4. Posets and Poset-Partitions

We recall that a "*Partially Ordered Set*", briefly a "*Poset*", is a pair $\mathcal{P} = (\Omega, \leq)$ where Ω is a finite set and "$\leq$" is a relation satisfying the following three properties

$$\begin{array}{llcll} i) & \quad x \leq x & & & \\ ii) & x \leq y \ \ \& \ \ y \leq z & \Longrightarrow & x \leq z & \qquad \forall\ x,y,z \in \Omega \\ iii) & x \leq y \ \ \& \ \ y \leq x & \Longrightarrow & x = y & \end{array}$$

We shall also write "$x < y$" to mean

$$x \leq y \quad \& \quad x \neq y$$

If $x < y$ and there is no element of Ω that lies between them then x is called a *predecessor* of y (y a *successor* of x) and we set "$x \rightarrow y$". In symbols

$$x \rightarrow y \quad \Longleftrightarrow \quad (x \leq z \leq y) \ \Longrightarrow (z = x) \ or \ (z = y)$$

For a given $x \in \Omega$ we shall set

$$\widehat{x} \ = \ \{ y \in \Omega \ : \ y \leq x \ \}$$

and refer to it as the "*principal order ideal below* x". A "*lower order ideal* " of $\mathcal{P}$ is a subset $I \subseteq \Omega$ such that

$$x \in I \quad \& \quad y \leq x \ \Longrightarrow \quad y \in I$$

Our finiteness assumption implies that every element of a lower order ideal I lies below a *maximal elememt* of I † Thus we se that every lower order ideal I may be expressed as a union of a finite number of principal lower order ideals. In symbols we have

$$I \ = \ \bigcup_{i=1}^{m} \widehat{x_i}$$

where $\{x_1, x_2, \ldots, x_m\}$ is the set of maximal elements of I. We should mention that there is also a parallel notion of "*upper order ideal*" where all the defining inequalities are reversed. In particular we set

$$\check{x} \ = \ \{ y \in \Omega \ : \ y \geq x \ \}$$

and call it a "*principal upper order ideal*". We can also express an upper order ideal in the form

$$I \ = \ \bigcup_{i=1}^{n} \check{y}_i$$

where $\{y_1, y_2, \ldots, y_n\}$ are the "*minimal*" elements of I.

A listing $x_1 \ , \ x_2 \ , \ x_3 \ , \ x_4 \ , \ \ldots \ , \ x_N$ of all the elements of Ω is called "*linear extension of* $\mathcal{P}$" if and only if for any pair $x_i \ , x_j$ we have $x_i < x_j$ in $\mathcal{P}$ only when $i < j$. There is a very simple recursive algorithm for constructing all linear extensions of a poset $\mathcal{P}$. We simply start with any one of the minimal elements of $\mathcal{P}$ then having constructed the first $k-1$ elements $x_1 \ , \ x_2 \ , \ \ldots \ , \ x_{k-1}$ of the linear extension we chose x_k to be any of the minimal elements of the poset obtained by removing $x_1 \ , \ x_2 \ , \ \ldots \ , \ x_{k-1}$ from Ω.

We recall that a "*chain*" c of $\mathcal{P}$ is subset $c = \{x_1, x_2, \ldots, x_k\}$ such that $x_1 < x_2 < \cdots < x_k$. The cardinality k of c is often referred to as the "*length*" of c. We say that c is "*unrefinable*" or "*locally maximal*" if

$$x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k$$

Our posets will sometimes have a unique minimal and a unique maximal element, these are usually denoted by the symbols $\widehat{0}$ and $\widehat{1}$. In a poset $\mathcal{P}$ with $\widehat{0}$ and $\widehat{1}$ a subset $m = \{x_1, x_2, \ldots, x_l\}$ is said to be a

† That is an element $x_o \in I$ such that $(y \geq x_o) \ \& \ (y \in I) \ \Longrightarrow \ \ y = x_o$

"*maximal chain* " if and only if

$$\widehat{0} \to x_1 \to x_2 \to \cdots \to x_l \to \widehat{1}$$

A poset $\mathcal{P}$ with $\widehat{0}$ and $\widehat{1}$ whose maximal chains have all the same length is said to be '"*ranked*". In such a poset the rank of an element x is taken to be one less than the length of a locally maximal chain that starts at $\widehat{0}$ and ends at x. In symbols, we set $r(x) = k$ if and only if we have elements $x_1, x_2, \ldots, x_k$ such that

$$\widehat{0} \to x_1 \to x_2 \to \cdots \to x_{k-1} \to x_k = x \ .$$

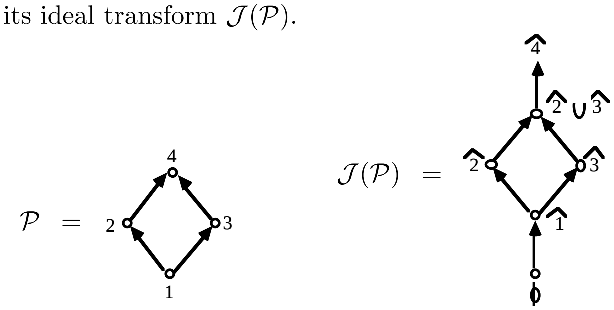
This notion of rank is well defined since it is easy to derive from our definitions that all the locally maximal chains joining $\widehat{0}$ to x must necessarily have the same length.

Given a poset $\mathcal{P} = (\Omega, \leq)$ we define "*the ideal transform of* $\mathcal{P}$" to be the poset $\mathcal{J}(\mathcal{P})$ which consists of the lower order ideals of $\mathcal{P}$ ordered by ordinary set inclusion. In symbols

$$\mathcal{J}(\mathcal{P}) \ = \ \big(\, \mathcal{I}(\mathcal{P}) \, , \ \text{“} \subseteq \text{”} \, \big)$$

where $\mathcal{I}(\mathcal{P})$ is the collection of all lower order ideals of $\mathcal{P}$.

Posets are usually drawn with arrows pointing upwards joining all the pairs "$x \to y$". We refer to these drawings as "*Hasse diagrams*". They are very effective in representing a poset since all the inequalities may be easily derived by "*following the arrows*". In the figure below we represent in this manner an example of a poset $\mathcal{P}$ and its ideal transform $\mathcal{J}(\mathcal{P})$.



Note that the elements of $\mathcal{I}(\mathcal{P})$ are †

$$\emptyset = \{\} \ , \ \widehat{1} = \{1\} \ , \ \widehat{2} = \{1,2\} \ , \ \widehat{3} = \{1,3\} \ , \ \widehat{2} \cup \widehat{2} = \{1,2,3\} \ , \ \widehat{4} = \{1,2,3,4\} \ .$$

From this we can easily infer that if $\mathcal{P} = (\Omega, \leq)$ is any poset, its ideal transform is a poset with a $\widehat{0}$ and $\widehat{1}$ given by $\emptyset$ and Ω respectively. $\mathcal{J}(\mathcal{P})$ is also is always ranked. In fact, the rank of an element $I \subseteq \mathcal{I}(\mathcal{P})$ is simply given by the cardinality of I as a subset of Ω. Most importantly we see that since to go from an ideal I to any of its successors in $\mathcal{J}(\mathcal{P})$ we need only add one of the minimal elements in the complement of I in Ω, the maximal chains of $\mathcal{J}(\mathcal{P})$ are into one to one correspondence with the linear extensions of $\mathcal{P}$.

A "*poset partition*" or briefly a"*$\mathcal{P}$-partition*" is a map f of a poset $\mathcal{P} = (\Omega, \leq)$ into the natural numbers which is weakly decreasing with respect to "$\leq$". The collection of all $\mathcal{P}$-partitions will be denoted by $\mathcal{F}(\mathcal{P})$. In symbols

$$\mathcal{F}(\mathcal{P}) \ = \ \Big\{ \ f \ : \ (x \leq y) \ \ \to \ \ \big(f(x) \geq f(y) \big) \ \Big\} \ .$$

We can easily see that this notion reduces to the ordinary notion of partition when $\mathcal{P}$ consists of a single chain. In analogy with our definitions for ordinary partitions we define the generating functions of $\mathcal{P}$-partitions to be the series

$$F_{\mathcal{F}(\mathcal{P})}(q) \ = \ \sum_{f \in \mathcal{F}(\mathcal{P})} q^{|f|} \tag{4.1}$$

† Since we have chosen the integers $1, 2, 3, 4$ to be the elements of Ω, we are forced to an abuse of notation which requires us to use the symbol $\widehat{1}$ for the top element of $\mathcal{J}(\mathcal{P})$ and the same symbol $\widehat{1}$ to denote the principal lower ideal consisting of the elements of $\mathcal{P}$ that are below the element 1.

where for convenience we let

$$|f| \;=\; \sum_{x\in\Omega} f(x)$$

Sometimes we also include an additional parameter t to take account of the maximum part and set

$$F_{\mathcal{F}(\mathcal{P})}(t;q) \;=\; \sum_{f\in\mathcal{F}(\mathcal{P})} t^{max(f)}\, q^{|f|} \;. \tag{4.2}$$

There are no simple formulas giving $F_{\mathcal{F}(\mathcal{P})}(q)$ or $F_{\mathcal{F}(\mathcal{P})}(t;q)$ in full generality. However we can give two algorithms for constructing them. One is almost immediate but yields a rather complicated final expression the other is a bit more complex but leads to some remarkably beautiful identities. To present them we need some definitions.

To begin with let $f\in\mathcal{F}(\mathcal{P})$ be given and suppose it takes $k+1$ distinct values $\lambda_1>\lambda_2>\cdots>\lambda_{k+1}\geq 0$. Set

$$A_i \;=\; \{\, x\in\Omega \;:\; f(x)=\lambda_i \,\} \qquad \text{and} \qquad B_i \;=\; \{\, x\in\Omega \;:\; f(x)\geq\lambda_i \,\} \tag{4.3}$$

It is clear that

$$B_i \;=\; A_1+A_2+\cdots+A_i \tag{4.4}$$

Note further that since (by the definition of $\mathcal{F}(\mathcal{P})$) $y\leq x$ implies $f(y)\geq f(x)$ we see that $f(x)\geq\lambda_i$ forces $f(y)\geq\lambda_i$. In other words $x\in B_i$ and $y\leq x$ forces $y\in B_i$, this shows that each B_i is a lower order ideal of $\mathcal{P}$. This means that the sequence

$$\emptyset\subset B_1\subset B_2\subset\cdots\subset B_k\subset B_{k+1}=\Omega$$

is a chain in $\mathcal{J}(\mathcal{P})$. In summary we can associate to each $f\in\mathcal{F}(\mathcal{P})$ a pair $\big(c(f),la(f)\big)$ consisting of a chain $c(f)$ in $\mathcal{J}(\mathcal{P})$ and a partition $\lambda(f)=(\lambda_1>\lambda_2>\cdots>\lambda_{k+1}\geq 0)$ with $k=length\big(c(f)\big)$. Note that since the number of times f takes the value λ_i is equal to the cardinality of the set A_i we see that

$$|f| \;=\; \lambda_1\,|A_1| \;+\; \lambda_2\,|A_2| \;+\;\cdots\;+\; \lambda_{k+1}\,|A_{k+1}| \;. \tag{4.5}$$

For convenience set

$$p_{k+1}=\lambda_{k+1} \quad , \quad p_i=\lambda_i-\lambda_{i+1} \quad (\, i=1,2,\cdots,k\,)$$

so that 4.4 may be converted to

$$|f| \;=\; (p_1+\cdots+p_{k+1})\,|A_1| \;+\; (p_2+\cdots+p_{k+1})\,|A_2| \;+\;\cdots\;+\; p_{k+1}\,|A_{k+1}|$$

which, together with 4.4 gives that

$$|f| \;=\; p_1\,|B_1| \;+\; p_2\,|B_2| \;+\;\cdots\;+\; p_{k+1}\,|B_{k+1}| \tag{4.5}$$

Note that we must have $p_i\geq 1$ for $i=1,2,\cdots,k$ and $p_{k+1}\geq 0$. Thus by summing over all the elements f of $\mathcal{F}(\mathcal{P})$ whose associated chain $c(f)$ is the chain $c=\big(\,\emptyset\subset B_1\subset B_2\subset\cdots\subset B_k\subset\Omega\,\big)$ we obtain

$$\begin{aligned}
\sum_{f\in\mathcal{F}(\mathcal{P});c(f)=c} q^{|f|} \;&=\; \sum_{p_1\geq 1}\sum_{p_2\geq 1}\cdots\sum_{p_k\geq 1}\sum_{p_{k+1}\geq 0} q^{p_1|B_1|}q^{p_2|B_2|}\cdots q^{p_k|B_k|}q^{p_{k+1}|B_{k+1}|}\\
&=\; \frac{q^{|B_1|}}{1-q^{|B_1|}}\frac{q^{|B_2|}}{1-q^{|B_2|}}\cdots\frac{q^{|B_k|}}{1-q^{|B_k|}}\frac{1}{1-q^{|B_{k+1}|}}\\
&=\; \frac{1}{1-q^n}\prod_{i=1}^{k}\frac{q^{|B_i|}}{1-q^{|B_i|}} \;=\; \frac{1}{1-q^n}\prod_{r\in r(c)}\frac{q^r}{1-q^r} \;.
\end{aligned}$$

Where for convenience we have set $|\Omega|=n$ and we have indicated by $r(c)$ the set of ranks of the components of c as elements of $\mathcal{J}(\mathcal{P})$. In this case of course we have

$$r(c) \;=\; \{\,|B_1|,|B_2|,\ldots,|B_k|\,\}$$

We are thus led to the following remarkable result

Theorem 4.1

Denoting by $\mathcal{C}(\mathcal{J}(\mathcal{P}))$ the collection of all chains o $\mathcal{J}(\mathcal{P})$ we have

$$F_{\mathcal{F}(\mathcal{P})}(q) \;=\; \frac{1}{1-q^n} \sum_{c\in\mathcal{C}(\mathcal{J}(\mathcal{P}))} \prod_{r\in r(c)} \frac{q^r}{1-q^r} \;. \qquad 4.6$$

Proof

Just group terms in 4.1 according to the value of $c(f)$.