5. Reverse plane partitions

A "*Reverse Plane Partition of shape* λ " (briefly an RPP) is a filling of the Ferrers diagram of λ with integers ≥ 0 which weakly increases as we go North or East. It will be convenient to denote the collection of all Reverse Plane Partitions of shape λ by the symbol "$\mathcal{RPP}(\lambda)$". In the diagram below we have illustrated such a filling of the Ferrers diagram of the partition $\lambda = (5,4,4,2)$

$$\pi = \begin{matrix} 3 & 4 & & & \\ 2 & 4 & 4 & 4 & \\ 1 & 2 & 3 & 3 & \\ 1 & 1 & 1 & 2 & 2 \end{matrix} \qquad 5.1$$

A "*reverse plane partition of* n" is simply a filling whose entries add up to n, and if π is such a filling we shall set $|\pi| = n$. Note that the generating function

$$F_{\mathcal{RPP}(\lambda)}(q) \;=\; \sum_{\pi \in \mathcal{RPP}(\lambda)} q^{|\pi|}$$

may be computed by means of the Stanley-MacMahon formula

$$F_{\mathcal{RPP}(\lambda)}(q) \;=\; \sum_{f \in \mathcal{F}(\mathcal{P}_\lambda)} q^{|f|} \;=\; \frac{\sum_{\sigma \in \mathcal{L}(\mathcal{P}_\lambda)} q^{maj(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^N)} \qquad 5.2$$

where N is the number of cells of λ and $\mathcal{P}_\lambda$ is the poset whose elements are the cells of the diagram of λ where for two cells (i,j) and (i',j') we set

$$(i',j') \leq (i,j) \quad \Longleftrightarrow \quad (i \leq i') \;\&\; (j \leq j')$$

Since the reversing of inequalities may be confusing it may be helpful to mention that the cells that are below (i,j) in this partial order are those that can be reached by steps that go East or North. In particular this means that the linear extensions of this order are given by the standard tableaux with the label i replaced by $N+1-i$. For instance, in the figure below we give the 5 linear extensions of the poset $\mathcal{P}_\lambda$ described above for the case $\lambda = (3,2)$.

$$\begin{matrix} 3 & 2 & \\ 5 & 4 & 1 \end{matrix} \;,\quad \begin{matrix} 4 & 2 & \\ 5 & 3 & 1 \end{matrix} \;,\quad \begin{matrix} 3 & 1 & \\ 5 & 4 & 2 \end{matrix} \;,\quad \begin{matrix} 4 & 1 & \\ 5 & 3 & 2 \end{matrix} \;,\quad \begin{matrix} 2 & 1 & \\ 5 & 4 & 3 \end{matrix} \;,$$

What we shall present here is a formula which is essential due to D. Littlewood which expresses $F_{\mathcal{RPP}(\lambda)}$ in terms of the hooks lengths of the cells of λ. † We recall that the hook of cell $c \in \lambda$ consists of c together with all the cells that are strictly North or strictly East of c in λ. Following Macdonald we shall refer to the cells that are East of c as the "*arm*" of c and to those that are North as the "*leg* " of c. If $a(c)$ and $l(c)$ are respectively the lengths of the arm and the leg of c, then the total number of cells in the hook of c is simply

$$h(c) \;=\; 1 + a(c) + l(c) \;.$$

These numbers are usually referred to as the "*hook lengths*" or just simply the "*hooks of* λ". In the figure below we have filled the cells of Ferrers diagram of $(5,4,4,2)$ with their respective hook lengths.

$$\begin{matrix} 2 & 1 & & & \\ 5 & 4 & 2 & 1 & \\ 6 & 5 & 3 & 2 & \\ 8 & 7 & 5 & 4 & 1 \end{matrix} \qquad 5.3$$

It develops that we have the following remarkably beautiful formula

† It will be convenient here after to identify a partition with its Ferrers diagram so that we not have to repeat saying "the Ferrers diagram of λ" each time we need to refer to it.

Theorem 5.1 (D. Littlewood)

$$F_{\mathcal{RPP}(\lambda)}(q) \;=\; \prod_{c\in\lambda} \frac{1}{1-q^{h(c)}} \tag{5.4}$$

Our main goal in this is to present the combinatorial proof of this identity given by Hillman and Grassl in J.C.T. (A) **21** 216-221 (1976).

To proceed we need some notation. To begin with we shall total order the cells of λ from right to left and from bottom to top. We give below this order for $\lambda = (5,4,4,2)$:

$$\begin{array}{ccccc} 15 & 11 & & & \\ 14 & 10 & 7 & 4 & \\ 13 & 9 & 6 & 3 & \\ 12 & 8 & 5 & 2 & 1 \end{array} \tag{5.5}$$

The main idea of the proof consists of interpreting each term of the series on the right of 5.4 as produced by a multiset of cells of λ. More precisely, if cell c_i (in the above order) has multiplicity m_i in a given multiset $\widehat{S}$ and the weight of this multiset is taken to be

$$w(\widehat{S}) \;=\; q^{\sum_i m_i h(c_i)}$$

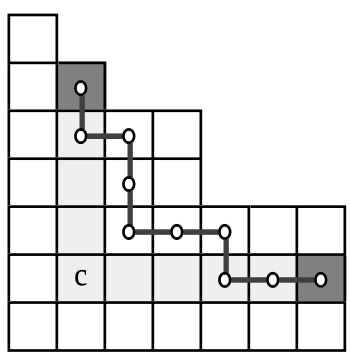
then we can easily see that we have

$$\prod_{c\in\lambda} \frac{1}{1-q^{h(c)}} \;=\; \sum_{\widehat{S}\subseteq\,\lambda} w(\widehat{S}) \;.$$

This given we may view 5.4 as saying that

$$\sum_{\pi\in\mathcal{RPP}(\lambda)} q^{|\pi|} \;=\; \sum_{\widehat{S}\subseteq\,\lambda} w(\widehat{S})$$

Thus to show 5.4 Hillman and Grassl construct a weight preserving bijection between reverse plane partitions of shape λ and multisubsets of cells of λ. Their construction is actually quite simple, since the weight preserving property is automatic, the only difficulty lies in grasping that it is indeed a bijection. Their basic observation is that hook lengths may also be obtained as "Taxi-cab" distances between the end points of a given hook. To make this point clear we only need to look at the figure below.



We have shaded the cells of the hook of $c = (2,2)$ for $\lambda = (7,7,7,4,4,2,1)$ and darkened its two ends. We can clearly see that any "taxi-cab" path (South & East steps) that joins the top end of the hook of c to the bottom end will necessarily have a number of steps equal to the length of the hook. Hillman-Grassl call c the "*pivot*" of the taxi-cap path, and their bijection consists of mapping a reverse plane partition π onto a multiset of "taxi-cab paths" and at the end only recording the corresponding multiset of "pivots". The crucial point is that, given the multiset of pivots, the original multiset of taxi-cab paths can be uniquely reconstructed.

In their construction, the taxi-cab paths are used to disassemble the given reverse plane partition π down to the zero partition by subtracting 1's from each of the entries that lie along a path. Of course, to assure that the process is recursive, they are forced to choose the successive paths so that after each step we are again left with a reverse plane partition. It develops that this condition almost uniquely determines each of the successive paths. The only thing that remains to describe is how to get started and how to organize the succession of steps in a manner that assures that they can be reversed.

The process is decomposed into a succession of stages, where each stage consists of a sequence of steps aimed at reducing to zero the entries of a given column. The organization of the stages consists in proceeding from left to right annihilating one column at each stage until all the columns are annihilated.

Each of the stages is identical:

(1) We locate the leftmost non zero column. Say it is column j.

(2) Each of the taxi-cab paths in this stage starts at the top cell of column j. It then proceeds with East or South steps according to the following rules.

   (i) If the path has reached cell c and c' is the cell immediately below, then:

      (a) proceeds by a South step if entry at c' is equal to the entry at c.

      (b) proceeds by an East step if entry at c' is smaller than the entry at c.

   (ii) the path stops when it reaches the end of a row. If that is row i then cell (i, j) is the corresponding pivot.

   (iii) we next subtract 1 from each of the entries along the path

   (iv) We repeat this process as long as there are non-zero entries in this column

   (v) The stage ends when all the entries in column j are equal to 0.

A few remarks are in order here. Firstly it is clear that rule (a) is forced by the requirement that we should be left with a reverse plane partition after we carry out (iii). Rule (b) on the other hand assures that the path remains as high as possible. The crucial fact here is that each path ends up acting as a "barrier" for the next path. Indeed, each path is stopped in its eastbound motion only by vertical pairs of equal entries, but after we carry out (iii) we are still left with the same vertical pairs of equal entries and may actually create more. This results in the successive paths, within a stage, to remain each weakly above the next. As a consequence, the successive pivots created within a given stage will always form a sequence that weakly descends down the column.

Carrying out this algorithm for a special case is all is needed here to complete our description of the Hilmann-Grassl bijection. In our example we have a total of 9 steps and 4 stages., we have depicted the process by listing the stages in separate lines. At each step, the corresponding taxi-cab path is depicted by underlining the entries along the path. At the end of each stage we depict the multiplicities of the corresponding pivots.

$$
\begin{array}{lllll}
\underline{3} & \underline{4} \\
2 & \underline{4} & \underline{4} & \underline{4} \\
1 & 2 & 3 & 3 \\
1 & 1 & 1 & 2 & 2
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
\underline{2} & 3 \\
\underline{2} & \underline{3} & \underline{3} & 3 \\
1 & 2 & \underline{3} & \underline{3} \\
1 & 1 & 1 & 2 & 2
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
\underline{1} & 3 \\
\underline{1} & 2 & 2 & 3 \\
\underline{1} & 2 & 2 & 2 \\
\underline{1} & \underline{1} & \underline{1} & \underline{2} & \underline{2}
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
0 & 3 \\
0 & 2 & 2 & 3 \\
0 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 1
\end{array}
\quad pivots \Rightarrow \quad
\begin{array}{lllll}
0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{array}
$$

$$
\begin{array}{lllll}
0 & \underline{3} \\
0 & 2 & 2 & 3 \\
0 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 1
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
0 & \underline{2} \\
0 & \underline{2} & 2 & 3 \\
0 & \underline{2} & \underline{2} & \underline{2} \\
0 & 0 & 0 & 1 & 1
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
0 & \underline{1} \\
0 & \underline{1} & 2 & 3 \\
0 & \underline{1} & \underline{1} & \underline{1} \\
0 & 0 & 0 & \underline{1} & \underline{1}
\end{array}
\;\longrightarrow\;
\begin{array}{lllll}
0 & 0 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\quad pivots \Rightarrow \quad
\begin{array}{lllll}
0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{array}
$$

$$\begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & \underline{2} & \underline{3} & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & \underline{1} & \underline{2} & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array} \quad pivots \Rightarrow \begin{array}{ccccc} 0 & 1 & & & \\ 1 & 0 & 2 & 0 & \\ 1 & 2 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & \underline{1} & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array} \quad pivots \Rightarrow \begin{array}{ccccc} 0 & 1 & & & \\ 1 & 0 & 2 & 1 & \\ 1 & 2 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

The fact that the pivots at each stage form a weakly descending sequence, is a clue for the construction of the inverse bijection. To reverse the process we start with the zero partition and construct a sequence of reverse plane partitions by processing the pivots of a given multiset in the order illustrated in 5.5. Now a given pivot (i, j) will uniquely determine the corresponding path which must start at the end of row i and proceeds by West and North steps and stop at the top of column j. Reversing rules (a) and (b) given above, the path now, having reached cell c, must to go North when the entry in the cell c' immediately above is equal to the entry in c and otherwise go West or stop when it reaches the top cell of column j. Of course, to reverse the previous process, after the construction of a path, we must now add ones to the entries along the path.

Assuming, by induction that after we process all the pivots in column $j+1$ the resulting partition has nothing but 0's West of this column, it is not difficult to conclude that each path constructed by processing the pivots of column j will in fact end at the top of the column. The reason for this is that after we reach this column for the first time we will fill it with 1's from the point of impact to the top. Since now each successive path acts as a **lower** barrier for the next, each time we impact column j it will be at a point weakly higher than the previous time, and since all the entries above are equal, the path will be forced to go North all the rest of the way and leave untouched all entries west of this column.

To complete our presentation we need only carry out a special case of the reverse algorithm. Our starting point here will be the following multiset of pivots of the diagram of $\lambda = (5, 4, 4, 2)$:

$$\begin{array}{ccccc} 0 & 0 & & & \\ 1 & 0 & 1 & 0 & \\ 2 & 1 & 0 & 3 & \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

Note at the onset that we shall have 12 steps (one for each pivot) and 4 stages (one for each non-empty column). We begin by processing the rightmost non-empty column:

$$\begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & \underline{0} & \\ 0 & 0 & 0 & \underline{0} & \\ 0 & 0 & 0 & \underline{0} & \underline{0} \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & \underline{1} & \\ 0 & 0 & 0 & \underline{1} & \\ 0 & 0 & 0 & 1 & 1 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & \underline{2} & \\ 0 & 0 & 0 & \underline{2} & \\ 0 & 0 & 0 & 1 & 1 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & \underline{3} & \\ 0 & 0 & 0 & \underline{3} & \\ 0 & 0 & 0 & 1 & 1 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 0 & 4 & \\ 0 & 0 & 0 & 4 & \\ 0 & 0 & 0 & 1 & 1 \end{array}$$

That leaves us with the multiset of pivots:

$$\begin{array}{ccccc} 0 & 0 & & & \\ 1 & 0 & 1 & 0 & \\ 2 & 1 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 0 \end{array}$$

and the next stage gives

$$\begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & \underline{0} & 4 & \\ 0 & 0 & \underline{0} & 4 & \\ 0 & 0 & \underline{0} & \underline{1} & \underline{1} \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & \underline{1} & \underline{4} & \\ 0 & 0 & 1 & 4 & \\ 0 & 0 & 1 & 2 & 2 \end{array} \longrightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 0 & 0 & 2 & 5 & \\ 0 & 0 & 1 & 4 & \\ 0 & 0 & 1 & 2 & 2 \end{array} \quad pivots \Rightarrow \begin{array}{ccccc} 0 & 0 & & & \\ 1 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 \end{array}$$

Stage 3:

$$
\begin{array}{lllll}
0 & \underline{0} & & & \\
0 & \underline{0} & 2 & 5 & \\
0 & \underline{0} & \underline{1} & 4 & \\
0 & 0 & \underline{1} & \underline{2} & \underline{2}
\end{array}
\longrightarrow
\begin{array}{lllll}
0 & 1 & & & \\
0 & 1 & 2 & 5 & \\
0 & 1 & 2 & 4 & \\
0 & 0 & 2 & 3 & 3
\end{array}
\longrightarrow
\begin{array}{lllll}
0 & \underline{1} & & & \\
0 & \underline{1} & \underline{2} & 5 & \\
0 & 1 & \underline{2} & \underline{4} & \\
0 & 0 & 2 & 3 & 3
\end{array}
\longrightarrow
\begin{array}{lllll}
0 & 2 & & & \\
0 & 2 & 3 & 5 & \\
0 & 1 & 3 & 5 & \\
0 & 0 & 2 & 3 & 3
\end{array}
\quad pivots \Rightarrow
\begin{array}{lllll}
0 & 0 & & & \\
1 & 0 & 0 & 0 & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & 0
\end{array}
$$

Stage 4:

$$
\begin{array}{lllll}
\underline{0} & 2 & & & \\
\underline{0} & 2 & 3 & 5 & \\
\underline{0} & 1 & 3 & 5 & \\
\underline{0} & \underline{0} & \underline{2} & \underline{3} & \underline{3}
\end{array}
\longrightarrow
\begin{array}{lllll}
\underline{1} & \underline{2} & & & \\
1 & \underline{2} & \underline{3} & \underline{5} & \\
1 & 1 & 3 & \underline{5} & \\
1 & 1 & 3 & 4 & 4
\end{array}
\longrightarrow
\begin{array}{lllll}
\underline{2} & \underline{3} & & & \\
1 & \underline{3} & \underline{4} & \underline{6} & \\
1 & 1 & 3 & \underline{6} & \\
1 & 1 & 3 & 4 & 4
\end{array}
\longrightarrow
\begin{array}{lllll}
\underline{3} & \underline{4} & & & \\
1 & \underline{4} & \underline{5} & \underline{7} & \\
1 & 1 & 3 & 7 & \\
1 & 1 & 3 & 4 & 4
\end{array}
\longrightarrow
\begin{array}{lllll}
4 & 5 & & & \\
1 & 5 & 6 & 8 & \\
1 & 1 & 3 & 7 & \\
1 & 1 & 3 & 4 & 4
\end{array}
$$

This completes our presentation of the Hilman-Grassl bijection.

An important corollary of Theorem 5.1 is the famed "*hook formula*" for the number of standard tableaux.

Theorem 5.2(Frame, Robinson & Thrall)

For any $\lambda \vdash n$ we have

$$f_\lambda \;=\; \frac{n!}{\prod_{c\in\lambda} h(c)} \tag{5.6}$$

where f_λ denotes the number of standard tableaux of shape λ.

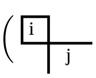
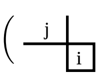
Proof

Combining 5.2 for $N = n$ and 5.4 we obtain that

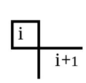
$$\sum_{\sigma\in\mathcal{L}(\mathcal{P}_\lambda)} q^{maj(\sigma)} \;=\; \frac{(1-q)(1-q^2)\cdots(1-q^n)}{\prod_{c\in\lambda}(1-q^{h(c)})} \tag{5.7}$$

and 5.6 then simply follows by letting $q \to 1$ and recalling that the linear extensions of the poset $\mathcal{P}_\lambda$ are in one to one correspondence with the standard tableaux of shape λ.

Formula 5.7 may be viewed as a q-analogue of 5.6. However, in this form, to compute the left hand side we must first chose a labeling of $\mathcal{P}_\lambda$, construct from each standard tableau of shape λ the corresponding linear extension and finally read the labels in the order given by this linear extension. Now it develops that there is a way to state the same result in a manner that makes a more direct use of standard tableaux. To this end let the symbols

$\chi_T\Big($  $\Big)$ or $\chi_T\Big($  $\Big)$ 5.8

respectively denote that the index j is strictly South-East or strictly North-Westof i in a tableau T. Given a standard tableau T, we shall say that the index i is a "*descent*" of T if and only if $i+1$ is strictly south of i in T. Note that, since in a standard tableau, $i+1$ cannot be weakly South-West of i, this condition results in the first of 5.8 being true for $j = i+1$. This given, the collection of all descents of T will be called the "*Descent Set* " of T and denoted by "$\mathcal{D}_T$". In symbols

$\mathcal{D}_T \;=\; \Big\{\, i \in T \;:\;$  $\Big\}$ 5.9

Accordingly, we set

$$d(T) \;=\; \#\,\mathcal{D}(T) \qquad , \qquad maj(T) \;=\; \sum_{i\in T} i\,\chi\big(i\;\in \mathcal{D}(T)\big)$$

and call them respectively the "*descent number*" and the "*major index*" of T. This notation permits us to state formula 5.7 in the following elegant form.

Theorem 5.3

For $\lambda \vdash n$ we have

$$\sum_{T\in ST(\lambda)} q^{nd(T)-maj(T)} \;=\; \frac{(1-q)(1-q^2)\cdots(1-q^n)}{\prod_{c\in\lambda}\,(1-q^{h(c)})} \qquad 5.10$$

where $ST(\lambda)$ denotes the collection of standard tableaux of shape λ.

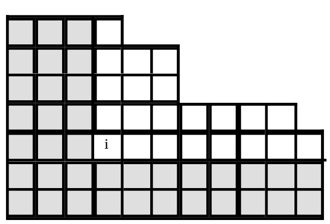
Proof

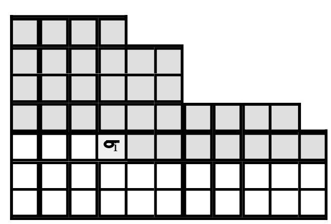
For a given standard tableau T of shape $\lambda \vdash n$, let T^* denote the tableau obtained by replacing i by $n+1-i$. From our definition of $\mathcal{P}_\lambda$ we see that T^* is a linear extension of the partial order of $\mathcal{P}_\lambda$. To obtain the permutation $\sigma = \sigma(T^*)$ that yields the corresponding summand in 5.7 we need to decide once and for all on a labeling of the cells of λ that is compatible with the partial order of $\mathcal{P}_\lambda$. The simplest choice is to label the cells with $1, 2, \ldots, n$ from right to left whithin rows starting from the top row and proceeding on down. In the figure below we give such a labeling for the partition $\lambda = (5, 4, 4, 2)$:

$$L \;=\; \begin{array}{ccccc} 2 & 1 & & & \\ 6 & 5 & 4 & 3 & \\ 10 & 9 & 8 & 7 & \\ 15 & 14 & 13 & 12 & 11 \end{array} \qquad 5.11$$

This given, the permutation $\sigma(T^*) = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is simply obtained by taking σ_i equal to the label of the cell that contains i in T^*. To do this in a more visual way, we may place T^* on top of the tableau L then take σ_i to be the label that lies below i.

This shows that we will have a descent $\sigma_i > \sigma_{i+1}$ if and only if the label σ_{i+1} falls in the region of the tableau L where all the labels less than σ_i are located. Equivalenty, the same must hold true for the entry $i+1$ of T^*. On the other hand we must not forget that since T^* comes from a standard tableau T there is also a restriction on where $i+1$ may be located. To visualize what these two conditions impose on the relative positions of i and $i+1$ we have depicted below what must happen in the generic case:

T*=


,
L=


We clearly see here that since all the entries of T^* that are North-East of i are necessarily smaller than i, the entry $i+1$ can only fall in the shaded area of the picture on the left. On the other hand the way we label our tableaux in order for σ_{i+1} to come out smaller than σ_i it must fall in the shaded area of the picture on the right. Thus we deduce that the descents of σ are the indices i of T^* such that $i+1$ is strictly North-West of i . We may thus conclude that

$$maj(\sigma) \;=\; \sum_{i=1}^{n-1} i\,\chi(\sigma_i > \sigma_{i+1}) \;=\; \sum_{i=1}^{n-1} i\,\chi_{T^*}\left(\begin{array}{c|c} i{+}1 & \\ \hline & \boxed{i} \end{array}\right)$$

Since the entry i of T^* comes from the entry $n+1-i$ in T we may rewrite this as

$$maj(\sigma) \;=\; \sum_{i=1}^{n-1} i\,\chi_T\left(\begin{array}{c|c} n\text{-}i & \\ \hline & \boxed{n\text{+}1\ \text{-}i} \end{array}\right)$$

Making the substitution $i \to n-i$ we come to the final conclusion that we have

$$maj(\sigma) \;=\; \sum_{i=1}^{n-1} (n-i)\,\chi_T\left(\begin{array}{c|c} \boxed{i} & \\ \hline & i\text{+}1 \end{array}\right) \;=\; n\,d(T) \;-\; maj(T)\;,$$

and 5.10 is obtained by substituting this in 5.7, completing our proof.

We shall terminate by verifying 5.10 in the case of the partition $\lambda = (2,2,2)$. Here, the hook numbers hare given by the tableau

$$H \;=\; \begin{array}{cc} 2 & 1 \\ 3 & 2 \\ 4 & 3 \end{array}$$

This gives that the righthand side of 5.10 is

$$RHS \;=\; \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)}{(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)}\;. \qquad 5.12$$

On the other and the five standard tableau of shape $(2,2,2)$ with underlined descents are

$$\begin{array}{cc} \underline{3} & 6 \\ 2 & 5 \\ 1 & 4 \end{array}\;,\qquad \begin{array}{cc} \underline{4} & 6 \\ \underline{2} & 5 \\ 1 & 3 \end{array}\;,\qquad \begin{array}{cc} 5 & 6 \\ \underline{2} & 4 \\ 1 & 3 \end{array}\;,\qquad \begin{array}{cc} \underline{4} & 6 \\ 3 & 5 \\ 1 & 2 \end{array}\;,\qquad \begin{array}{cc} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{array}\;.$$

Giving the statistics

$$\begin{array}{rcccccc}
T & \to & \begin{array}{cc} \underline{3} & 6 \\ 2 & 5 \\ 1 & 4 \end{array} & \begin{array}{cc} \underline{4} & 6 \\ \underline{2} & 5 \\ 1 & 3 \end{array} & \begin{array}{cc} 5 & 6 \\ \underline{2} & 4 \\ 1 & 3 \end{array} & \begin{array}{cc} \underline{4} & 6 \\ 3 & 5 \\ 1 & 2 \end{array} & \begin{array}{cc} 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{array} \\
nd(T) & \to & 6 & 12 & 6 & 6 & 0 \\
maj(T) & \to & 3 & 2+4 & 2 & 4 & 0 \\
nd(T)-maj(T) & \to & 3 & 6 & 4 & 2 & 0
\end{array}$$

Thus the lefthand side of 5.10 is

$$LHS \;=\; 1 \;+\; q^2 \;+\; q^3 \;+\; q^4 \;+\; q^6$$

Yielding that we must have

$$1+q^2+q^3+q^4+q^6 \;=\; \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)}{(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)}$$

which we can easily check to be true!