Introduction
to
The Representation Theory of Finite Groups

1. Introduction

In a February 1896 letter to Frobenius, Dedekind revealed a puzzling discovery he had made which had left him with a conjecture and a variety of questions which he could not resolve. The construct that Dedekind had created is the so called "*Group Determinant*". To be specific, given a finite group

$$G \;=\; \{g_1, g_2, \ldots, g_N\}$$

Dedekind sets

$$D(G) \;=\; det \left\| x_{g_i^{-1} g_j} \right\|_{i,j=1}^{N},$$

where

$$\{x_{g_1}, x_{g_2}, \ldots, x_{g_n}\}$$

are auxiliary indeterminates. The evaluation of this determinant, even for relatively small groups, results in a polynomial in $x_{g_1}, x_{g_2}, \ldots, x_{g_n}$ of considerable size. This notwithstanding Dedeking was able to experiment with a variety of examples and discovered that when G is abelian, $D(G)$ always decomposes into a product of linear factors. He then worked out a number of non abelian cases. In particular he noted that for the symmetric group S_3 (the permutations of 3 letters) $D(S_3)$ decomposes into the product of two linear factors and two irreducible quadratic factors. Remarkably, he had factored, by hand, a 6×6 determinant which yielded $6! = 720$ different monomials. But if this sounds impressive, note that he was also able to deal with the case when G is the semidirect product of cyclic group C_2 by the cyclic group C_5 and this involved calculating the determinant of a 10×10 matrix. Again he was able to factor $D(G)$ into linear and irreducible quadratic factors. Out of these experimentations Dedekind was led to conjecture that the number of linear factor in $D(G)$ is equal to $[G : G']$, i.e. the degree of the abelian group G/G', where G' denotes the commutator subgroup of G. However he was unable to identify the number of the irreducible factors of higher degree nor explain their nature. The crucial question he asked Frobenius in his letter was whether there was an extension of the notion of "*Group Characters*" that could account for the presence of the mysterious factors. At that time a character of a Group G was defined as a map $\chi \,:\, G\to\mathbf{C}$ satisfying the requirement

$$\chi(\alpha\beta) \;=\; \chi(\alpha)\,\chi(\beta)\,. \qquad \text{I.1}$$

The question intrigued Frobenius and in a few months he was able to completely unravel the mystery. In July 30 and December 3 1896 Frobenius presented his results to the Berlin Academy under the titles "*Over Group Characters*" and "*Over the Prime Factors of the Group Determi-*

nant". In the process he had come to discover one of the most fascinating and useful of mathematical constructs. The "*Representation Theory of Groups*", together with its fundamental theorems.

In these notes we shall develop Frobenius theory very much as it was done at the time of Schur †. The advantage of following the original setting versus the modern "module" setting is that it is is more conducive to an algorithmic development of the material which is essential if we are to perform explicit calculations whether by hand or by computer.

The definition of a representation is quite simple. It is none other than a matrix version of I.1. To be precise let $M_n(\mathbf{C})$ denote the algebra of all non-singular $n \times n$ matrices with complex entries. This given, map A of a group G into $M_n(\mathbf{C})$ which satisfies the condition

$$A(\alpha)A(\beta) = A(\alpha\,\beta) \quad \forall \quad \alpha, \beta \in G, \tag{I.2}$$

is said to be a "*Representation of G of degree n*".

Note that with $\beta = \alpha^{-1}$ I.2 implies that

$$A(\alpha)A(\alpha^{-1}) = A(\epsilon) \quad \text{(with ϵ the identity of G)} \tag{I.3}$$

Moreover taking $\alpha = \beta = \epsilon$ I.2 gives

$$A(\epsilon)A(\epsilon) = A(\epsilon) \tag{I.4}$$

Using the invertibility of $A(\epsilon)$ we see that I.3 and I.4 combine to yield

$$\begin{array}{ll} \text{a)} \;\; A(\epsilon) = I & \qquad \text{(with I is the $n \times n$ identity matrix)} \\ \text{b)} \;\; A(\alpha^{-1}) = (A(\alpha))^{-1} & \end{array} \tag{I.5}$$

In summary we see that a representation of degree n is simply a homomorphism of G into $M_n(\mathbf{C})$. Before we can proceed to study representations we need some conventions concerning the various constructs that we need in our computations. This will be done in the first section where we also present a variety of examples that show the richness of the subject we are about to develop.

1. Group actions

In these notes we shall systematically adhere to the convention of representing bases by column vectors and coefficients by row vectors. More precisely if $\mathbf{V}$ is a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of $\mathbf{V}$ then any element of $\mathbf{v} \in \mathbf{V}$ has the expansion.

$$\mathbf{v} \;\; = \;\; x_1\,\mathbf{v}_1 + x_2\,\mathbf{v}_2 + \cdots + x_n\,\mathbf{v}_n\,.$$

† Frobenius was Schur's thesis advisor

Using matrix notation we will write this identity in the form

$$\mathbf{v} \;=\; \big\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \big\rangle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} . \tag{1.1}$$

If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is another basis of $\mathbf{V}$, the expansion of the $\mathbf{u}_j$ in terms of the $\mathbf{v}_i$ will always be given in the form

$$\mathbf{u}_j \;=\; \sum_{i=1}^{n} \mathbf{v}_i \; a_{ij} \, . \qquad\qquad (\text{for } j = 1, \ldots, n) \tag{1.2}$$

With this convention, these n equations may be written as one single matrix equation. Namely

$$\big\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \big\rangle \;=\; \big\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \big\rangle \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} .$$

Moreover, if we let A be the matrix $\|a_{ij}\|_{i,j=1}^{n}$ that is

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then the transition from the basis $\mathbf{V} = \big\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \big\rangle$ to the basis $\mathbf{U} = \big\langle \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \big\rangle$ may simply be written as

$$\mathbf{U} \;=\; \mathbf{V}\, A \, . \tag{1.3}$$

In particular, we also obtain that

$$\mathbf{V} \;=\; \mathbf{U}\, A^{-1} \, , \tag{1.4}$$

where A^{-1} denotes the inverse of A. In the same vein, letting X represent the column vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We may rewrite 1.1 as

$$\mathbf{v} \;=\; \mathbf{V}\, X \, .$$

Now using 1.4 we immediately obtain that

$$\mathbf{v} \;=\; \mathbf{V}\, X \;=\; \mathbf{U}\, A^{-1}\, X \;=\; \mathbf{U}\, Y \, , \tag{1.5}$$

with

$$Y \;=\; A^{-1}\,X$$

and if

$$Y \;=\; \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then 1.5 simply says that

$$\mathbf{v} \;=\; y_1\,\mathbf{u}_1 \,+\, y_2\,\mathbf{u}_2 \,+\, \cdots + y_n\,\mathbf{u}_n\,.$$

In summary, with our conventions we derive that if the change of basis is expressed by right mutiplication by the matrix A, the change in the coefficients is expressed by left multiplication by A^{-1}. In time it will be seen that a strict adherence to these conventions not only simplifies our calculations but will avoid running into the pitfalls of confusing changes of bases with changes of coordinates.

Let $\mathcal{F}$ be a family of objects and Γ be a group whose elements are permutations of the elements of $\mathcal{F}$. For $f \in \mathcal{F}$ and $\gamma \in \Gamma$ let us denote by "$\gamma\, f$" the image of f by γ. This given we say that Γ "*acts on* $\mathcal{F}$" if

$$\alpha\big(\beta f\big) \;=\; (\alpha\beta) f \qquad \text{(for all } \alpha, \beta \in \Gamma \text{ and } f \in \mathcal{F}\,) \tag{1.6}$$

In other words, we may say that a group action is simply a homomorphism ν of a group Γ into the symmetric group $\mathcal{S}_{\mathcal{F}}$ of all permutations of $\mathcal{F}$. Under this view point, if $\nu : \Gamma \rightarrow \mathcal{S}_{\mathcal{F}}$ is the homomorphism yielding the group action then, for a given $\gamma \in \Gamma$, the image of an element f of $\mathcal{F}$ under the permutation $\nu(\gamma) \in \mathcal{S}_{\mathcal{F}}$ should perhaps more precisely denoted $\nu(\gamma)f$. However, when there is no danger of confusion we shall simply use the symbol γf and the underlying homomorphism ν will not be explicitly mentioned.

The notion of group action is better understood after a few examples. We give below a few basic ones that have occured in enumeration problems as well as in a variety of theoretical constructions.

(a) Polya actions: We have a set Ω (places) and a set R (colors) and $\mathcal{F}$ is the set of all maps $f : \Omega \rightarrow R$. We may view $\mathcal{F}$ as the collection of all different ways of coloring the places. For a given subgroup Γ of the symmetric group $\mathcal{S}_\Omega$, the Polya action induced by Γ on $\mathcal{F}$ is simply given by

$$\nu(\gamma) f = f \circ \gamma^{-1}$$

where the symbol "$\circ$" denotes functional composition.

(b) De-Bruijn actions: Ω, R and $\mathcal{F}$ are as before, Γ is a subgroup of the symmetric group $\mathcal{S}_R$ of R and the action of Γ on $\mathcal{F}$ is given by

$$\nu(\gamma)f = \gamma \circ f.$$

(c) Left multiplication : A group Γ acts on itself by left multiplication. Here $\mathcal{F} = \Gamma$ and for $f \in \Gamma$ we set

$$\nu(\gamma)f = \gamma \cdot f$$

where " $\cdot$ " denotes group multiplication.

(d) Conjugacy: A group Γ acts on itself by conjugation. Here again $\mathcal{F} = \Gamma$ and for $f \in \Gamma$ we set

$$\nu(\gamma)f = \gamma \cdot f \cdot \gamma^{-1}.$$

Further examples of group actions will be given in the sequel.

Given a group action $\nu(\gamma)$ on $\mathcal{F}$ we define an equivalence relation "$\sim_\Gamma$" on $\mathcal{F}$ by saying that

$$f_1 \sim_\Gamma f_2$$

if and only if we have

$$f_2 = \gamma f_1\,, \qquad \text{(for some } \gamma \in \Gamma).$$

The equivalence classes of $\mathcal{F}$ under $\sim_\Gamma$ are called "Orbits" and for $g \in \mathcal{F}$ we set

$$0_g = \{f \in \mathcal{F} : f \sim g\} = \{f \in \mathcal{F} : \exists\, \gamma \in \Gamma \quad \text{such that} \quad f = \gamma g\}\,,$$

and call it the "orbit" of g.

In the cases (a) and (b) of the example 1.1 the equivalence classes are called "*patterns*" and in case (d) the equivalence classes are called the "*conjugacy" classes*" of Γ.

If we have an action of a group Γ on a family $\mathcal{F}$ here and in the following we shall use the symbol Δ to denote a complete system of representatives for the equivalence classes of $\mathcal{F}$ under $\sim_\Gamma$. Thus if

$$\Delta = \{f_1, f_2, \ldots, f_k\}$$

then

$$\mathcal{F} = 0_{f_1} + 0_{f_2} + \cdots + 0_{f_k}. \tag{1.7}$$

A subset A of $\mathcal{F}$ which is a union of orbits is said to define an "*orbit property*" and we shall say that 0_f has property A if and only if $0_f \subset A$.

This means that if we set

$$\Phi(f) = \chi(f \in A)\ , \quad (*)$$

then the quantity

$$Q = \sum_{f \in \Delta} \Phi(f) \tag{1.8}$$

gives the number of orbits having property A. It is easily seen that this quantity can also be expresses in the form

$$Q = \sum_{f \in \mathcal{F}} \frac{\Phi(f)}{|0f|} \tag{1.9}$$

The collection of all the elements of Γ that leave an element f of $\mathcal{F}$ invariant will be called the "*stabilizer of f*". It will be denoted by the symbol Γ_f. Thus

$$\Gamma_f = \{\gamma \in \Gamma : \gamma f = f\}\,.$$

Fixed $f \in \mathcal{F}$, we introduce an equivalence relation "$\sim_f$" on Γ itself by setting

$$\gamma_1 \sim_f \gamma_2$$

if and only if

$$\gamma_1 f = \gamma_2 f.$$

That means we have $\gamma_1 \sim_f \gamma_2$ if and only if

$$\gamma_1^{-1}\gamma_2 \in \Gamma_f.$$

Thus if $T_f = \{\tau_1, \tau_2, \ldots, \tau_\ell\}$ is a complete set of representatives for the equivalence classes $\sim_f$ (τ_1 being the identity in Γ) then we have a decomposition of Γ into the disjoint union

$$\Gamma = \Gamma_f + \tau_2\Gamma_f + \cdots + \tau_\ell\Gamma_f. \tag{1.10}$$

This identity is usually referred to as the "*decomposition of Γ into left Γ_f-cosets*"

This implies that the orbit of f consists of the elements

$$f, \tau_2 f, \ldots, \tau_\ell f.$$

So we see them that 1.10 implies

$$|0_f| = |T_f| = \frac{|\Gamma|}{|\Gamma_f|}. \tag{1.11}$$

(*) Here and in the following if S is a statement then $\chi(S)$ is to be taken equal to one when S is true and zero when S is false.

The identity in 1.11 yields a fundamental formula of the theory namely:

Theorem 1.1

$$|\Delta| = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \sum_{f\in\mathcal{F}} \chi(\gamma f = f). \tag{1.12}$$

Proof.

From 1.7 we easily derive that

$$|\Delta| = \sum_{f\in\mathcal{F}} \frac{1}{|0_f|}.$$

Thus using 1.11 we get

$$|\Delta| = \sum_{f\in\mathcal{F}} \frac{1}{|\Gamma|}\, |\Gamma_f| = \frac{1}{|\Gamma|} \sum_{f\in\mathcal{F}} \sum_{\gamma\in\Gamma} \chi(\gamma f = f)$$

and 1.12 is obtained by a change of order of summation.

Theorem 1.1 is usually referred to as "*Burnside's identity*". There are several generalizations of it all of which can be reduced to the basic identity 1.11. The simplest version is given by

Theorem 1.2

If $\Phi(f)$ *is constant on orbits, that is*

$$\Phi(\gamma f) = \Phi(f) \quad \forall \quad \gamma\in\Gamma \tag{1.13}$$

then

$$\sum_{f\in\Delta} \Phi(f) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \sum_{f\in\mathcal{F}} \Phi(f)\chi(\gamma f = f). \tag{1.14}$$

Proof.

We have

$$\begin{aligned}
\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \sum_{f\in\mathcal{F}} \Phi(f)\,\chi(\gamma f = f) \;&=\; \frac{1}{|\Gamma|} \sum_{f\in\mathcal{F}} \Phi(f) \sum_{\gamma\in\Gamma} \chi(\gamma f = f)\\
\text{(by 1.11)} \;&=\; \frac{1}{|\Gamma|} \sum_{f\in\mathcal{F}} \Phi(f)\,|\Gamma_f| \;=\; \sum_{f\in\mathcal{F}} \Phi(f)\,\frac{1}{|0_f|}\\
\text{(by 1.7)} \;&=\; \sum_{i=1}^{k} \sum_{f\in O_{f_i}} \frac{\Phi(f)}{|O_f|}\\
&=\; \sum_{i=1}^{k} \frac{\Phi(f_i)}{|O_{f_i}|} \sum_{f\in O_{f_i}} 1 \;=\; \sum_{i=1}^{k} \Phi(f_i)\,. \qquad \textbf{Q.E.D.}
\end{aligned}$$

We thus have the following important:

Corollary 1.1

If A is an orbit property then the number Q of orbits having property A is given by the formula

$$\sum_{f\in\Delta}\chi(f\in A) \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{f\in A}\chi(\gamma f=f). \qquad 1.15$$

Proof.

Clearly, by the definition of orbit property, the function

$$\Phi(f)=\chi(f\in A)$$

is constant on orbits. Thus we may apply Theorem 1.2 and obtain

$$\sum_{f\in\Delta}\chi(f\in A) \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{f\in\mathcal{F}}\chi(f\in A)\;\chi(\gamma f=f)$$

and this is simply another way of writing 1.15

An applied branch of Group Theory (commonly referred to as "Polya theory ") is concerned with the practical calculation of expressions of the form 1.8 or 1.9. We shall see that the problem of obtaining explicit expressions for these quantities leads to some highly non trivial beautiful problems of algebra and the theory of special functions.

We shall return to this particular topic later in these notes. For the moment we need only show how group actions can be used to construct representations. To this end let our family $\mathcal{F}$ consist of the elements

$$f_1, f_2, \ldots, f_N\,.$$

Interpreting these elements as forming a basis of a vector space, we can represent the action of an element $\gamma\in\Gamma$ on this basis by right multiplication of this basis by a matrix. More precisely we are to find the matrix $A(\gamma)$ yielding the identity

$$\gamma\,\big\langle f_1\,,\,f_2\,,\,\ldots,f_N\big\rangle \;=\; \big\langle\gamma f_1\,,\,\gamma f_2\,,\,\ldots,\gamma f_N\big\rangle \;=\; \big\langle f_1\,,\,f_2\,,\,\ldots,f_N\big\rangle\,A(\gamma)\,. \qquad 1.16$$

Now if $A(\gamma)=\big\|a_{i\,j}(\gamma)\big\|_{i,j=1}^{N}$ then we must have

$$\gamma\,f_j \;=\; \sum_{i=1}^{N} f_i\;a_{i\,j}(\gamma)$$

this forces the identities

$$a_{i\,j}(\gamma) \;=\; \begin{cases}1 & \text{if } f_i=\gamma\,f_j\\[2ex] 0 & \text{otherwise}\end{cases}$$

Using the "χ" symbol we may thus write

$$A(\gamma) \;=\; \big\|\chi(f_i=\gamma f_j)\big\|_{i,j=1}^{N}\,. \qquad 1.17$$

This given note that the condition in 1.6 immediately implies the sequence of equalities

$$\begin{aligned} \alpha\big\langle \beta f_1\,,\,\beta f_2\,,\,\ldots,\beta f_N\big\rangle &= \big\langle \alpha\,(\beta f_1)\,,\,\alpha\,(\beta f_2)\,,\,\ldots,\alpha\,(\beta f_N)\big\rangle \\ &= \big\langle \alpha\beta f_1\,,\,\alpha\beta f_2\,,\,\ldots,\alpha\beta f_N\big\rangle \\ &= \big\langle f_1\,,\,f_2\,,\,\ldots,f_N\big\rangle A(\alpha\beta)\big)\,. \end{aligned} \qquad 1.18$$

On the other hand, by linearity and associativity of matrix multiplication we also obtain that

$$\begin{aligned} \alpha\,\big(\beta\langle f_1\,,\,f_2\,,\,\ldots,f_N\rangle\big) &= \alpha\big(\langle f_1\,,\,f_2\,,\,\ldots,f_N\rangle A(\beta)\big) \\ &= \big\langle \alpha f_1\,,\,\alpha f_2\,,\,\ldots,\alpha f_N\big\rangle A(\beta) \\ &= \big(\langle f_1\,,\,f_2\,,\,\ldots,f_N\rangle A(\alpha)\big)A(\beta) \\ &= \big\langle f_1\,,\,f_2\,,\,\ldots,f_N\big\rangle\big(A(\alpha)A(\beta)\big) \end{aligned} \qquad 1.19$$

Combining 1.18 and 1.19 we derive that

$$A(\alpha\beta) \;=\; A(\alpha)A(\beta)$$

Proving that the matrix function defined in 1.17 is in fact a representation of Γ.

The simplest non trivial example of this type of construction is when $\Gamma=S_3$ and $\mathcal{F}$ consists of the elements f_1,f_2,f_3. Representing, a element $\sigma\in S_3$ in the form

$$\sigma \;=\; \begin{bmatrix} 1 & 2 & 3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}$$

we get that

$$\begin{aligned} S_3 \;=\; \Big\{ &\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \\ &, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \Big\} \end{aligned} \qquad 1.20$$

Note that if we set

$$\sigma\big\langle f_1\,,\,f_2\,,\,f_3\big\rangle \;=\; \big\langle f_{\sigma_1}\,,\,f_{\sigma_2}\,,\,f_{\sigma_3}\big\rangle \;=\; \big\langle f_1\,,\,f_2\,,\,f_3\big\rangle\,\big\|\chi(i=\sigma_j)\big\|_{i,j=1}^{3}$$

then in particular

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}\big\langle f_1\,,\,f_2\,,\,f_3\big\rangle \;=\; \big\langle f_2,f_1,f_3\big\rangle \;=\; \big\langle f_1\,,\,f_2\,,\,f_3\big\rangle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying this construction to every element in 1.20 we obtain the following representation of S_3

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\ \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Giving a closer look at the group determinant reveals the process that may have led Frobenius to the discovery of the basic theorems of Group Representation Theory. The notion of group action which was quite familiar to Frobenius after the work of Lagrange and Galois naturally leads to the unraveling some of the complexity of the matrix

$$M[G] \;=\; \big\| x_{g_i^{-1} g_j} \big\|_{i,j=1}^{N} \tag{1.21}$$

whose determinant gives $D[G]$. To begin, the very definition in 1.21 leads us to focus on the representation corresponding the action of G on itself by left multiplication. Indeed, for

$$G = \{g_1, g_2, \ldots, g_N\} \tag{1.22}$$

the formula in 1.17 gives

$$\sigma\big\langle g_1, g_2, \ldots, g_N\big\rangle \;=\; \big\langle \sigma g_1, \sigma g_2, \ldots, \sigma g_N\big\rangle \;=\; \big\langle g_1, g_2, \ldots, g_N\big\rangle\, L(\sigma) \tag{1.23}$$

with

$$L(\sigma) \;=\; \big\| \chi(g_i = \sigma g_j)\big\|_{i,j=1}^{N}\,. \tag{1.24}$$

We used the letter "L" here instead of "A" since the map

$$\sigma \;\to\; L(\sigma)$$

is nowdays called "*the Left Regular Representation of* G". This given, we have the following basic identity

Proposition 1.1

For G given by 1.22 we have

$$D(G) = det M[G] = \det\big(\sum_{\sigma\in G} x_\sigma\, L(\sigma)\big) \tag{1.25}$$

Proof

Denoting the i,j-element of a matrix M by the symbol $M\big|_{i,j}$, from 1.24 we derive that

$$\begin{aligned}\sum_{\sigma\in G} x_\sigma\, L(\sigma)\big|_{i,j} \;&=\; \sum_{\sigma\in G} x_\sigma\, \chi(g_i = \sigma g_j)\\ &=\; \sum_{\sigma\in G} x_\sigma\, \chi\big(g_i\, g_j^{-1} = \sigma\big) \;=\; x_{g_i\, g_j^{-1}}\end{aligned}$$

and this gives

$$det\big(\sum_{\sigma\in G} x_\sigma\, L(\sigma)\big) \;=\; \det \big\|x_{g_i\, g_j^{-1}}\big\|_{i,j=1}^{N}\,. \qquad 1.26$$

Since the operation $g_i \rightarrow g_i^{-1}$ permutes the elements of G, it follows that we have

$$\big\|x_{g_i\, g_j^{-1}}\big\|_{i,j=1}^{N} \;=\; P\, M[G\,]\, P^{-1}\,.$$

for some $N\times N$ permutation matrix P. Using this in 1.2 yields 1.25 and completes the proof of the proposition.

Once in possession of 1.25 it compels us to look at the determinants produced by other representations. That is, determinants of the form

$$D(A) \;=\; \det\Big(\sum_{\sigma\in G} x_\sigma\, A(\sigma)\Big) \qquad 1.26$$

with any given representation of G. Clearly all of these determinants are polynomials in the variables

$$x_{g_1}\;,\; x_{g_2}\;,\;\ldots\;,\; x_{g_N}\;,$$

If we calculate a few of them (as we shall experiment in class) we are led to the surprising discovery that all their factors are also factors of the group determinant. We shall soon see how this fact leads to one of the fundamental theorems of Group Representation Theory.

Before we terminate this section we should point out that the sum in 1.26 leads to another bold step that is crucial in the development of representation theory. Indeed, the map $\sigma \rightarrow x_\sigma$ may be viewed as a function on the Group. This leads to study expressions of the form

$$A(f) = \sum_{\sigma\in G} f(\sigma)\, A(\sigma)$$

for an arbitrary function $f(\sigma)$. As we shall see this leads two important developments, the notion of "*Group Algebra*" and a corresponding natural extension of a representation of a Group to a representation of its Group algebra.

2. The Group Algebra.

If Γ is a group and f, g are two complex valued functions on Γ we define as the "convolution" of f and g and denote it by $f * g$ (sometimes simply fg) the function h whose value at the element $\gamma \in \Gamma$ is given by

$$h(\gamma) = \sum_{\alpha \in \Gamma} \sum_{\beta \in \Gamma} f(\alpha) g(\beta) \chi(\alpha\beta = \gamma). \qquad 2.1$$

Clearly we can write

$$h(\gamma) = \sum_{\alpha \in \Gamma} f(\alpha) g(\alpha^{-1}\gamma) = \sum_{\beta \in \Gamma} f(\gamma_\beta^{-1}) g(\beta). \qquad 2.2$$

The collection of all complex valued functions on Γ with this convolution operation as "**product**" and the usual **pointwise addition and multiplication by a constant** is an algebra. We shall call it the "group algebra" of Γ and shall denote it by the symbol $\mathcal{A}(\Gamma)$. Formulas 2.1 and 2.2 are quite clumsy to work with. For this reason it is more convenient to represent the elements of $\mathcal{A}(\Gamma)$ as formal sums

$$f = \sum_{\alpha \in \Gamma} f(\alpha)\alpha \qquad 2.3$$

and consider $\mathcal{A}(\Gamma)$ as the set of all linear combinations of the elements of Γ, the latter being interpreted as purely formal symbols. If we then simply multiply the expression in 2.3 by

$$\sum_{\beta \in \Gamma} g(\beta)\beta \qquad 2.4$$

and group terms using the multiplication table of the group we get

$$f \cdot g = \sum_{\gamma \in \Gamma} \gamma \sum_{\alpha \in \Gamma} \sum_{\beta \in \Gamma} f(\alpha) g(\beta)\ \chi(\alpha\beta = \gamma) = \sum_{\gamma \in \Gamma} h(\gamma)\gamma.$$

Thus formal multiplication of the expressions in 2.3 and 2.4 corresponds precisely to the multiplication defined by 2.1. In the following we shall work with $\mathcal{A}(\Gamma)$ entirely in this manner.

To illustrate this point we observe that the function

$$g(\gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha \\ 0 & \text{otherwise} \end{cases}$$

can be represented by the symbol "α" and thus we can write

$$\alpha f = \sum_{\gamma \in \Gamma} f(\gamma)\alpha\gamma = \sum_{\beta \in \Gamma} f(\alpha^{-1}\beta)\beta. \qquad 2.5$$

Similarly we get

$$f\alpha = \sum_{\beta \in \Gamma} f(\beta\alpha^{-1})\beta. \qquad 2.6$$

It is clear that the convolution product is commutative if and only if Γ is Abelian. Nevertheless, there is an interesting subalgebra of $\mathcal{A}(\Gamma)$ whose elements commute with every element of $\mathcal{A}(\Gamma)$. To introduce it we need some further notation.

An element f of $\mathcal{A}(\Gamma)$ is said to be a "*class function*" if and only if f is constant on the conjugacy classes of Γ. From the definition given in example **(d)** of section 1 it immediately follows that f is a class function if and only if the values taken by f satisfy the condition

$$f(\sigma\gamma\sigma^{-1}) = f(\gamma) \quad \forall \quad \sigma, \gamma \in \Gamma. \tag{2.7}$$

Making the replacements

$$\gamma \to \beta\alpha \quad \text{and} \quad \sigma \to \alpha$$

we see that 2.7 is equivalent to

$$f(\alpha\beta) = f(\beta\alpha) \quad \forall \quad \alpha, \beta \in \Gamma. \tag{2.8}$$

We then have the following basic result:

Theroem 2.1

An element $f \in \mathcal{A}(\Gamma)$ commutes with every element of $\mathcal{A}(\Gamma)$ if and only if f is a class function.

Proof.

Clearly f commutes with every element of $\mathcal{A}(\Gamma)$ if and only if we have

$$\alpha f = f\alpha \quad \forall \quad \alpha \in \Gamma. \tag{2.9}$$

In view of 2.5 and 2.6 we see that this holds if and only if

$$f(\alpha^{-1}\beta) = f(\beta\alpha^{-1}) \quad \forall \quad \beta \in \Gamma.$$

Since α and β can be chosen arbitrarily we see that 2.9 and 2.8 are equivalent. This establishes the result since we have shown that 2.8 is equivalent to 2.7 and the latter identity characterizes class functions.

The collection of all class functions shall be denoted by $\mathcal{C}(\Gamma)$ and can also be referred to as the "*Center*" of $\mathcal{A}(\Gamma)$.

Remark 2.1

Note that if $f \in \mathcal{A}(\Gamma)$ then the function

$$\widehat{f} = \sum_{\sigma \in \Gamma} \sigma f \sigma^{-1} \tag{2.10}$$

is in $\mathcal{C}(\Gamma)$. Indeed, we see that for any $\alpha \in \Gamma$ we have

$$\begin{aligned} \alpha \widehat{f} \; &= \; \sum_{\sigma \in \Gamma} \alpha \, \sigma f \sigma^{-1} \\ (\text{setting } \alpha\sigma = \gamma) \; &= \; \sum_{\gamma \in \Gamma} \gamma \, f \gamma^{-1} \alpha \; = \; \widehat{f} \, \alpha \end{aligned}$$

This proves that $\widehat{f}$ is a class function. The operation which sends f into $\widehat{f}$ will be referred to as "*symmetrization*" and $\widehat{f}$ will be called the "symmetric extension of f".

The group algebra $\mathcal{A}(\Gamma)$ is also made into a Hilbert space by means of the inner product

$$< f, g > = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma) \overline{g(\gamma)}. \tag{2.11}$$

Note that from 2.5 we get

$$\begin{aligned} < \alpha f, g > &= \frac{1}{|\Gamma|} \sum_{\beta \in \Gamma} f(\alpha^{-1}\beta) \overline{g(\beta)} \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma) \overline{g(\alpha\sigma)} = < f, \alpha^{-1} g > . \end{aligned} \tag{2.12}$$

Similarly we show that

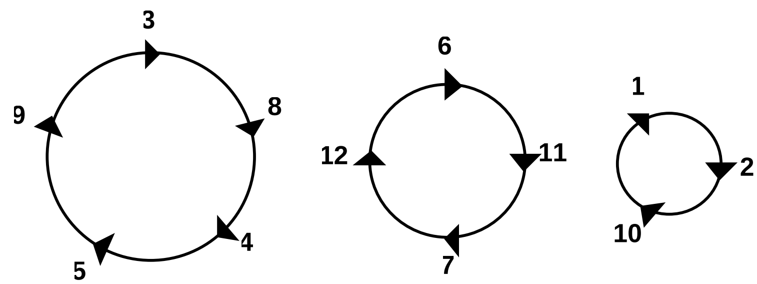
$$< f\alpha, g > = < f, g\alpha^{-1} > . \tag{2.13}$$

This can be interpreted as saying that the "adjoint" of the operation of "left multiplication" by α is "left multiplication" by α^{-1}. This means that left multiplications are unitary operators on $\mathcal{A}(\Gamma)$ with respect to the inner product 2.11. Of course, the same holds true for right multiplications.

To familiarize ourselves with the notion of class function, it is helpful to study in detail the paricular case $\Gamma = S_n$. Let us study the particular case of a permutation $\sigma \in S_{12}$. For instance let us take

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 10 & 8 & 5 & 9 & 11 & 12 & 4 & 3 & 1 & 7 & 6 \end{bmatrix}$$

This permutation may be graphically represented by the following diagram



This is constructed by simply following the trajectories of the elements of set

$$\{1,2,3,4,5,6,7,8,9,11,12\}$$

under the action of the successive powers of σ. We may also represent this diagram by the symbol

$$3{\to}8{\to}4{\to}5{\to}{\to}9{\to}3 \quad 6{\to}11{\to}7{\to}12{\to}6 \quad 1{\to}2{\to}10{\to}1 \tag{2.14}$$

In current litterature, this decomposition usually expressed by writing

$$\sigma \;\;=\;\; (3,8,4,5,9)\,(6,11,7,12)\,(1,2,10) \tag{2.15}$$

and the factors in 2.15 are called the "*cycles of*" σ. Note that each of these factors may also be interpreted as a permutation. For instance the first factor is to represent the permutation

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 8 & 5 & 9 & 6 & 7 & 8 & 3 & 10 & 11 & 12 \end{bmatrix}$$

That is the permutation that moves $3,8,4,5,9$ as indicated by 2.14 and leaves $2,7,8,10,11,12$ fixed. The letters $2,7,8,10,11,12$ are also called the fixed points of $(3,8,4,5,9)$. With theses conventions, these symbols may be multiplied following the rules (†) that govern permutation multiplication.

In the same vein the relation in 2.15 is called "*the cycle factorization of* σ" and the partition $(5,4,3)$ is usually referred to as the "*cycle structure of* σ".

Taking accout of this terminology, we have the following basic fact.

(†) Perhaps we should recall that these are the rules of composition of functions. Thus for $\alpha,\beta\in S_n$ the pemutation $\gamma=\alpha\beta$ simply represents the function $\alpha(\beta)$. That is for any i we set $\gamma_i=\alpha_{\beta_i}$

Proposition 2.1

Two elements $\alpha, \beta \in S_n$ are in the same conjugacy class if and only if they have the same cycle structure. Since cycle structures may be represented by partitions we see that S_n has as many different conjugacy classes as there are partitions of n. Here and after we shall denote the conjugacy class consisting of all permutations of cycle structure λ by the symbol "C_λ".

Proof

The result follows immediately by noting that if a permutation γ has the cycle $(i_1, i_2, \cdots, i_k)$ in its cycle factorization, then the permutation $\sigma\,\gamma\,\sigma^{-1}$ has the cycle

$$(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k})$$

Problems

1. *Use formula 1.11 to prove that the number of permutations $\sigma \in S_n$ with cycle structure*

$$\rho \;\; = \;\; 1^{\alpha_1}\, 2^{\alpha_2}\, \cdots n^{\alpha_n} \;\; (\dagger)$$

is given by the formula

$$z_\rho = 1^{\alpha_1} \times 2^{\alpha_2} \times \cdots \times n^{\alpha_n} \quad \times \alpha_1! \times \alpha_2! \cdots \times \times \alpha_n!$$

2. *Prove that if f is a nilpotent element of the group algebra of Γ then the coefficient of the identity in f must necessarily vanish.*

Note: There is a short proof that uses the left regular representation of Γ. But the result can also be proved without it.

3. Unitary Representations.

A representation A is said to be "unitary" if and only if $A(\gamma)$ is a unitary matrix for all $\gamma \in \Gamma$. It is easy to show from 1.17 that all the representations induced by group actions are unitary. In particular the left regular representation of any finite group is unitary.

Two representations A and B of the same degree are said to be "similar" if and only if there is a non singular matrix θ such that

$$B(\gamma) = \theta A(\gamma)\theta^{-1} \quad \forall \quad \alpha \in \Gamma.$$

In other words σ has α_i cycles of length i .

Our object here is to obtain a recipe for constructing all possible representations of a given group Γ. Of course, to do this we need only produce a complete set of representatives for the equivalence classes of similar representations.

It turns out that such representatives can be put together using only **unitary representations**. Indeed, we have the following remarkable fact:

Theorem 3.1

Every representation is similar to a unitary representation.

Proof.

Let $A(\alpha)$ be a representation of Γ of degree n. We introduce in $\mathbf{C}_n$ (the n-dimensional vector space over the complex numbers) the scalar product

$$\big\langle u\,, v\big\rangle_A \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\big(A(\gamma)\,u\,, A(\gamma)\,v\big)\,. \tag{3.1}$$

Here $(u\,,v)$ denotes the standard scalar product in $\mathbf{C}_n$. In other words for any two column vectors U, V we set

$$(u,v) \;=\; u^T\,\overline{v}\,. \tag{3.2}$$

Now let T_α be the linear transformation of $\mathbf{C}_n$ into itself defined by setting for each $u\in\mathbf{C}_n$

$$T_\alpha u = A(\alpha)u.$$

We see that

$$\begin{aligned}\big\langle T_\alpha\,u\,, v\big\rangle_A &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\big(A(\gamma\,\alpha)u\,, A(\gamma)\,v\big) = \\ &= \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}\big(A(\sigma)u\,, A(\sigma\alpha^{-1})\,v\big) \;=\; \big\langle u\,, T_\alpha^{-1}v\big\rangle_A.\end{aligned} \tag{3.3}$$

This means that T_α is a unitary operator on $\mathbf{C}_n$ with respect to the scalar product 3.1. This given, let $u^{(1)},u^{(2)},\ldots,u^{(n)}$ be an orthonormal basis for $\mathbf{C}_n$ with respect to the scalar product 3.3. Then the matrix $B(\alpha)$ of T_α with respect to $u^{(1)},u^{(2)},\ldots,u^{(n)}$, must necessarily be unitary. However, this matrix is none other than

$$B(\alpha) = \mathcal{U}^{-1}A(\alpha)\,\mathcal{U} \tag{3.4}$$

where $\mathcal{U}$ is the matrix whose columns are $u^{(1)},u^{(2)},\ldots,u^{(n)}$. This gives our desired result.

Remark 3.1

The proof that the represntation B defined by 3.4 is unitary can be made more explicit. In fact, the scalar product in 3.1 may also be rewritten in the form

$$\langle u\,,\,v\rangle_A \;=\; u^T M\,\overline{v}$$

with

$$M \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} A^T(\gamma)\,\overline{A}(\gamma) \tag{3.5}$$

an d it is easy to see that

$$M^\top \;=\; \overline{M} \tag{3.6}$$

Thus the orthonormality of the basis $u^{(1)},u^{(2)},\ldots,u^{(n)}$ consists of the identities

$$u^{(i)\,\top} M\,\overline{u}^{(j)} \;=\; \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \tag{3.7}$$

So if U is the matrix with columns $u^{(1)},u^{(2)},\ldots,u^{(n)}$ then all the relations in 3.7 may simply be written as the single identity

$$U^\top M\,\overline{U} \;=\; I \tag{3.8}$$

Now note that in the same way we obtained 3.3 we get from 3.5

$$\begin{aligned} A^\top(\alpha)M &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} A^\top(\alpha)A^T(\gamma)\,\overline{A}(\gamma) \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} A^\top(\gamma\alpha)\,\overline{A}(\gamma) \\ &= \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} A^\top(\beta)\,\overline{A}(\beta\alpha^{-1}) \;=\; \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma} A^\top(\beta)\,\overline{A}(\beta)\overline{A}(\alpha^{-1}) \;=\; M\,\overline{A}(\alpha^{-1}) \end{aligned}$$

Conjugating both sides of this identity and both sides of 3.8 gives

$$a)\quad \overline{A}^\top(\alpha)\overline{M} \;=\; \overline{M}\,A(\alpha^{-1}) \qquad \text{and} \qquad b)\quad U^{-1}=\overline{U}^\top\overline{M} \tag{3.9}$$

Thus if we set

$$B(\sigma) \;=\; U^{-1}A(\sigma)\,U \;=\; \overline{U}^\top\overline{M}\,A(\sigma)\,U$$

then

$$\begin{aligned} \overline{B}(\sigma)^\top &= \overline{U}^\top\,\overline{A}^\top(\sigma)M^\top U \\ \big(\text{by } 3.6\big) &= \overline{U}^\top\,\overline{A}^\top(\sigma)\,\overline{M}\,U \\ \big(\text{by } 3.7\text{ a)}\big) &= \overline{U}^\top\,\overline{M}\,A(\sigma^{-1})U \\ \big(\text{by } 3.7\text{ b)}\big) &= U^{-1}\,A(\sigma^{-1})U \;=\; B(\sigma^{-1}) \end{aligned}$$

as desired.

Here and in the following, unless expressly stated otherwise the representations we work with will be assumed unitary.

We shall say that a subspace $\mathcal{S}$ of $\mathbf{C}_n$ is "*invariant*" under a representation A of degree n if and only if

$$A(\alpha)\,\mathcal{S} = \mathcal{S} \quad \forall \quad \alpha \in \Gamma.$$

By this we mean that each vector in $\mathcal{S}$ is transformed by $A(\alpha)$ into a vector of $\mathcal{S}$. Since A is unitary if and only if it leaves invariant the standard scalar product

$$(x,y) = \sum_{i=1}^{n} \; x_i \overline{y}_i$$

We see that the orthogonal complement $\mathcal{S}^{\perp}$ of an invariant subspace $\mathcal{S}$ is also invariant. Indeed, for every $x \in \mathcal{S}$ and $y \in \mathcal{S}^{\perp}$ we have

$$0 \;=\; (A(\alpha)x, y) \;=\; (x, A(\alpha^{-1})y).$$

This means that for each $\alpha \in \Gamma$

$$A(\alpha^{-1})\,\mathcal{S}^{\perp} = \mathcal{S}^{\perp}.$$

I.e., $\mathcal{S}^{\perp}$ is invariant under A as we asserted. We shall say that a representation A is **reducible** if and only if A leaves invariant a subspace $\mathcal{S}$ that is nontrivial. (i.e. $1 \leq \dim \mathcal{S} \leq n-1$.)

Let then A be reducible and $\langle \mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots, \mathcal{U}^{(h)} \rangle$ be an orthonormal basis for the invariant subspace $\mathcal{S}$ and $\langle \mathcal{U}^{(h+1)}, \mathcal{U}^{(h+2)}, \ldots, \mathcal{U}^{(n)} \rangle$ be an orthonormal basis for $\mathcal{S}^{\perp}$ and let $\mathcal{U}$ be the matrix whose columns are $\mathcal{U}^{(1)} \ldots \mathcal{U}^{(n)}$. We see from the invariance of $\mathcal{S}$ and $\mathcal{S}^{\perp}$ that for each $\alpha \in \Gamma$ there are two matrices $B_1(\alpha), B_2(\alpha)$ such that

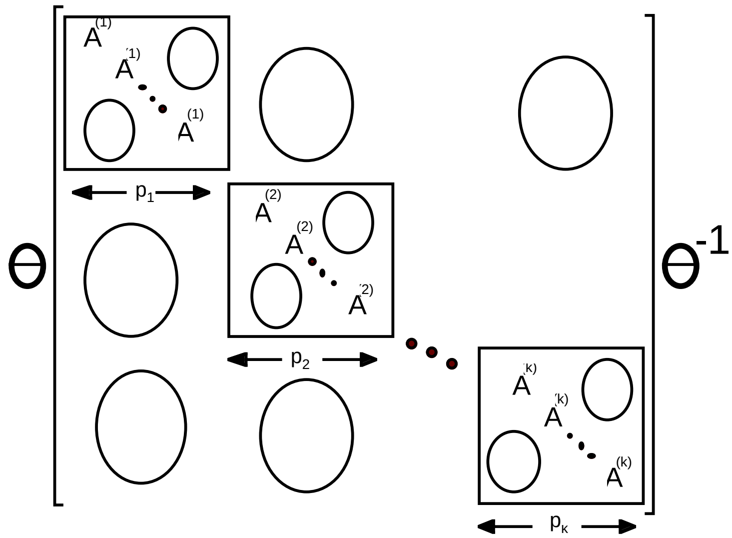
$$A(\alpha)\,\mathcal{U} = \mathcal{U} \begin{bmatrix} B_1(\alpha) & \bigcirc \\ \bigcirc & B_2(\alpha) \end{bmatrix}$$

Since $\mathcal{U}$ is a unitary matrix we see that both $B_1(\alpha), B_2(\alpha)$ are unitary representations of Γ of degrees h and n-h respectively. Thus every reducible representation is similar to a "direct sum" of two representations of smaller degrees. We shall refer to these two representations as "constituents" of A. By repeating this procedure to each of the constituents and then to the constituents of the constituents we see that each reducible representation is similar to one of the form

$$B(\alpha) = \begin{bmatrix} B^{(1)}(\alpha) & \bigcirc & \cdots & \bigcirc \\ \bigcirc & B^{(2)}(\alpha) & \cdots & \bigcirc \\ \bigcirc & \bigcirc & \ddots & \bigcirc \\ \bigcirc & \bigcirc & \cdots & B^{(h)}(\alpha) \end{bmatrix}$$

where each $B^{(i)}(\alpha)$ is irreducible. That is, each $B^{(i)}(\alpha)$ leaves no subspace invariant other than the trivial ones.

Observe further that, in general, some of the pairs $B^{(i)}$, $B^{(j)}$ (for $i \neq j$) may be similar. Thus, if we pick, once and for all, one representative $A^{(i)}$ for each similarity class of irreducilble constituents of the given representation we see that we can write A in the form



3.10

where our display represents that in the block matrix between θ and θ^{-1} the matrix $A^{(i)}$ appears repeated p_i times.

We shall call $A^{(1)} \ldots A^{(k)}$ a canonical set of irreducible constitutents for A and refer to p_i as the *"multiplicity" of $A^{(i)}$ in A''.*

Thus our goal of providing a recipe for constructing all possible representations of Γ will be reached as soon as we put together a complete set of representatives for the similarity classes of irreducible representations of Γ. This will be one of the many byproducts of the basic result presented in the next section.

Remark 3.2

Applying the decomposition in 3.10 to the left regular representation we derive that the group determinant must factor in the form

$$D(\Gamma) \;=\; \prod_{i=1}^{k} D_i(x)^{p_i} \qquad 3.11$$

with

$$D_i(x) \;=\; \det\Big(\sum_{\alpha\in\Gamma} x_\alpha A^i(\alpha)\Big).$$

This is essentally the solution of the Group determinant problem posed by Dedekind.

Problems

1. Show that the representations induced by group actions are unitary.

2. For a representation A set

$$D_A(x) \;=\; \det\Big(\sum_{\alpha\in\Gamma} x_\alpha\, A(\alpha)\Big) \tag{3.12}$$

In computer experimentations we find that the factors of the polynomials in 3.12 are invariably also factors of the group determinant. Assuming that this is true in general, what can you conclude about the nature of the irreducible constituents of the left regular representation?

4. The Fundamental Theorem of Group Representation Theory.

Let $A(\sigma)$ and $B(\sigma)$ be two representations of Γ of degrees n and m respectively. Assume that $m \leq n$. For a given $n\times m$ matrix $\mathcal{C}$ set

$$\theta(\mathcal{C}) = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma} A(\sigma)\,\mathcal{C}B(\sigma^{-1}). \tag{4.1}$$

Note that $\theta(\mathcal{C})$ is also $n\times m$ and that for all $\alpha\in\Gamma$

$$\begin{aligned} A(\alpha)\theta(\mathcal{C}) &= \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma} A(\alpha\sigma)\,\mathcal{C}B(\sigma^{-1}) = \\ &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} A(\gamma)\,\mathcal{C}\;B(\gamma^{-1}\alpha) = \theta(\mathcal{C})\mathcal{B}(\alpha). \end{aligned} \tag{4.2}$$

In other words, if $\theta^{(1)},\theta^{(2)},\ldots,\theta^{(m)}$ denote the columns he matrix $\theta(\mathcal{C})$ then for each $j = 1,2,\ldots,m$ we have:

$$A(\alpha)\theta^{(j)} = \sum_{i=1}^{m}\theta^{(i)} b_{ij}(\alpha)\;\;(*) \tag{4.3}$$

This identity implies that the columns of $\theta(\mathcal{C})$ span a subspace of $\mathbf{C}^n$ that is **invariant** under A.

This fact has remarkable consequences. To present them we need some further notation. Given a representation A let us denote by a_{ij} the element of $\mathcal{A}(\Gamma)$ which at σ takes the value $a_{ij}(\sigma)$. We shall denote by n_A the degree of the representation A and set

$$\chi^A = a_{11} + a_{22} + \ldots + a_{n_A n_A}. \tag{4.4}$$

(*) Here and in the following if B is a matrix b_{ij} will denote the entry in the i^{th} row and j^{th} column of B.

The latter function on Γ is actually an element of $\mathcal{C}(\Gamma)$. Indeed, since

$$\chi^A(\sigma) = \text{trace } A(\sigma)$$

we have for all $\alpha, \beta \in \Gamma$

$$\chi^A(\alpha\beta) = \text{trace } A(\alpha)A(\beta) = \text{trace } A(\beta)A(\alpha) = \chi^A(\beta\alpha).$$

That means χ_A is a class function as asserted. We call the function χ^A the "*character*" of the representation A. This given we have

Theorem 4.1

If A is an irreducible unitary representation then the functions $\{\sqrt{n_A}\ a_{ij}\}_{ij=1}^{n_A}$ are orthonormal with respect to the scalar product 2.11. Furthermore, if B is any irreducible unitary representation that is not similar to A, then the sets $\{a_{ij}\}_{ij=1}^{n_A}, \{b_{ij}\}_{ij=1}^{n_B}$ are orthogonal to each other.

Proof.

It is sufficient to show that

$$< \sqrt{n_A}\ a_{i_1 j_1}, \sqrt{n_A} a_{i_2 j_2} >= \begin{cases} 1 & \text{if } i_1 = i_2 \ \&\ j_1 = j_2, \\ 0 & \text{otherwise,} \end{cases} \qquad 4.5$$

and that if $m \leq n, A$ is irreducible and B is not similar to A then

$$< a_{i_1 j_1}, b_{i_2 j_2} > = 0 \ \text{ for all } \ i_1, j_1; i_2, j_2. \qquad 4.6$$

Both these relations are immediate consequences of 4.2 and 4.3 upon suitable choices of A, B and C. We can quickly dispose of 4.6. Indeed, if A is irreducible and $m \leq n$, 4.3 implies that the vectors $\theta^{(1)}, \ldots, \theta^{(m)}$ span either $\{0\}$ or the whole of $\mathbf{C}_n$. That is, they are either all zero or they are independent and $m = n$. However, in the latter case θ is non-singular and we have

$$A(\alpha) = \theta(\mathcal{C}) B(\alpha) \theta^{-1}(\mathcal{C}) \quad \forall \quad \alpha \in \Gamma,$$

which is excluded if A and B are not similar. Thus, under these circumstances, we must have $\theta(\mathcal{C}) = 0$ for any choice of $\mathcal{C}$. Setting

$$c_{ij} = \chi(i = i_1)\chi(j = j_2) \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix},$$

from 4.1 we get

$$\begin{aligned} 0 = \theta_{h,k}(\mathcal{C}) &= \frac{1}{|\Gamma|} \sum_{\sigma} a_{h,j_1}(\sigma) b_{j_2,k}(\sigma^{-1}) = \\ &= \frac{1}{|\Gamma|} \sum_{\sigma} a_{h,j_1}(\sigma) \overline{b}_{k,j_2}(\sigma) \end{aligned} \qquad (*)$$

(*) Here we use fact that B is unitary.

but this is 4.6 when $h = i_1$ and $k = j_2$.

To obtain 4.5 we set $B = A$ in 4.2 and get

$$A(\alpha)\theta(\mathcal{C}) = \theta(\mathcal{C})A(\alpha) \tag{4.7}$$

for any $\mathcal{C}$. Again, since A is supposed to be irreducible we must have that $\theta(\mathcal{C})$ is either non-singular or zero. Of course, we should observe that the same must hold for any $n \times n$ matrix θ such that

$$A(\alpha)\theta = \theta A(\alpha)$$

holds for all $\alpha \in \Gamma$. However, from 4.7 we get that for any λ

$$A(\alpha)(\theta(\mathcal{C}) - \lambda I) = (\theta(\mathcal{C}) - \lambda I)A(\alpha),$$

by choosing λ to be an eigenvalue of $\theta(\mathcal{C})$ we can **guarantee** that $\theta(\mathcal{C}) - \lambda I$ is singular. The above observation then yields that we must have

$$\theta(\mathcal{C}) = \lambda I. \tag{4.8}$$

Thus, $\theta(\mathcal{C})$ can only have a single eigenvalue, this eigenvalue can easily be determined. Indeed, from 4.8 we get

$$\text{trace } \theta(\mathcal{C}) = n_A \lambda.$$

On the other hand, in this case 4.1 gives

$$\begin{aligned} \text{trace } \theta(\mathcal{C}) &= \frac{1}{|\Gamma|} \sum_{\sigma} \text{ trace } [A(\sigma)\, \mathcal{C} A^{-1}(\sigma)] \\ &= \text{trace } \mathcal{C}. \end{aligned}$$

In other words, we can conclude that for any $\mathcal{C}$ we must have

$$\frac{1}{|\Gamma|} \sum_{\sigma} A(\sigma)\, \mathcal{C} A^{-1}(\sigma) = \left(\frac{\text{trace } \mathcal{C}}{n_A} \right) I.$$

This relation implies 4.5 upon choosing as before

$$c_{ij} = \chi(i = j_1)\chi(j = j_2).$$

Remark 4.1

We should take note that the above argument proves that a matrix which commutes with every matrix of an irreducible representation is necessarily a multiple of the identity.

Theorem 4.1 has the following remarkable corollary.

Theorem 4.2

If $A^{(1)}(\alpha), A^{(2)}(\alpha), \cdots, A^{(h)}(\alpha)$ are irreducible representations, no two of which are similar, then the functions $\left\{\chi^{A^{(i)}}\right\}_{i=1}^{h}$ are an orthonormal set in $\mathcal{A}(\Gamma)$ and

$$h \leq k = \# \text{ conjugacy classes of } \Gamma. \tag{4.9}$$

Proof.

Set $n_\nu = n_{A^{(\nu)}}$. Since

$$\langle \chi^{A^{(\nu)}}, \chi^{A^{(\mu)}} \rangle = \sum_{i=1}^{n_\nu} \sum_{j=1}^{n_\mu} \langle a_{ii}^{(\nu)}, a_{jj}^{(\mu)} \rangle,$$

formulas 4.5 and 4.6 give

$$\langle \chi^{A^{(\nu)}}, \chi^{A^{(\mu)}} \rangle = \begin{cases} \sum_{i=1}^{n_\nu} \frac{1}{n_\nu} = 1 & \text{if } \nu = \mu. \\ 0 & \text{otherwise.} \end{cases}$$

To prove 4.9 observe first that if $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ denote the conjugacy classes of Γ then each class function f can be written in the form

$$f = \sum_{i=1}^{k} f_i \chi(\gamma \in \mathcal{C}_i).$$

Thus $\mathcal{C}(\Gamma)$ is a k-dimensional space. As we have pointed out each $\chi^{A^{(\nu)}}$ is a class function, thus 4.9 is a simple consequence of the fact that the cardinality of an orthonormal set of vectors never exceeds the dimension of the space to which these vectors belong.

Remark 4.1

Note that our proof uses the fact that the representations $A^{(\nu)}$ are unitary. Yet the unitary hypothesis was not included in the statement of the theorem. Complete the proof of the theorem by showing that the othonormality of the functions $\left\{\chi^{A^{(i)}}\right\}_{i=1}^{h}$ remains valid without this assumption.

We are finally in a position to establish the fundamental theorem of group representation theory. This can be stated as follows:

Theorem 4.3

Let $A^{(1)}(\alpha),\ldots,A^{(h)}(\alpha)$ be a complete set of representatives of the similarity classes of irreducible constituents of the left regular representation of the group Γ. Set for convenience

$$n_\nu = n_{A^{(\nu)}}\ , \quad \chi^{(\nu)} = \chi^{A^{(\nu)}}.$$

Then under the assumption that $A^{(1)},\ldots,A^{(h)}$ are unitary

(a) The group algebra elements $\bigcup_{\nu=1}^{h}\left\{\sqrt{n_\nu}\ a_{ij}^{(\nu)}\right\}_{ij=1}^{n_\nu}$ form a complete orthonormal system for $\mathcal{A}(\Gamma)$, that is

$$n_1^2+n_2^2+\cdots+n_h^2 = N = |\Gamma|. \tag{4.10}$$

(b) The group algebra elements $\{\chi^{(\nu)}\}_{\nu=1}^{h}$ form a complete orthonormal system for $\mathcal{C}(\Gamma)$, that is,

$$h = k = \ \#\ \text{conjugacy classes of } \Gamma. \tag{4.11}$$

(c) Every irreducible representation of Γ is similar to one of the $A^{(\nu)}(\alpha)$.

Proof.

As before we let $L(\alpha)$ denote the left regular representation of Γ. For a moment, let us denote by p_r the number of occurrences of $A^{(r)}(\alpha)$ in $L(\alpha)$. We then have

$$\chi^L(\alpha) = \text{trace } L(\alpha) = \sum_{r=1}^{h} p_r\chi^{(r)}(\alpha).$$

The orthonormality of $\left\{\chi^{(r)}\right\}_{r=1}^{h}$ gives

$$p_r = \langle \chi^L, \chi^{(r)}\rangle.$$

On the other hand, from 2.14 we get

$$\text{trace } L(\alpha) = \sum_{i=1}^{N}\ \chi(\gamma_i = \alpha\gamma_i) = \begin{cases} N & \text{if } \alpha = e = \text{identity,} \\ 0 & \text{otherwise.}\end{cases} \tag{4.12}$$

Thus,

$$p_r = \frac{1}{N}\ N\chi^{(r)}(e) = n_r.$$

However, 4.12 and the orthonormality of $\left\{\chi^{(r)}\right\}_{r=1}^{n}$ then gives

$$N = \langle \chi^L, \chi^L\rangle = \sum_{r=1}^{h}\ p_r^2 = \sum_{r=1}^{h}\ n_r^2.$$

This completes the proof of part (a). Our program in proving (b) is to show that every $f \in \mathcal{C}(\Gamma)$ can be written as linear combination of $\chi^{(1)}, \chi^{(2)}, \cdots, \chi^{(h)}$.

To this end we observe that the completeness of the orthonormal system $\bigcup_{r=1}^{h} \left\{ \sqrt{n_r}\ a_{ij}^{(r)} \right\}_{ij=1}^{n_r}$ yields

$$f = \sum_{r=1}^{h} n_r \sum_{i,j=1}^{n} \langle f, a_{ij}^{(\nu)} \rangle a_{ij}^{(r)} .$$

Thus (b) will be established if we show that for every r

$$\langle f, a_{ij}^{(r)} \rangle = \begin{cases} \frac{1}{n_r}\ \langle f, \chi^{(r)} \rangle & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad 4.13$$

Consider then the matrix

$$\theta = \frac{1}{N} \sum_{\sigma} \overline{f(\sigma)}\ A^{(r)}(\sigma) \qquad 4.14$$

and note that for each α we have

$$A^{(r)}(\alpha)\theta = \frac{1}{N} \sum_{\sigma} \overline{f(\sigma)}\ A^{(r)}(\alpha\sigma) = \frac{1}{N} \sum_{\gamma} \overline{f(\alpha^{-1}\gamma)}\ A^{(r)}(\gamma)$$

and similarly we derive that

$$\theta A^{(r)}(\alpha) = \frac{1}{N} \sum_{\gamma} \overline{f(\gamma\alpha^{-1})}\ A^{(r)}(\gamma).$$

Thus when f is a class function we can conclude that

$$A^{(r)}(\alpha)\theta = \theta A^{(r)}(\alpha).$$

Since $A^{(r)}(\alpha)$ is supposed to be irreducible, we can use Remark 4.1 and conclude that θ must be a multiple of the identity. Setting

$$\theta = \lambda I$$

in 4.14 and equating the i,j-entries in both sides we finally get

$$\frac{1}{N} \sum_{\sigma} \overline{f(\sigma)}\ a_{ij}^{(r)}(\sigma) = \begin{cases} \lambda & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

These relations immediately yield 4.13 as desired. (*)

It remains to prove (c), but this is an immediate consequence of the completeness of the system $\bigcup_{r=1}^{k} \left\{ a_{ij}^{(r)} \right\}_{i,j=1}^{n_r}$. Indeed, if B is irreducible, unitary and not similar to any of the $A^{(r)}$ then its entries (by Theorem 4.1) would give an orthonormal system orthogonal to all the $a_{ij}^{(r)}$'s which is impossible.

(*) Equate the traces of both sides of 4.14

Remark 4.2

The notation of theorem 4.3 will be used throughout the rest of these notes. That is, given a group Γ, k will denote the number of conjugacy classes of Γ and

$$A^{(\lambda)}\,, \quad n_\lambda\,, \quad \chi^\lambda \quad (\lambda = 1, 2, \cdots, k)$$

will have the meaning given to them in the statement of Theorem 4.3.

It is clear that since the $A^{(\lambda)}$'s are unique up to a similarity and labelling the χ^λ themselves are unique up to labelling.

The fundamental theorem completely resolves the problem of constructing all representations of a given group Γ. Indeed, since each irreducible representation is similar to one of the $A^{(\lambda)}$ then every representation of Γ is of the form given in 3.10 where the $A^{(\lambda)}$ have the meaning presently given to them. To simplify our displays we shall refer to a representation of the form 3.10 by the symbol

$$A = \theta \left[\bigoplus_{\lambda=1}^{k} \oplus_{p_\lambda} A^{(\lambda)} \right] \theta^{-1}. \qquad 4.15$$

Note that the character of such representation is given by the formula

$$\chi^A = \sum_{\lambda=1}^{k} p_\lambda \chi^\lambda. \qquad 4.16$$

We thus obtain the following important corollary of Theorem 4.3.

Theorem 4.4

Let A be a representation of Γ, then 4.15 holds with a suitable θ and the multiplicity of $A^{(\lambda)}$ in A is given by the formula

$$p_\lambda = \langle \chi^A, \chi^\lambda \rangle\,.$$

Proof.

We need only observe that this identity is an immediate consequence of 4.16 and the orthonormality of the χ^λ's.

The fundamental theorem has several immediate consequences. The reader may enjoy discovering some of them . Here we wish to point out only those we shall use in our further developments. To do this we need some further notation.

Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k$ be the conjugacy classes of Γ. For convenience we shall use the same symbols for the classes as well as for the corresponding elements of the group algebra of Γ but with a superscript rather than a subscript, that is we set

$$\mathcal{C}^\lambda = \sum_{\gamma \in \Gamma} \gamma\, \chi(\gamma \in \mathcal{C}_\lambda).$$

Note that since each χ^λ is a class function we shall have

$$\chi^\lambda = \sum_{\mu=1}^{k} \chi^\lambda_\mu \, \mathcal{C}^\mu \qquad 4.17$$

for some suitable complex numbers χ^λ_μ. Here and in the following the $k \times k$ matrix whose (i,j) entry is χ_i^j will be denoted by the symbol χ. Thus 4.17 in matrix notation is

$$< \chi^1, \cdots, \chi^k > = < \mathcal{C}^1, \mathcal{C}^2, \cdots, \mathcal{C}^k > \chi \, . \quad (\dagger) \qquad 4.18$$

Setting

$$R_\alpha = |\mathcal{C}_\alpha|, \;\; N = |\Gamma|,$$

we see that

$$\langle \mathcal{C}^\alpha, \mathcal{C}^\beta \rangle_\Gamma = \begin{cases} 0 & \text{of } \alpha \neq \beta \\ \frac{R_\alpha}{N} & \text{if } \alpha = \beta. \end{cases} \qquad 4.19$$

This given, from 4.17 and the orthonormality of the χ^λ's we get

$$\begin{aligned} \chi(\lambda = \mu) &= \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \chi^\lambda_\alpha \, \overline{\chi}^\mu_\beta \, \langle \mathcal{C}^\alpha, \mathcal{C}^\beta \rangle \\ \text{(by 4.19)} &= \sum_{\alpha=1}^{k} \chi^\lambda_\alpha \, \overline{\chi}^\mu_\alpha \, R_\alpha / N \end{aligned} \, . \qquad 4.20$$

In matrix notation, these relations reduce to the single equation

$$I \;=\; \chi^T \begin{bmatrix} R_1/N & \bigcirc & \cdots & \bigcirc \\ \bigcirc & R_2/N & \cdots & \bigcirc \\ \bigcirc & \cdots & \ddots & \bigcirc \\ \bigcirc & \cdots & \cdots & R_k/N \end{bmatrix} \overline{\chi}. \qquad 4.21$$

where I denotes the $k \times k$ identity matrix.

If $d_1 \ldots, d_k$ are given complex numbers the diagonal matrix

$$\begin{bmatrix} d_1 & \bigcirc & \cdots & & \bigcirc \\ \bigcirc & d_2 & \cdots & & \bigcirc \\ \bigcirc & \cdots & \ddots & & \bigcirc \\ \bigcirc & \bigcirc & \cdots & \bigcirc & d_k \end{bmatrix}$$

(†) In this formula as well as in several instances in the following we can interpret elements of the group algebra as column vectors.

will be denoted by $\|d\|$. Thus if we introduce the ratios

$$\rho_\lambda = N/R_\lambda \,,$$

the identity in 4.21 may be rewritten as

$$\chi^T \;\; \|1/\rho\| \;\; \overline{\chi} \;\; = \;\; I\,.$$

Now transposing both sides gives

$$\overline{\chi}^T \;\; \|1/\rho\| \;\; \chi \;\; = \;\; I\,. \tag{4.22}$$

Thus

$$\chi^{-1} \;\; = \;\; \overline{\chi}^T \;\; \|1/\rho\|$$

and this gives

$$\chi \;\; \overline{\chi}^T \;\; \|1/\rho\| \;\; = \;\; I$$

or, better yet

$$\chi \;\; \overline{\chi}^T = \|\rho\| \tag{4.23}$$

The identities in 4.22 and 4.23 are usually referred to as the "*character relations*" and the matrix χ is sometimes called the "*character table*" of Γ.

Remark 4.3

It is to be noted that the ratios $\rho_\lambda = N/R_\lambda$ are also integers. Indeed, ρ_λ represents the number of elements of Γ which commute with any given element of the class $\mathcal{C}_\lambda$. That is, for any $\sigma \in \mathcal{C}_\lambda$ we have

$$\sum_{\gamma\in\Gamma} \chi(\gamma\,\sigma\gamma^{-1} = \sigma) = \rho_\lambda. \tag{4.24}$$

To see this note first that the quantity

$$\rho(\sigma) = \sum_{\gamma\in\Gamma} \chi(\gamma\,\sigma\gamma)^{-1}$$

depends only on the conjugacy class of σ that is if $\sigma_2 = \theta\sigma_1\theta^{-1}$ then

$$\rho(\sigma_2) = \rho(\sigma_1)\;.$$

This given, using Theorem 1.1 with $\mathcal{F} = \mathcal{C}_\lambda$ and with the action $\nu(\gamma)\sigma = \gamma\,\sigma\gamma^{-1}$ for any $\sigma \in \mathcal{C}_\lambda$ we get

$$\begin{aligned} 1 = |\Delta| &= \frac{1}{N} \sum_{\gamma\in\Gamma} \sum_{\sigma\in\mathcal{C}_\lambda} \chi(\gamma\sigma\gamma^{-1} = \sigma) = \\ &= \frac{1}{N} \sum_{\sigma\in\mathcal{C}_\lambda} \rho(\sigma) = \frac{R_\lambda}{N}\;\rho(\sigma_1) \end{aligned}$$

where σ_1 is any element of $\mathcal{C}_\lambda$.

The functions $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ will sometimes be referred to as the *fundamental characters of* Γ, but more often as **the** *characters* of Γ.

An element χ of $\mathcal{C}(\Gamma)$ which is of the form

$$\chi = \sum_{\lambda=1}^{k} p_\lambda \chi^\lambda$$

with $p_1, p_2, \ldots, p_k$ all non-negative integers will be referred to as a "*generalized* character of Γ" or briefly **a character of** Γ. From Theorem 4.4 we see that given any generalized character χ there are several representations whose characters are equal to χ.

Problems

1. Let Γ act on a Family $\mathcal{F}$ and let A be the representation of Γ induced by this action. Show that the multiplicity of the trivial representation in A is equal to the number of orbits of $\mathcal{F}$ under the action of Γ.

2. Let Γ act on a Family $\mathcal{F}$ and, as before, let A be the representation of Γ induced by this action. Denoting by χ^A the character of A, show that the integer

$$\big\langle \chi^A \, , \, \chi^A \big\rangle_\Gamma$$

counts the number of orbits in the action of Γ on the family of ordered pairs

$$\mathcal{F} \times \mathcal{F} \;\; = \;\; \big\{ \, (f \, , \, g) \, : \, f, g \in \mathcal{F} \, \big\}$$

3. For $\sigma \in S_n$ let $\Pi(\sigma)$ denote the corresponding permutation matrix. That is

$$\Pi(\sigma) \;\; = \;\; \big\| \, \chi(i = \sigma_j) \, \big\|_{i,j=1}^{n}$$

Let χ^Π denote the character of Π. Show that

$$\big\langle \chi^\Pi \, , \, \chi^\Pi \big\rangle_{S_n} \;\; = \;\; 2 \, .$$

4. The Dihedral group $\mathcal{D}_4$ is the group generated by the two matrices

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

sometimes referred to as the "4-*bead necklace group*" because it consists of all the symmetries of a 4-bead necklace (four 90^o rotations and four 90^o rotations followed by a reflection). The elements of the group are (with $\overline{1} = -1$)

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} D_1 &= \begin{bmatrix} 1 & 0 \\ 0 & \overline{1} \end{bmatrix} , \; D_2D_1 = \begin{bmatrix} 0 & \overline{1} \\ 1 & 0 \end{bmatrix} , \; D_1D_2D_1 = \begin{bmatrix} 0 & \overline{1} \\ \overline{1} & 0 \end{bmatrix} , \\ D_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \; D_1D_2 = \begin{bmatrix} 0 & 1 \\ \overline{1} & 0 \end{bmatrix} , \; D_2D_1D_2 = \begin{bmatrix} \overline{1} & 0 \\ 0 & 1 \end{bmatrix} , \end{aligned} \tag{4.25}$$

and

$$D_1D_2D_1D_2 = \begin{bmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{bmatrix} = D_2D_1D_2D_1 \, .$$

a) Determine, all the irreducible representgations of $\mathcal{D}_4$.
b) Compute, all its characters.
b) Determine all its conjugacy classes.

5. The family of the 2-subsets of $\{1, 2, 3, 4\}$ is

$$\mathcal{F}_{2,4} \;\; = \;\; \{\{1, 2\} \, , \, \{1, 3\} \, , \, \{1, 4\} \, , \, \{2, 3\} \, , \, \{2, 4\} \, , \, \{3, 4\}\}$$

Here for $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in S_4$ we set $\sigma\{i, j\} = \{\sigma_i, \sigma_j\}$. It is easily seen that this action induces a 6-dimensional transitive permutation representation. Denoting this representation by $\sigma \rightarrow A(\sigma)$, determine the dimensions of the irreducible constituents of A and their respective multiplicities.

6. Recall that the even permutations of the Symmetric group S_n (that is the permutations with an even number of inversions) form a subgroup of S_n called the "*alternating group*" usually denoted A_n. For a given even permutation α let $(\alpha)_{A_n}$ and $(\alpha)_{S_n}$ denote the conjugacy classes of α in A_n and S_n respectively.
   a) Show that if $\alpha \in A_n$ commutes with some odd permutation then $(\alpha)_{A_n} = (\alpha)_{S_n}$
   b) Show that if $\alpha \in A_n$ does not commute with any odd permutation then $(\alpha)_{S_n}$ splits into two parts of equal cardinality according to the identity

$$(\alpha)_{S_n} \;\; = \;\; (\alpha)_{A_n} \;\; + \;\; (1, 2)(\alpha)_{A_n}(1, 2)$$

   c) Use a) and b) to determine which partitions correspond to conjugacy classes of S_n that do not split and which correspond to conjugacy classes that split.