A rewriting of Jennifer's derivation of the action of the Macdonald operator D_1 on $S_\lambda[X_n \frac{1-t}{1-q}]$

1. Plethystic formulas for a family of Macdonald-like operators.

For every integer $k \geq 0$ set

$$D_1^{(k)} \;=\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i - x_j}{x_i - x_j}\Big)\, x_i^k\, T_q^{(i)} \qquad 1.1$$

Where $T_q^{(i)}$ acts on polynomials in the alphabet $X_n = x_1 + x_2 + \cdots + x_n$ by replacing x_i with qx_i. That is

$$T_q^{(i)}\, P[X_n] = P[X_n + (q-1)x_i] \qquad 1.2$$

We have the following basic identities

Theorem 1.1

For all symmetric polynomials in X_n we have

$$D_1^{(k)} P[X_n] \;=\; \chi(k=0)\,\frac{P[X_n]}{1-t} \;+\; \frac{t^{n-k}}{t-1} P\big[X_n + \tfrac{q-1}{t\,z}\big]\,\Omega\big[(t-1)X_n z\big]\,\Big|_{z^k} \;. \qquad 1.3$$

where the symbol "$\Big|_{z^k}$" means that we must take the coefficient of z^k in the preceding expression.

Proof

Using the Cauchy kernel

$$\Omega\big[X_n Y_n\big] \;=\; \sum_\lambda S_\lambda[X_n] S_\lambda[Y_n]$$

as a generating function of the Schur basis we can establish 1.3 for all Schur functions (and therefore for all symmetric polynomials) by showing that

$$D_1^{(k)}\Omega\big[X_n Y_n\big] \;=\; \chi(k=0)\,\frac{\Omega\big[X_n Y_n\big]}{1-t} \;+\; \frac{t^{n-k}}{t-1}\Omega\big[\big(X_n + \tfrac{q-1}{t\,z}\big)Y_n\big]\,\Omega\big[(t-1)X_n z\big]\,\Big|_{z^k} \;. \qquad 1.4$$

To this end note that from the definition 1.1 we derive that

$$D_1^{(k)}\Omega\big[X_n Y_n\big] \;=\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i - x_j}{x_i - x_j}\Big)\, x_i^k\, \Omega\big[\big(X_n + (q-1)x_i\big)Y_n\big]$$

or equivalently

$$\frac{D_1^{(k)}\Omega\big[X_n Y_n\big]}{\Omega\big[X_n Y_n\big]} \;=\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i - x_j}{x_i - x_j}\Big)\, x_i^k\, \Omega\big[(q-1)x_i Y_n\big] \;. \qquad 1.5$$

Expanding $\Omega\big[(q-1)x_iY_n\big]$ in terms of the homogeneous basis gives

$$\Omega\big[(q-1)x_iY_n\big] \;=\; \sum_{m\geq 0} h_m[(q-1)Y_n]\; x_i^m$$

Substituting this in 1.5 gives

$$\frac{D_1^{(k)}\Omega\big[X_nY_n\big]}{\Omega\big[X_nY_n\big]} \;=\; \sum_{m\geq 0} h_m[(q-1)Y_n]\;\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i-x_j}{x_i-x_j}\Big)\, x_i^{k+m}\;. \qquad 1.6$$

Note now that the kernel

$$\Omega\big[(t-1)X_nz\big] \;=\; \prod_{i=1}^n \frac{1-z\,x_i}{1-z\,t\,x_i}$$

has a partial fraction expansion of the form

$$\Omega\big[(t-1)X_nz\big] \;=\; \frac{1}{t^n} \;+\; \sum_{i=1}^n A_i(x;t)\,\frac{1}{1-z\,t\,x_i} \qquad 1.7$$

Multiplying both sides by $1-z\,t\,x_i$ and passing to the limit as $z{\rightarrow}1/tx_i$ gives

$$A_i(x;t) \;=\; \frac{t-1}{t^n}\Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i-x_j}{x_i-x_j}\Big)\;.$$

Substituting in 1.7 and equating coefficients of z^m on both sides we get that

$$\sum_{i=1}^n \Big(\prod_{i=1}^n \frac{tx_i-x_j}{x_i-x_j}\Big)\, x_i^m \;=\; \frac{1}{t^m}\left(\frac{t^n\Omega\big[(t-1)X_n\,z\big]}{t-1} \;-\; \frac{1}{t-1}\right)\Big|_{z^m}\;.$$

Using this with $m{\rightarrow}m+k$ we may rewrite 1.6 as

$$\frac{D_1^{(k)}\Omega\big[X_nY_n\big]}{\Omega\big[X_nY_n\big]} \;=\; \sum_{m\geq 0} h_m[(q-1)Y_n]\;\frac{1}{t^{m+k}}\left(\frac{t^n\Omega\big[(t-1)X_n\,z\big]}{t-1} \;-\; \frac{1}{t-1}\right)\Big|_{z^{m+k}}\;.$$

Since the term $1/(t-1)$ inside the big parentheses makes no contribution when $m+k>0$, we derive that

$$\begin{aligned}\frac{D_1^{(k)}\Omega\big[X_nY_n\big]}{\Omega\big[X_nY_n\big]} \;&=\; \chi(k=0)\,\frac{1}{1-t} \;+\; \sum_{m\geq 0} h_m\Big[\frac{(q-1)}{t\,z}Y_n\Big]\;\frac{1}{t^k}\;\frac{t^n\Omega\big[(t-1)X_n\,z\big]}{t-1}\;\Big|_{z^k}\\ &=\; \chi(k=0)\,\frac{1}{1-t} \;+\; \frac{t^{n-k}}{t-1}\sum_{m\geq 0} h_m\Big[\frac{(q-1)}{t\,z}Y_n\Big]\;\;\Omega\big[(t-1)X_n\,z\big]\;\Big|_{z^k}\;.\end{aligned}$$

In summary we get

$$\frac{D_1^{(k)}\Omega\big[X_nY_n\big]}{\Omega\big[X_nY_n\big]} \;=\; \chi(k=0)\,\frac{1}{1-t} \;+\; \frac{t^{n-k}}{t-1}\Omega\Big[\frac{(q-1)}{t\,z}Y_n\Big]\;\;\Omega\big[(t-1)X_n\,z\big]\;\Big|_{z^k}\;.$$

Multiplying both sides by $\Omega\big[X_nY_n\big]$ gives 1.4 and completes the proof of the theorem.

From this result we may easily derive the action of $D_q^{(k)}$ on the complete homogeneous symmetric functions in the alphabet $X_n\frac{t-1}{q-1}$. This may be stated as follows

Proposition 1.1

$$\begin{aligned} D_1^{(k)}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] \;&=\; \tfrac{\chi(k=0)}{1-t}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1} h_m\big[X_n\tfrac{t-1}{q-1}\big] h_k\big[(t-1)X_n\big] \;+ \\ &\qquad + t^{n-k-1}\sum_{r=1}^{m} h_{m-r}[X_n\tfrac{t-1}{q-1}]\; h_{k+r}\big[(t-1)X_n\big] \end{aligned} \tag{1.8}$$

Proof

Using 1.3 with $P[X_n]=h_m\big[X_n\frac{t-1}{q-1}\big]$ gives

$$\begin{aligned} D_1^{(k)}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] \;&=\; \tfrac{\chi(k=0)}{1-t}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1}\, h_m\big[X_n\tfrac{t-1}{q-1} + \tfrac{t-1}{t\,z}\big]\,\Omega\big[(t-1)X_nz\big]\,\Big|_{z^k} \\ &=\; \tfrac{\chi(k=0)}{1-t}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1}\sum_{r=0}^{m} h_{m-r}\big[X_n\tfrac{t-1}{q-1}\big] h_r\big[\tfrac{t-1}{t\,z}\big]\,\Omega\big[(t-1)X_nz\big]\,\Big|_{z^k}\;. \end{aligned} \tag{1.9}$$

Since

$$h_r\big[\tfrac{t-1}{t\,z}\big] \;=\; \begin{cases} 1 & \text{if } r=0 \\ \frac{1}{z^r}\frac{(t-1)}{t} & \text{if } r>1 \end{cases}$$

we can rewrite 1.9 as

$$\begin{aligned} D_1^{(k)} h_m\big[X_n\tfrac{t-1}{q-1}\big] &= \tfrac{\chi(k=0)}{1-t}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1}\, h_m\big[X_n\tfrac{t-1}{q-1}\big]\, h_k\big[(t-1)X_n\big] \\ &\qquad + \tfrac{t^{n-k}}{t-1}\sum_{r=1}^{m} h_{m-r}\big[X_n\tfrac{t-1}{q-1}\big]\tfrac{1}{z^r}\,\tfrac{t-1}{t}\,\Omega\big[(t-1)X_nz\big]\,\Big|_{z^k} \\ &= \tfrac{\chi(k=0)}{1-t}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1}\, h_m\big[X_n\tfrac{t-1}{q-1}\big]\, h_k\big[(t-1)X_n\big] \\ &\qquad + t^{n-k-1}\sum_{r=1}^{m} h_{m-r}\big[X_n\tfrac{t-1}{q-1}\big]\,\big]\; h_{r+k}\big[(t-1)X_n\big]\;\;, \end{aligned}$$

and this proves 1.8.

Corollary 1.1

$$D_1^{(0)}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] \;=\; \big(1+t+\cdots+t^{n-2}+t^{n-1}q^m\big)\; h_m\big[X_n\tfrac{t-1}{q-1}\big] \tag{1.10}$$

and for $k\geq 1$*:*

$$\begin{aligned} D_1^{(k)}\, h_m\big[X_n\tfrac{t-1}{q-1}\big] &= t^{n-k-1}\, h_{m+k}\big[qX_n\tfrac{t-1}{q-1}\big] + \tfrac{t^{n-k}}{t-1} h_m\big[X_n\tfrac{t-1}{q-1}\big]\, h_k\big[(t-1)X_n\big] \\ &\qquad -\;\; t^{n-k-1}\sum_{r=0}^{k} h_{m+k-r}\big[X_n\tfrac{t-1}{q-1}\big]\; h_r\big[(t-1)X_n\big]\;\;. \end{aligned} \tag{1.11}$$

Proof

For $k=0$ 1.8 gives

$$\begin{aligned} D_1^{(0)}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] &= \tfrac{1-t^n}{1-t}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] \;+\; t^{n-1}\sum_{r=1}^{m} h_{m-r}\big[X_n \tfrac{t-1}{q-1}\big] h_r\big[(t-1)X_n\big] \\ &= \tfrac{1-t^{n-1}}{1-t}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] \;+\; t^{n-1}\sum_{r=0}^{m} h_{m-r}\big[X_n \tfrac{t-1}{q-1}\big] h_r\big[(t-1)X_n\big] \\ &= \tfrac{1-t^{n-1}}{1-t}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] \;+\; t^{n-1}\, h_m\big[X_n \tfrac{t-1}{q-1}+(t-1)X_n\big] \end{aligned}$$

and 1.9 follows since

$$X_n \tfrac{t-1}{q-1}+(t-1)X_n \;=\; q\, X_n \tfrac{t-1}{q-1}\;. \tag{1.12}$$

On the other hand for $k\geq 1$ we can rewrite 1.8 as

$$\begin{aligned} D_1^{(k)}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] &= \tfrac{t^{n-k}}{t-1}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] h_k\big[(t-1)X_n\big] \;+\; t^{n-k-1}\sum_{r=k+1}^{k+m} h_{m+k-r}\big[X_n \tfrac{t-1}{q-1}\big] h_r\big[(t-1)X_n\big] \\ &= \tfrac{t^{n-k}}{t-1}\, h_m\big[X_n \tfrac{t-1}{q-1}\big] h_k\big[(t-1)X_n\big] \;+\; t^{n-k-1} h_{m+k}\big[X_n \tfrac{t-1}{q-1}+(t-1)X_n\big] \\ &\qquad -\; t^{n-k-1}\sum_{r=0}^{k} h_{m+k-r}\big[X_n \tfrac{t-1}{q-1}\big] h_r\big[(t-1)X_n\big] \end{aligned}$$

and 1.11 follows again because of 1.12.

2. The action of the Macdonald operator $\mathbf{D_1}$ on $\mathbf{S}_\lambda\big[\mathbf{X_n}\frac{\mathbf{t-1}}{\mathbf{q-1}}\big]$.

To economize on space we shall use the abvreviations

$$X_n \tfrac{t-1}{q-1} \to X_n^{tq} \quad , \qquad X_n(t-1)\to X_n^t \quad , \qquad x_i(t-1)\to x_i^t$$

This given, we may write for $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n)$

$$S_\lambda\big[X_n \tfrac{t-1}{q-1}\big] \;=\; det\| h_{\alpha_r}[X_n^{tq}]\,,\; h_{\alpha_r+1}[X_n^{tq}]\,,\;\cdots\;,\; h_{\alpha_r+n-1}[X_n^{tq}]\;\|_{r=1}^n \tag{2.1}$$

where for convenience we have set

$$\alpha_r=\lambda_r-r+1 \tag{2.2}$$

We shall also denote by $\xi_r^{(k)}$ the operation of adding r to the index of every entry in the k^{th} column of the Jacobi-Trudi matrix in 2.1 and likewise by $\mathbf{T}^{(k)}$ the operation of making the replacement $X_n\to qX_n$ in every entry in the k^{th} column.

With these conventions we have

Theorem 2.1

$$D_1\, S_\lambda[X_n^{tq}] \;=\; \sum_{k=1}^{n} t^{n-k}\, det\, \mathbf{T}^{(k)}\, \| h_{\alpha_j}[X_n^{tq}]\,,\; h_{\alpha_j+1}[X_n^{tq}]\,,\;\cdots\;,\; h_{\alpha_j+n-1}[X_n^{tq}]\;\|_{j=1}^n \tag{2.3}$$

Proof

By definition $D_1 = D_1^{(0)}$ thus

$$D_1\ S_\lambda[X_n^{tq}] \;=\; \sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n \frac{tx_i - x_j}{x_i - x_j}\Big) T_q^{(i)}\ S_\lambda[X_n^{tq}]\ . \tag{2.4}$$

However, from 2.1 we derive that

$$T_q^{(i)}\ S_\lambda[X_n^{tq}] \;=\; det\|h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+1}[X_n^{tq}+x_i^t]\,,\ \cdots\,,\ h_{\alpha_r+n-1}[X_n^{tq}+x_i^t]\,\|_{r=1}^n\ . \tag{2.5}$$

Now the relation

$$h_m[X_n^{tq}+x_i^t] \;=\; h_m[X_n^{tq}+tx_i-x_i] \;=\; h_m[X_n^{tq}-x_i] + tx_i\, h_{m-1}[X_n^{tq}+x_i^t]$$

allows us to write 2.5 in the form

$$T_q^{(i)}\ S_\lambda[X_n^{tq}] \;=\; det\Big\|h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+1}[X_n^{tq}-x_i]\,,\ \cdots\,,\ h_{\alpha_r+n-1}[X_n^{tq}-x_i]\ \Big\|_{r=1}^n\ . \tag{2.6}$$

Furthermore using the relation

$$h_m[X_n^{tq}-x_i] \;=\; h_m[X_n^{tq}] \;-\; x_i\, h_{m-1}[X_n^{tq}]$$

for each entry in columns $2, 3, \ldots, n$ of the matrix in 2.6 we may yet rewrite 2.6 in the form

$$T_q^{(i)}\ S_\lambda[X_n^{tq}] \;=\; det\ \left(\Big(\prod_{s=2}^n \big(1 - x_i\ \xi_{-1}^{(s)}\big)\Big)\Big\|h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+1}[X_n^{tq}]\,,\ \cdots\,,\ h_{\alpha_r+n-1}[X_n^{tq}]\ \Big\|_{r=1}^n\right)\ . \tag{2.7}$$

Now note that, for $2 \leq s_1 < s_2 < \cdots < s_{k-1} \leq n$, we have

$$det\ \Big(\xi_{-1}^{(s_1)}\xi_{-1}^{(s_2)}\cdots\xi_{-1}^{(s_{k-1})}\Big\|h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+1}[X_n^{tq}]\,,\ \cdots\,,\ h_{\alpha_r+n-1}[X_n^{tq}]\ \Big\|_{r=1}^n\Big) \neq 0$$

only when

$$s_1 = 2\ ,\ s_2 = 3\ ,\ \ldots\ ,\ s_{k-1} = k\ .$$

Thus 2.7 reduces to

$$T_q^{(i)}\ S_\lambda[X_n^{tq}] \;=\; \sum_{k=1}^n (-x_i)^{k-1} det\ \Big(\ \xi_{-1}^{(2)}\xi_{-1}^{(3)}\cdots\xi_{-1}^{(k)}\ \Big\|h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+1}[X_n^{tq}]\,,\ \cdots\,,\ h_{\alpha_r+n-1}[X_n^{tq}]\ \Big\|_{r=1}^n\Big)\ ,$$

and this is easily seen to be equivalent to

$$T_q^{(i)}\ S_\lambda[X_n^{tq}] = \sum_{k=1}^n det\ \Big\|h_{\alpha_r}[X_n^{tq}]\,,\cdot\cdot\,,h_{\alpha_r+k-2}[X_n^{tq}]\,,\ x_i^{k-1}h_{\alpha_r}[X_n^{tq}+x_i^t]\,,\ h_{\alpha_r+k}[X_n^{tq}]\,,\cdot\cdot\,,h_{\alpha_r+n-1}[X_n^{tq}]\ \Big\|_{r=1}^n\ .$$

Using this we may rewrite 2.4 in the remarkable form

$$D_1\, S_\lambda[X_n^{tq}] = \sum_{k=1}^{n} det\left\| h_{\alpha_r}[X_n^{tq}]\,, \cdot\cdot\,, h_{\alpha_r+k-2}[X_n^{tq}]\ ,\ D_1^{(k-1)} h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+k}[X_n^{tq}]\,, \cdot\cdot\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n} \quad 2.8$$

Note that the first term in this sum is

$$det\left\| D_1^{(0)} h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}$$

and the substitution of 1.10 with $m = \alpha_s$ reduces it to

$$\begin{aligned} &\tfrac{1-t^{n-1}}{1-t}\, det\left\| h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}\ + \\ &\qquad +\ t^{n-1} det\left\| q^{\alpha_r} h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n} \qquad 2.9 \\ &\qquad = \tfrac{1-t^{n-1}}{1-t} S_\lambda[X_n^{tq}]\ +\ t^{n-1} det\left(\mathbf{T}^{(1)} \left\| h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}\right. . \end{aligned}$$

For the remaining terms in 2.8 we use 1.11 with $k \rightarrow k-1$ and $m = \alpha_r$ and obtain that

$$\begin{aligned} D_1^{(k)} h_{\alpha_r}[X_n^{tq}] \ &=\ t^{n-k} h_{\alpha_r+k-1}[qX_n^{tq}]\ +\ \tfrac{t^{n-k+1}}{t-1} h_{\alpha_r}[X_n^{tq}]\, h_{k-1}[X_n^t] \\ &\qquad\qquad -\ t^{n-k} \sum_{r=0}^{k-1} h_{\alpha_r+k-1-r}[X_n^{tq}]\ h_r[X_n^t] \\ &=\ t^{n-k}\, h_{\alpha_r+k-1}[qX_n^{tq}]\ -\ t^{n-k}\, h_{\alpha_r+k-1}[X_n^{tq}] \\ &\qquad\qquad +\ \tfrac{t^{n-k+1}}{t-1} h_{\alpha_r}[X_n^{tq}]\, h_{k-1}[X_n^t]\ -\ t^{n-k} \sum_{r=1}^{k-1} h_{\alpha_r+k-1-r}[X_n^{tq}]\ h_r[X_n^t]\ . \end{aligned}$$

Note that using this relation in k^{th} summand of 2.8 produces $2+k$ determinants the last k of which vanish identically by the presence of two equal columns. Combining the resulting terms with 2.9, we are led to the final conclusion that

$$\begin{aligned} D_1\, S_\lambda[X_n^{tq}] \ &=\ \tfrac{1-t^{n-1}}{1-t} S_\lambda[X_n^{tq}]\ + \\ &\quad t^{n-1} det\left(\mathbf{T}^{(1)} \left\| h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}\right.\ + \\ &\quad \sum_{k=2}^{n} t^{n-k} det\left(\mathbf{T}^{(k)} \left\| h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}\right. \\ &\quad -\ \sum_{k=2}^{n} t^{n-k} det\left(\left\| h_{\alpha_r}[X_n^{tq}]\ ,\ h_{\alpha_r+1}[X_n^{tq}]\,, \ldots\,, h_{\alpha_r+n-1}[X_n^{tq}]\ \right\|_{r=1}^{n}\right.\ . \end{aligned}$$

which proves 2.3 precisely as asserted since the first and last terms here exactly cancel each other.