4}$$

and letting $D(u)$ act on a polynomial $P[X_n]$ according to the formula

$$D(u)\, P[X_n] \;=\; \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma\delta} \prod_{i=1}^{n} (1+u\, t^{n-\sigma_i}\, T^q_{x_i})\, P[X_n]\ . \tag{3.5}$$

Th following identity shows that $D(u)$ does transform a symmetric polynomial in $x_1, x_2, \ldots, x_n$ into another such polynomial.

Proposition 3.1

For every partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0)$ we have

$$D(u)\, m_\lambda[X_n] \;=\; \sum_{\lambda(p)=\lambda} \prod_{i=1}^{n} \left(1+u\, t^{n-i}\, q^{p_i}\right) S_p[X_n] \tag{3.6}$$

Proof

We recall that the monomial symmetric function indexed by λ may be written in the form

$$m_\lambda[X_n] \;=\; \sum_{\lambda(p)=\lambda} x^p \tag{3.7}$$

where the condition $\lambda(p)=\lambda$ is to mean that the sum is carried out over all distinct compositions $p=(p_1,p_2,\cdots,p_n)$ that rearrange to λ. Substituting this expression for P in 3.5 and interchanging the order of summation gives

$$D(u)\, m_\lambda[X_n] \;=\; \sum_{\lambda(p)=\lambda} \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma\delta} \prod_{i=1}^{n} (1+u\, t^{n-\sigma_i}\, T^q_{x_i})\, x^{\sigma p}\ , \tag{3.8}$$

where we have taken the liberty of writing $x^{\sigma p}$ instead of x^p because of the symmetry of m_λ. Now it is easy to see that we have

$$x^{\sigma\delta} \prod_{i=1}^{n} (1+u\, t^{n-\sigma_i}\, T^q_{x_i})\, x^{\sigma p} = x^{\sigma(p+\delta)} \prod_{i=1}^{n} \left(1+u\, t^{n-\sigma_i}\, q^{p_{\sigma_i}}\right) = x^{\sigma(p+\delta)} \prod_{i=1}^{n} \left(1+u\, t^{n-i}\, q^{p_i}\right)\ .$$

Since the last factor here is independent of σ it can be taken out of the inner sum in 3.8 and obtain

$$D(u)\, m_\lambda[X_n] \;=\; \sum_{\lambda(p)=\lambda} \prod_{i=1}^{n} \left(1+u\, t^{n-i}\, q^{p_i}\right) \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma(p+\delta)} \;\; ,$$

and 3.6 immediately follows since

$$\frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma(p+\delta)} \;=\; \frac{det\|x_i^{p_j+n-j}\|_{i,j=1}^{n}}{det\|x_i^{n-j}\|_{i,j=1}^{n}} \;=\; S_p[X_n] \;\; . \qquad 3.9$$

Formula 3.6 may be made more explicit. For a given partition μ and composition p set

$$\epsilon_\mu(p) \;=\; \begin{cases} \mathrm{sign}(\sigma) & \text{if } p_{\sigma_i}+n-\sigma_i=\mu_i+n-i \;\; , \\ 0 & \text{otherwise} \;\; . \end{cases} \qquad 3.10$$

In other words, $\epsilon_\mu(p)$ doesnot vanish only if $p+\delta$ may be rearranged to $\mu+\delta$, in which case $\epsilon_\mu(p)$ gives the sign of permutation that does the rearrangement. Note that in this case we have

$$det\|x_i^{p_j+n-j}\|_{i,j=1}^{n} \;=\; \mathrm{sign}\,(\sigma) det\|x_i^{\mu_j+n-j}\|_{i,j=1}^{n}$$

This given we may write

$$S_p[X_n] \;=\; \sum_{\mu} \epsilon_\mu(p)\, S_\mu[X_n]$$

Substituting this in 3.6 and interchanging order of summation we get

$$D(u)\, m_\lambda[X_n] \;=\; \sum_{\mu} S_\mu[X_n] \sum_{\lambda(p)=\lambda} \epsilon_\mu(p) \prod_{i=1}^{n} \left(1+u\, t^{n-i}\, q^{p_i}\right) \;\; ,$$

Note further that if $p_{\sigma_i}+n-\sigma_i=\mu_i+n-i$ and p rearranges to λ we necessarily have

$$\begin{aligned} \mu_1+\mu_2+\cdots+\mu_i-1-2-\cdots-i \;&=\; p_{\sigma_1}+p_{\sigma_1}+\cdots+p_{\sigma_i}-\sigma_1-\sigma_2-\cdots-\sigma_i \\ &\le\; \lambda_1+\lambda_2+\cdots+\lambda_i \;-\; 1-2-\cdots-i \end{aligned}$$

With equality for all $i=1..n$ only if σ is the identity permutation and $\lambda=p=\mu$. Thus we may write

$$D(u)\, m_\lambda[X_n] \;=\; \omega_\lambda(u;q,t)\, S_\lambda[X_n] \;+\; \sum_{\mu<\lambda} S_\mu[X_n]\, \omega_{\mu\lambda}(u;q,t) \;\; , \qquad 3.11$$

where for convenience we have set

$$\omega_\lambda(u;q,t)=\prod_{i=1}^{n}\left(1+u\, t^{n-i}\, q^{\lambda_i}\right) \quad \text{and} \quad \omega_{\mu\lambda}(u;q,t)=\sum_{\lambda(p)=\lambda} \epsilon_\mu(p) \prod_{i=1}^{n}\left(1+u\, t^{n-i}\, q^{p_i}\right) \;\; . \qquad 3.12$$

An important consequence of this formula is the following basic result of Macdonald.

Theorem 3.3

All the operators D_k act on the Schur basis upper triangularly in the dominance partial order. More precisely for each $k = 1..n$ we have

$$D_k\, S_\lambda[X_n] \;=\; e_k\Big[\sum_{i=1}^{n} t^{n-i} q^{\lambda_i}\Big] S_\lambda[X_n] \;+\; \sum_{\mu<\lambda} S_\mu[X_n]\, \widetilde{\omega}^{(k)}_{\mu\lambda}(q,t) \qquad 3.13$$

with

$$\widetilde{\omega}^{(k)}_{\mu\lambda}(q,t) \;=\; \sum_{\mu<\nu\le\lambda} \omega_{\mu\nu}(u;q,t)\,\Big|_{u^k} K_{\lambda\nu} \qquad 3.14$$

Proof

If we rewrite formula 3.11 with ν replacing λ multiply both sides by the Kostka number $K_{\lambda\nu}$ and sum over ν we get (since $K_{\lambda\lambda} = 1$)

$$D(u)\, S_\lambda[X_n] \;=\; \omega_\lambda(u;q,t)\, S_\lambda[X_n] \;+\; \sum_{\mu} S_\mu[X_n] \sum_{\mu<\nu\le\lambda} \omega_{\mu\nu}(u;q,t)\, K_{\lambda\nu}$$

and 3.13 (with 3.14) follows by equating coefficients of u^k.

Our next task is to show that all of the operators D_k are self-adjoint with respect to the Macdonald scalar product $\langle\ ,\ \rangle_{q,t}$. To this end we need the following auxiliary result.

Proposition 3.2

An operator T acting on symmetric polynomials in $x_1, x_2, \ldots, x_n$ is self-adjoint with respect to the scalar product $\langle\ ,\ \rangle_f$ defined in 1.6 if and only if we have

$$T^x\, \Omega_f[X_nY_n] \;=\; T^y\, \Omega_f[X_nY_n] \qquad 3.15$$

where $\Omega_f[X_nY_n]$ is given by 1.7, and T^x and T^y respectively denote T operating on polynomials in $x_1, x_2, \ldots, x_n$ and T operating on polynomials in $y_1, y_2, \ldots, y_n$.

Proof

We shall use matrix notation here. So let $\langle p\rangle$ denote the the power basis $\{p_\lambda\}_\lambda$ in some total order that is compatible with increasing values of $\sum_{i=1}^n \lambda_i$ and let M the matrix expressing the action of T in terms of $\langle p\rangle$. We are to show that 3.15 is equivalent to the identities

$$\big\langle T\, p_\lambda\ ,\ p_\mu \big\rangle_f \;=\; \big\langle p_\lambda\ ,\ T\, p_\mu \big\rangle_f \qquad (\ \forall\quad \lambda,\mu\ )$$

Now we can express this condition in matrix form as

$$\big(\langle p\rangle M\big)^T \odot_f\ \langle p\rangle \;=\; \langle p\rangle^T \odot_f\ \langle p\rangle M \qquad 3.16$$

where $\langle p\rangle$ is to be viewed as a row vector and the symbol $\odot_f$ is to represent that in carrying out that particular matrix product we must replace the ordinary multiplication by the scalar product $\langle\ ,\ \rangle_f$.

In the same vein we can express the definition in 1.7 in the form

$$\Omega[X_nY_n] \;=\; \big\langle p[X_n]\big\rangle\, D[\tfrac{1}{f}]\, \big\langle p[Y_n]\big\rangle^T$$

where where $D[1/f]$ denotes the diagonal matrix with diagonal elements $1/f_\lambda$. This given, 3.15 may be written as

$$\big(\langle p[X_n]\rangle M\big)\; D[\frac{1}{f}]\; \langle p[Y_n]\rangle^T \;\;=\;\; \langle p[X_n]\rangle\; D[\frac{1}{f}]\; \big(\langle p[Y_n]\rangle M\big)^T$$

which reduces to the matrix identity

$$M\; D[\frac{1}{f}] \;\;=\;\; D[\frac{1}{f}]\; M^T \quad . \qquad 3.17$$

Now we may rewrite 3.16 as

$$M^T \langle p\rangle^T \odot_f \; \langle p\rangle \;\;=\;\; \langle p\rangle^T \odot_f \; \langle p\rangle M \quad ,$$

better yet, taking account of the definition 1.6, we see that 3.16 is equivalent to

$$M^T D[f] \;\;=\;\; D[f]\; M \qquad 3.18$$

where $D[f]$ denotes the diagonal matrix with diagonal elements f_λ. This completes our proof since 3.17 and 3.18 are easily seen to be equivalent.

For some calculations the following alternate formulas for the operators $D(u)$ and D_k turn out to be more convenient.

Proposition 3.3

$$D(u) \;\;=\;\; \sum_{I\subseteq[1,n]} u^{|I|}\; A_I(x;t)\; T_I^q \qquad 3.19$$

where

$$T^q_{X_I} = \prod_{i\in I} T^q_{x_i} \qquad \text{and} \qquad A_I(x;t) = t^{\binom{k}{2}} \prod_{i\in I;\, j\notin I} \frac{t\; x_i - x_j}{x_i - x_j} \quad . \qquad 3.20$$

In particular we have

$$D_k \;\;=\;\; \sum_{I\subseteq[1,n],|I|=k} A_I(x;t)\; T_I^q \qquad 3.21$$

Proof

Expanding the product in 3.5 we get

$$\begin{aligned} D(u) \;\;&=\;\; \sum_{I\subseteq[1,n]} u^{|I|}\; \frac{1}{\Delta[X_n]}\; \sum_{\sigma\in S_n} \epsilon(\sigma)\; x^{\sigma\delta} \Big(\prod_{i\in I} t^{n-\sigma_i}\Big)\; T^q_{X_I} \\ &=\;\; \sum_{I\subseteq[1,n]} u^{|I|}\; \frac{1}{\Delta[X_n]}\; \Big(T^t_{X_I} \sum_{\sigma\in S_n} \epsilon(\sigma)\; x^{\sigma\delta}\Big)\; T^q_{X_I} \\ &=\;\; \sum_{I\subseteq[1,n]} u^{|I|}\; \frac{1}{\Delta[X_n]}\; \Big(T^t_{X_I} \Delta[X_n]\Big)\; T^q_{X_I} \end{aligned}$$

This shows 3.19 with

$$A_I(x,t) \;=\; \frac{1}{\Delta[X_n]}\Big(T^t_{X_I}\Delta[X_n]\Big)\ .$$

However, we see that

$$\frac{1}{\Delta[X_n]}\Big(T^t_{X_I}\Delta[X_n]\Big) \;=\; \prod_{\substack{i\in I;j\in I\\ i<j}}\frac{t\,x_i-t\,x_j}{x_i-x_j}\prod_{\substack{i\in I;j\notin I\\ i<j}}\frac{t\,x_i-tx_j}{x_i-x_j}\prod_{\substack{i\notin I;j\in I\\ i<j}}\frac{x_i-t\,x_j}{x_i-x_j}\prod_{\substack{i\notin I;j\notin I\\ i<j}}\frac{x_i-x_j}{x_i-x_j}$$

$$=\; t^{\binom{k}{2}}\prod_{\substack{i\in I;j\notin I\\ i<j}}\frac{t\,x_i-tx_j}{x_i-x_j}\prod_{\substack{i\notin I;j\in I\\ i<j}}\frac{x_i-t\,x_j}{x_i-x_j} \;=\; t^{\binom{k}{2}}\prod_{i\in I;j\notin I}\frac{t\,x_i-tx_j}{x_i-x_j}$$

This proves 3.19 with $A_I(x,t)$ as given by 3.20. Formula 3.21 is then obtained by equating coefficients of u^k on both sides of 3.19.

Theorem 3.4

The operators $D(u)$ and D_k are all self-adjoint with respect to the Macdonald scalar product $\langle\ ,\ \rangle_{q,t}$.

Proof

In view of Proposition 3.2 we need only verify that

$$D^x(u)\,\Omega\big[\tfrac{1-t}{1-q}X_nY_n\big] \;=\; D^y(u)\,\Omega\big[\tfrac{1-t}{1-q}X_nY_n\big]\ . \tag{3.22}$$

Note that formula 3.19 yields that

$$D^x(u)\Omega[\tfrac{1-t}{1-q}X_nY_n] \;=\; \sum_{I\subseteq[1,n]}u^{|I|}\,A_I(x;t)\,\Omega\big[\tfrac{1-t}{1-q}\big(X_n-(1-q)X_I\big)Y_n\big]$$

$$=\; \Omega[\tfrac{1-t}{1-q}X_nY_n]\sum_{I\subseteq[1,n]}u^{|I|}\,A_I(x;t)\,\Omega\big[-(1-t)X_IY_n\big]$$

where for convenience we have set $X_I=\sum_{i\in I}x_i$. The crucial consequence of this identity is that the ratio

$$\frac{D^x(u)\Omega[\frac{1-t}{1-q}X_nY_n]}{\Omega[\frac{1-t}{1-q}X_nY_n]} \;=\; \sum_{I\subseteq[1,n]}u^{|I|}\,A_I(x;t)\,\Omega\big[-(1-t)X_IY_n\big] \tag{3.23}$$

is independent of q. In particular it may be computed by setting $q=t$. That is we have

$$\sum_{I\subseteq[1,n]}u^{|I|}\,A_I(x;t)\,\Omega\big[-(1-t)X_IY_n\big] \;=\; \frac{D^x_{q=t}(u)\Omega[X_nY_n]}{\Omega[X_nY_n]}\ .$$

Now setting $q=t$ in 3.5 gives that

$$D^x_{q=t}(u)P[X_n] \;=\; \frac{1}{\Delta[X_n]}\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_{i=1}^n(1{+}u\,T^t_{x_i})\,x^{\sigma\delta}P[x_n] \;=\; \frac{1}{\Delta[X_n]}\prod_{i=1}^n(1{+}u\,T^t_{x_i})\,\Delta[X_n]P[X_n]\ .$$

In particular we get that

$$\begin{aligned} D^x_{q=t}(u) S_\lambda[X_n] &= \frac{1}{\Delta[X_n]} \prod_{i=1}^n (1+u\, T^t_{x_i})\; det\|x_i^{\lambda_j+n-j}\|_{i,j=1}^n \\ &= \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{i=1}^n (1+u\, T^t_{x_i}) x^{\sigma(\lambda+\delta)} \\ &= \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma(\lambda+\delta)} \prod_{i=1}^n \big(1+u\, t^{\lambda_{\sigma_i}+n-\sigma_i}\big) \\ &= \Big(\prod_{i=1}^n \big(1+u\, t^{\lambda_i+n-i}\big)\Big) \frac{1}{\Delta[X_n]} \sum_{\sigma\in S_n} \epsilon(\sigma)\, x^{\sigma(\lambda+\delta)} \ . \end{aligned}$$

That is we have

$$D^x_{q=t}(u) S_\lambda[X_n] \;=\; \Big(\prod_{i=1}^n \big(1+u\, t^{\lambda_i+n-i}\big)\Big)\; S_\lambda[X_n] \ , \qquad 3.24$$

and from the Cauchy identity we deduce that

$$D^x_{q=t}(u)\Omega[X_nY_n] \;=\; \sum_\lambda \omega_\lambda(u,t)\; S_\lambda[X_n] S_\lambda[Y_n] \ , \qquad 3.25$$

where for convenience we have set

$$\omega_\lambda(u,t) \;=\; \prod_{i=1}^n \big(1+u\, t^{\lambda_i+n-i}\big) \ .$$

Now the symmetry of the right-hand side of 3.25 with respect to the X_n and Y_n alphabets yields that we must have just as well that

$$D^y_{q=t}(u)\Omega[X_nY_n] \;=\; \sum_\lambda \omega_\lambda(u,t)\; S_\lambda[X_n] S_\lambda[Y_n] \ .$$

In conclusion, we have shown that

$$\sum_{I\subseteq[1,n]} u^{|I|}\, A_I(x;t)\, \Omega\big[-(1-t)X_IY_n\big] \;=\; \frac{D^y_{q=t}(u)\Omega[X_nY_n]}{\Omega[X_nY_n]} \;=\; \sum_{I\subseteq[1,n]} u^{|I|}\, A_I(y;t)\, \Omega\big[-(1-t)X_nY_I\big] \ ..$$

This proves 3.22 and completes the proof of the Theorem.

To proceed with our existence proof we need the following general result.

Proposition 3.5

Let $\mathbf{V}$ *be a finite dimensional real vector space with a scalar product* $\langle\ ,\ \rangle$ *and a basis* $\mathcal{B}=\{b_1,b_2,\ldots,b_m\}$ *and let* $\mathbf{T}$ *be a linear operator on* $\mathbf{V}$ *that is self-adjoind with respect to this scalar product. Suppose further that* $\mathbf{T}$ *acts upper triangularly on the basis* $\mathcal{B}$*, that is for some constants* λ_i , c_{ij} *we have*

$$\mathbf{T}\, b_j \;=\; \lambda_i\, b_j \;+\; \sum_{i=1}^{j-1} b_i\, c_{ij} \ . \qquad 3.26$$

Then, if the λ_i are distinct, there is a unique basis $\mathcal{F} = \{f_1, f_2, \ldots, f_M\}$ of $\mathbf{V}$ *such that*

$$\begin{aligned} a) &\quad \mathbf{T}\, f_j \;=\; \lambda_j\, f_j \ , \\ b) &\quad f_j \;=\; b_j \;+\; \sum_{i=1}^{j-1} b_i\, \xi_{ij} \ , \\ c) &\quad \big\langle f_i \ , \ f_j \big\rangle = 0 \quad \forall \quad i \neq j \ . \end{aligned} \tag{3.27}$$

Proof

Note first that the condition in 3.26 implies in particular that the scalars $\lambda_1, \lambda_2, \ldots, \lambda_M$ are the eigenvalues of $\mathbf{T}$. Thus $\mathbf{T}$ satisfies the characteristic equation

$$\prod_{i=1}^{M} (\mathbf{T} - \lambda_i I) \;=\; 0 \tag{3.28}$$

Set

$$\mathbf{Z}_j(\mathbf{T}) \;=\; \prod_{i=1, i\neq j}^{M} \frac{\mathbf{T} - \lambda_i\, I}{\lambda_j - \lambda_i} \ . \tag{3.29}$$

Note that because of 3.28 we necessarily have that

$$\mathbf{T}\, \mathbf{Z}_j(\mathbf{T}) \;=\; \lambda_j\, \mathbf{Z}_j(\mathbf{T}) \ . \tag{3.30}$$

We claim that our desired vectors f_i are simply given by the formula

$$f_j \;=\; \mathbf{Z}_j(\mathbf{T})\, b_j \ . \tag{3.31}$$

In fact, we can easily see from the form of 3.26 and the definition in 3.29 that the matrix of $\mathbf{Z}_j(\mathbf{T})$ in terms of the basis is, upper triangular, with diagonal element in position j equal to 1 and all the other diagonal elements equal to zero. Thus we must have constants ξ_{ij} yielding

$$f_j \;=\; b_j \;+\; \sum_{i=1}^{j-1} b_i\, \xi_{ij} \ .$$

This not only gives 3.27 b) but also assures that $f_j \neq 0$. This given, 3.31 yields that f_j is an eigenvector of $\mathbf{T}$ with eigenvalue λ_j. We are left with verifying 3.27 c). However this is immediate since the self-adjointness of $\mathbf{T}$ yields that for any pair i, j we have

$$\lambda_i \big\langle f_i \ , \ f_j \big\rangle \;=\; \big\langle \mathbf{T}\, f_i \ , \ f_j \big\rangle \;=\; \big\langle f_i \ , \ \mathbf{T}\, f_j \big\rangle \;=\; \lambda_j \big\langle f_i \ , \ f_j \big\rangle \ .$$

Thus 3.27 c) is forced by $\lambda_i \neq \lambda_j$. This completes the proof.

We are finally in a position to give the

Proof of Theorem 3.1

Note that the special case $k=1$ of 3.13 gives that

$$D_1\, S_\lambda \;=\; \Big(\sum_{i=1}^{n} t^{n-i} q^{\lambda_i}\Big)\, S_\lambda \;+\; \sum_{\mu<\lambda} S_\mu\, \widetilde{\omega}^{(1)}_{\mu\lambda}\;.$$

This assures the triangularity of D_1 with respect to the Schur basis in any order that linearly extends dominance. Theorem 3.4 assures that D_1 is self-adjoint with respect to the Macdonald scalar product. Finally note that we cannot have

$$\sum_{i=1}^{n} t^{n-i} q^{\lambda_i} \;=\; \sum_{i=1}^{n} t^{n-i} q^{\mu_i}$$

without $\lambda=\mu$. Thus the eigenvalues of D_1 are all distinct. Thus all the the hypotheses of Proposition 3.5 are satisfied when $\mathbf{V}$ consists of a subspace of symmetric polynomials which are homogeneous of a fixed degree, with $\mathcal{B}$ the collection of Schur functions of that degree, $\langle\;,\;\rangle$ the Macdonald scalar product and $\mathbf{T}=D_1$. Thus the Macdonald polynomials $P_\mu(x;q,t)$ may be simply constructed by means of the formula

$$P_\lambda \;=\; \Bigg(\prod_{\substack{\mu\neq\lambda\\ |\mu|=|\lambda|}} \frac{D_1-\omega_\mu I}{\omega_\lambda-\omega_\mu}\Bigg)\, S_\lambda\;, \qquad 3.32$$

where for convenience we have set

$$\sum_{i=1}^{n} t^{n-i} q^{\lambda_i} = \omega_\lambda \qquad \text{and} \qquad \sum_{i=1}^{n} t^{n-i} q^{\mu_i} = \omega_\mu \qquad 3.33$$

In summary, Proposition 3.5 gives that

$$\begin{aligned} &a) \quad D_1\, P_\lambda \;=\; \Big(\sum_{i=1}^{n} t^{n-i} q^{\lambda_i}\Big)\, P_\lambda \quad, \\ &b) \quad \big\langle P_\lambda\,,\, P_\mu\big\rangle_{q,t} \;=\; 0 \quad for \quad \lambda\neq\mu\;, \\ &c) \quad P_\lambda \;=\; S_\lambda \;+\; \sum_{\mu<\lambda} S_\mu\, \xi_{\mu\lambda}\;, \end{aligned} \qquad 3.34$$

where as we have seen $a)\to b)$ and the self-adjointness of D_1 with respect to $\langle\;,\;\rangle_{q,t}$, while c) follows from the fact that Proposition 3,5 assures that the basis $\{P_\lambda\}_\lambda$ is triangularly related to the Schur basis in **any** total order that linearly extends the dominance partial order. This completes the proof of Theorem 3.1.

Computer experimentation quickly reveals that 3.32 is not a very practical way of computing the polynomials P_λ. One of the difficulties results from the fact that formulas expressing a symmetric polynomial $P[X_n]$ in terms of the variables themselves are quite impractical for all but relatively trivial cases. In particular, neither the definition in 3.5 nor its alternate form in 3.19 are much help in computing the action of any of the D_k on a general symmetric polynomial. Now it develops that, at least for D_1, we do have a plethystic formula which is computationally as well as theoretically very convenient. To state and prove this result we need a close variant of Propositions 1.3 and 2.3.

Proposition 3.6

For any formal series $F(x)=\sum_{k\geq 0} c_k\, x^k$ and for all integers $m\geq 0$ we have

$$F(1/z)\,\Omega[(t-1)zX_n]\Big|_{z^m} = \frac{\chi(m=0)}{t^n}F(0) \;+\; \frac{t-1}{t^n}\sum_{i=1}^{n}\Big(\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)(tx_i)^m\,F(tx_i) \tag{3.35}$$

Proof

Starting from the partial fraction expansion

$$\Omega[(t-1)zX_n] \;=\; \frac{1}{t^n} \;+\; \frac{t-1}{t^n}\sum_{i=1}^{n}\Big(\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\frac{1}{1-ztx_i}\;\;,$$

the same argument that we used in the proof of Proposition 2.3 yields 2.7.

As a corollary we obtain the following basic identity.

Theorem 3.5

For any symmetric polynomial $P[X_n]$ we have

$$D_1\;P[X_n] \;=\; \frac{P[X_n]}{1-t} \;+\; \frac{t^n}{t-1}\;P\big[X_n-\tfrac{1-q}{tz}\big]\;\Omega[(t-1)zX_n]\;\big|_{z^0} \tag{3.36}$$

Proof

Note that 3.21 for $k=1$ gives that

$$D_1 \;=\; \sum_{i=1}^{n}\Big(\prod_{j=1,j\neq i}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\,T_{x_i}^{q} \tag{3.37}$$

Thus

$$\begin{aligned} D_1\,\Omega\big[\tfrac{1-t}{1-q}X_nY_n\big] \;&=\; \sum_{i=1}^{n}\Big(\prod_{j=1,j\neq i}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\Omega\big[\tfrac{1-t}{1-q}\big(X_n-(1-q)x_i\big)Y_n\big] \\ &=\; \Omega\big[\tfrac{1-t}{1-q}X_nY_n\big]\;\sum_{i=1}^{n}\Big(\prod_{j=1,j\neq i}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\Omega\big[-(1-t)x_iY_n\big]\;\;. \end{aligned} \tag{3.38}$$

On the other hand formula 3.35 for $m=0$ and $F(z)\to F(z/t)$ may be rewritten as

$$\sum_{i=1}^{n}\Big(\prod_{j=1,j\neq i}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\,F(x_i) \;=\; \frac{F(0)}{1-t} \;+\; \frac{t^n}{t-1}\,F(1/tz)\,\Omega\big[(t-1)zX_n\big]\;\big|_{z^0}\;\;.$$

Using this with $F(z)=\Omega[-(1-t)Y_n z]$ gives that

$$\sum_{i=1}^{n}\Big(\prod_{j=1,j\neq i}^{n}\frac{tx_i-x_j}{x_i-x_j}\Big)\Omega\big[(-(1-t)x_iY_n\big]=\frac{1}{1-t}+\frac{t^n}{t-1}\,\Omega[-(1-t)Y_n/tz]\,\Omega\big[(t-1)zX_n\big]\,\big|_{z^0}\ .$$

Substituting this in 3.38 gives

$$D_1\,\Omega\big[\tfrac{1-t}{1-q}X_nY_n\big]\ =\ \frac{\Omega\big[\frac{1-t}{1-q}X_nY_n\big]}{1-t}+\frac{t^n}{t-1}\,\Omega\big[\tfrac{1-t}{1-q}\big(X_n-\tfrac{1-q}{tz}\big)Y_n\big]\,\Omega\big[[(t-1)zX_n\big]\,\big|_{z^0}\ .$$

Equating coefficients of $S_\lambda\big[\frac{1-t}{1-q}Y_n\big]$ on both sides of this identity yields that

$$D_1\,S_\lambda[X_n]\ =\ \frac{S_\lambda[X_n]}{1-t}+\frac{t^n}{t-1}\,S_\lambda\big[X_n-\tfrac{1-q}{tz}\big]\,\Omega\big[[(t-1)zX_n\big]\,\big|_{z^0}\ ,$$

which is 3.36 for the Schur basis. This completes our argument.

Note that by combining 3.34 a) with 3.36 we obtain that

$$\frac{P_\lambda[X_n;q,t]}{1-t}+\frac{t^n}{t-1}P_\lambda\big[X_n-\tfrac{1-q}{tz};q,t\big]\Omega\big[(t-1)zX_n\big]\,\big|_{z^0}=\Big(\sum_{i=1}^{n}t^{n-i}q^{\lambda_i}\Big)\,P_\lambda[X_n;q,t]\ . \qquad 3.39$$

Now it is easy to see that the left hand side may be computed without involving any of the variables $x_1,x_2,\ldots,x_n$, by simply replacing X_n by the the symbol p_1 and interpreting the resulting identity as a relation between two polynomials in the variables $p_1,p_2,p_3\ldots$.

This done, we can transform 3.39 into a relation which contains no explicit dependence on n. To this end, for a given partition $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)$, set

$$B_\lambda(q,t)\ =\ \sum_{i=1}^{n}t^{i-1}(1+q+\cdots+q^{\lambda_i-1})\ =\ \sum_{i=1}^{n}t^{i-1}\ \frac{1-q^{\lambda_i}}{1-q} \qquad 3.40$$

This polynomial, usually called the *biexponent generator of* λ, plays an essential role in the Theory of Macdonald polynomials, this results from the following basic identity:

Theorem 3.6

$$P_\lambda\big[X_n-\tfrac{1-q}{tz};q,t\big]\Omega\big[(t-1)zX_n\big]\,\big|_{z^0}\ =\ \big(1\ -\ (1-\tfrac{1}{t})(1-q)B_\lambda(q,\tfrac{1}{t})\big)\,P_\lambda\big[X_n;q,t\big] \qquad 3.41$$

Proof

Note that from 3.40 we derive that

$$(1-t)(1-q)B_\lambda(q,t)\ =\ 1-t^n\ -\ (1-t)\sum_{i=1}^{n}t^{i-1}q^{\lambda_i}$$

Changing t into $1/t$ and multiplying by t^n we can rewrite this in the form

$$1\ -\ (1-t)\sum_{i=1}^{n}t^{n-i}q^{\lambda_i}\ =\ t^n\,\Big(1-(1-\tfrac{1}{t})(1-q)B_\lambda(q,\tfrac{1}{t})\Big)\ . \qquad 3.42$$

On the other hand, multiplying both sides of 3.39 by (1-t) we can rearrange it to

$$t^n \; P_\lambda\big[X_n - \tfrac{1-q}{tz}; q,t\big]\Omega\big[(t-1)zX_n\big]\,\big|_{z^0} \;=\; \Big(1 \;-\; (1-t)\sum_{i=1}^{n} t^{n-i}q^{\lambda_i}\Big)\, P_\lambda\big[X_n; q,t\big] \;.$$

Using 3.42 and cancelling the factor t^n this relation reduces to 3.41.

Remark 3.1

Recall that the length of a partition λ is the number of its non zero components, we shall denote it by $l(\lambda)$. Note that from 3.40 we immediately get that

$$B_\lambda(q,t) \;=\; \sum_{i=1}^{l(\lambda)} t^{i-1}\, \frac{1-q^{\lambda_i}}{1-q}$$

Thus adding extra zero components to λ doesnot change B_λ. This given, an important consequence of 3.41 is that, as long as $n \geq l(\lambda)$, the polynomial $P_\lambda[X_n; q,t]$ depends only on the non zero components of λ and not on n.

Theorems 3.3 and 3.4 combined yield us the following important result of Macdonald:

Theorem 3.7

For any λ of length $\leq n$

$$D(u)\; P_\lambda[X_n; q,t] \;=\; \Big(\prod_{i=1}^{n} (1 + u\, t^{n-i}q^{\lambda_i})\Big)\; P_\lambda(X_n; q,t] \qquad 3.43$$

in particular for all $k = 1,..,n$ we have

$$D_k\; P_\lambda[X_n; q,t] \;=\; e_k\Big[\sum_{i=1}^{n} t^{n-i}q^{\lambda_i}\Big]\; P_\lambda(X_n; q,t] \qquad 3.44$$

Proof

We have seen from Theorem 3.3 that $D(u)$ acts upper-triangularly on the Schur basis and with diagonal elements $\prod_{i=1}^{n} (1 + u\, t^{n-i}q^{\lambda_i})$. Since 3.34 c) says that the Macdonald basis $\{P_\lambda\}_\lambda$ is unitriangularly related to the Schur basis, we necessarily must have coefficients $m_{\mu\lambda}$ such that

$$D(u)\; P_\lambda \;=\; \Big(\prod_{i=1}^{n} (1 + u\, t^{n-i}q^{\lambda_i})\Big)\; P_\lambda \;+\; \sum_{\mu<\lambda} P_\mu\; m_{\mu\lambda} \;.$$

Using the orthogonality result in 3.34 b) we derive that

$$\big\langle D(u)\; P_\lambda \;,\; P_\mu \big\rangle_{q,t} \;=\; d_\mu(q,t)\; m_{\mu\lambda} \;,$$

with

$$d_\mu(q,t) \;=\; \big\langle P_\mu \;,\; P_\mu \big\rangle_{q,t} \;. \qquad 3.45$$

However, the self-adjointness expressed by Theorem 3.4 yields us

$$\big\langle D(u)\,P_\lambda\;,\;P_\mu\,\big\rangle_{q,t}\;=\;\big\langle\,P_\lambda\;,\;D(u)\,P_\mu\big\rangle_{q,t}$$

Now observe that we must also have

$$D(u)\,P_\mu\;=\;\Big(\prod_{i=1}^{n}\,(1+u\,t^{n-i}q^{\mu_i})\Big)\,P_\mu\;+\;\sum_{\nu<\mu}P_\nu\,m_{\nu\mu}\;\;,$$

but the orthogonality yields us that for $\nu<\mu<\lambda$

$$\big\langle P_\lambda\;,\;P_\mu\big\rangle_{q,t}=\big\langle P_\lambda\;,\;P_\nu\big\rangle_{q,t}=0\;\;,$$

thus

$$d_\mu(q,t)\;\;m_{\lambda\mu}\;\;=\;\;\big\langle\,P_\lambda\;,\;D(u)\,P_\mu\big\rangle_{q,t}\;\;=\;\;0\;\;.$$

We cannot have $d_\mu(q,t)=0$ for otherwise P_μ itself would vanish and this would contraddict 32.34 c). Thus we must conclude that $m_{\mu\lambda}=0$ for all $\mu<\lambda$ and 3.43 necessarily follows. The result in 3.44 follows by equating coefficients of u^k in both sides of 3.43. This completes our proof.

There are a number of remarkable identities that may be derived from 3.44. We shall present here those that can be obtained by setting $q=0$. To this end note that since for $q=0$ the Macdonald scalar product reduces the Hall-Littlewood scalar product, from 3.34 b) and c) we derive that

$$b)\qquad\big\langle P_\lambda[X_n;0,t]\;,\;P_\mu[X_n;0,t]\big\rangle_{q,t}\;\;=\;\;0\quad for\quad\lambda\neq\mu\;\;,$$

$$c)\qquad P_\lambda[X_n;0,t]\;\;=\;\;S_\lambda[X_n]\;\;+\;\sum_{\mu<\lambda}S_\mu[X_n]\,\xi_{\mu\lambda}\;\;,$$

Comparing with 3.3 we deduce that $P_\lambda[X_n,0,t]$ must be the Hall-Littlewood polynomial indexed by λ. To avoid confusions we shall denote the latter $P^{HL}[X_n;t]$ or simply P^{HL}. Denoting by $D^{q=0}(u)$ and $D_k^{q=0}$ the Macdonald operators with $q=0$ we deduce from 3.43 and 3.44 that

$$D^{q=0}(u)\,P_\lambda^{HL}[X_n;t]\;\;=\;\;\Big(\prod_{i=l(\lambda)+1}^{n}\,(1+u\,t^{n-i})\Big)\,P_\lambda^{HL}(X_n;t]\qquad 3.46$$

and

$$D_k^{q=0}\,P_\lambda^{HL}[X_n;t]\;\;=\;\;e_k\Big[\sum_{i=l(\lambda)+1}^{n}\,t^{n-i}\Big]\,P_\lambda^{HL}(X_n;t]\;\;.\qquad 3.47$$

Looking closer at the implications of this identity we derive the follow important result.

Theorem 3.8

For any partition λ of length l and for any $k=1,..,n$ we have

$$\sum_{\substack{I\subseteq[1,n]\\|I|=k}}\Big(\prod_{\substack{i\in I\\j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\Big)\,P^{HL}[X_I;1/t]\;\;=\;\;t^{l(n-k)}\begin{bmatrix}n-l\\n-k\end{bmatrix}_t\;P_\lambda^{HL}[X-n;t]\qquad 3.48$$

In particular for $\lambda = 1^k$ we deduce that

$$\sum_{\substack{I\subseteq[1,n]\\ |I|=k}} \Big(\prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\Big)\ \widetilde{X}_I \quad = \quad t^{k(n-k)}\ \ e_k[X] \tag{3.49}$$

where for convenience we have set

$$X_I = \sum_{i\in I} x_i \qquad \text{and} \qquad \widetilde{X}_I = \prod_{i\in I} x_i$$

Proof

Recall from 3.21 that for any symmetric polynomial P we have

$$D_k\ P[X_n] \quad = \quad \sum_{\substack{I\subseteq[1,n]\\ |I|=k}} A_I(x,t)\ P[X_n + (q-1)X_I]$$

Thus setting $q = 0$ in 3.47 yields that

$$\sum_{\substack{I\subseteq[1,n]\\ |I|=k}} A_I(x,t)\ P_\lambda^{HL}[X_n - X_I; t] \quad = \quad e_k[1 + t + \cdots + t^{n-l-1}]\ P_\lambda[x_n; t]\ \ . \tag{3.50}$$

Now it is well known that for any pair k, a we have

$$e_k[1 + t + \cdots + t^{a-1}] \quad = \quad t^{\binom{k}{2}} \begin{bmatrix} a \\ k \end{bmatrix}_t\ \ .$$

Thus recalling the definition of A_i given in 3.20 we may rewrite 3.50 as

$$t^{\binom{k}{2}} \sum_{\substack{I\subseteq[1,n]\\ |I|=k}} \Big(\prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\Big)\ \ P_\lambda^{HL}[X_{\,{}^cI}; t] \quad = \quad t^{\binom{k}{2}} \begin{bmatrix} n-l \\ k \end{bmatrix}_t P_\lambda[x_n; t] \tag{3.51}$$

where for convenience we have set

$$X_{\,{}^cI} \quad = \quad X_n - X_I = \sum_{i\notin I} x_i\ \ .$$

Cancelling the common factor $t^{\binom{k}{2}}$ and summing over $J = {}^cI$ rather than over I we may yet rewrite 3.51 with k replaced by $n-k$ as

$$\sum_{\substack{J\subseteq[1,n]\\ |J|=k}} \Big(\prod_{\substack{j\in J\\ i\notin J}} \frac{x_j - tx_i}{x_j - x_i}\Big)\ \ P_\lambda^{HL}[X_J; t] \quad = \quad \begin{bmatrix} n-l \\ n-k \end{bmatrix}_t P_\lambda[x_n; t]\ \ .$$

We now change t into $1/t$ and get

$$t^{-k(n-k)} \quad \sum_{\substack{J\subseteq[1,n]\\ |J|=k}} \Big(\prod_{\substack{j\in J\\ i\notin J}} \frac{tx_j - x_i}{x_j - x_i}\Big)\ \ P_\lambda^{HL}[X_J; 1/t] \quad = \quad \begin{bmatrix} n-l \\ n-k \end{bmatrix}_{1/t} P_\lambda[x_n; 1/t]\ \ . \tag{3.52}$$

Form the definition

$$\begin{bmatrix} a \\ b \end{bmatrix}_t = \frac{(1-t)(1-t^2)\cdots(1-t^a)}{(1-t)(1-t^2)\cdots(1-t^b)(1-t)(1-t^2)\cdots(1-t^{a-b})}$$

we easily derive that

$$\begin{bmatrix} a \\ b \end{bmatrix}_{1/t} = t^{-b(a-b)} \begin{bmatrix} a \\ b \end{bmatrix}_t$$

Using this in 3.52 with $a = n-l$ and $b = n-k$ yields

$$t^{-k(n-k)} \sum_{\substack{J \subseteq [1,n] \\ |J|=k}} \Big(\prod_{\substack{j\in J \\ i \notin J}} \frac{tx_j - x_i}{x_j - x_i} \Big) \; P_\lambda^{HL}[X_J; 1/t] \;\; = \;\; t^{-(n-k)(k-l)} \; \begin{bmatrix} n-l \\ n-k \end{bmatrix}_t P_\lambda[x_n; 1/t] \;\; .$$

and 3.48 follows by cancelling the factor $t^{-k(n-k)}$, replacing J by I and interchanging the roles of i and j. Finally, the identity in 3.49 follows by setting $\lambda = 1^k$ in 3.48 because the triangularity in 3.3 a) yields that P_{1^k} must necessarily be the elementary symmetric function e_k and thus in this case $P_\lambda[X_I]$ simply reduces to the product $\widetilde{X}_i = \prod_{i\in I} x_i$.