I. PARTITIONS

1. Definitions

A partition is a vector with weakly decreasing positive integer components. We shall use throughout this notes greek letters to denote partitions. The collection of all partitions will be denoted by $\mathcal{P}$. In symbols:

$$\mathcal{P} \;=\; \big\{\, \lambda \;:\; \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0) \;\; \textit{for some } k \,\big\}$$

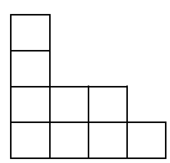
If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0)$ with

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \;=\; n \qquad 1.1$$

then we shall say that λ is "*a partition of n with k parts*" . We shall also refer to k as the "*length* " of λ and denote it by $l(k)$. It will be convenient to use the symbol " $\lambda \vdash n$ " as a shorthand for 1.1. The collection of all partitions of n will be denoted $\mathcal{P}_n$ and the collection of partitions of n of length k will be denoted $\mathcal{P}_{n,k}$. In symbols

$$\mathcal{P}_n \;=\; \big\{\, \lambda \,:\;\; \lambda \vdash n \,\big\} \;, \qquad \mathcal{P}_{n,.k} \;=\; \big\{\, \lambda \,:\;\; \lambda \vdash n \;\;\&\;\; l(\lambda) = k \big\}$$

The figure consisting of a number of left justified rows of squares is usually referred to as a Ferrers diagram. We use Ferrers diagrams to depict partitions. The Ferrers diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, is denoted by $\mathcal{F}(\lambda)$ and is constructed by placing λ_i squares in the i^{th} row. We shall use the French convention here and, given that the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrer's diagram have μ_i lattice squares in the i^{th} row (counting from the bottom up), hence the diagram of 4311 is:



We shall make use of a symbolism to represent collections of partitions by means of formal series whose terms are Ferrers diagrams. The basic principle we shall use is that to represent the collection $\mathcal{C}$ of pictures obtained by combining a number of copies of a fixed given object we use a geometric series based on the given object. That is we set

$$\mathcal{C} \;=\; 1 \;+\; \textit{object} \;+\; \textit{object}^{\,2} \;+\; \textit{object}^{\,3} \;+\; \textit{object}^{\,4} \;+\; \cdots \;. \qquad 1.2$$

Alternatively, we may also write

$$\mathcal{C} \;=\; \frac{1}{1 \;-\; \textit{object}}$$

which should only be understood as an abreviation for what is written in 1.2. With this convention the collection of Ferrers diagrams all whose rows have length one may be represented by the symbol

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 \;\cdots$$

What is to be understood, although not explicitely expressed by the symbol, is that we are to pick none, or one, or two, or three, etc... copies of a single square and setting them one on top of the other to represent a diagram with rows of length one. Thus

$$\square^4 \;=\; \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

In the same vein we can write

$$\frac{\square}{1-\square} = \square + \square^2 + \square^3 + \bullet\bullet\bullet$$

to represent taking a number greater or equal to one of single squares, and the symbol

$$1 + \square$$

to express that we must "not take" or "take" a single square.

This symbolism can be further extended by interpreting the product of two objects as the composite object obtained by glueing them together side by side or one on top of the other according to the context. More generally, in order not to clutter our synbols with too much information, we shall have two infer from the context how we must interpret the imagery we use to represent our objects. These conventions are best communicated by means of examples. For instance the collection of all Ferrers diagrams of partitions, namely the formal sum

$$GS(\mathcal{P}) \;=\; \sum_{\lambda\in\mathcal{P}} \mathcal{FD}(\lambda)$$

may be represented symbolically by the series

$$GS(\mathcal{P}) = \frac{1}{1-\square} \times \frac{1}{1-\square\square} \times \frac{1}{1-\square\square\square} \times \bullet\bullet\bullet \qquad 1.3$$

What has to be understood is that we are to pick:

1) one object from the first factor, (that is a number of rows of length one),

2) one object from the second factor, (that is a number of rows of length two),

3) then one object from the third factor (that is a number of rows of length three),

4) etc

and then stick them together one on top of the other to build a Ferrers diagram.

Similarly we may refer to the collection of all partitions with distinct parts by writing

$$GS(\mathcal{P}_D) = (1+\square)(1+\square\square)(1+\square\square\square)\,\bullet\bullet\bullet \qquad 1.4$$

Indeed, in building a partition with distict parts we must either take or not take a row of length one, either take or not take a row of length two, either take or not take a row of length three, etc... Note next that the collection of 4-part partitions

$$GS(\mathcal{P}_{4\ parts}) \;=\; \sum_{\lambda\ with\ 4-parts} \mathcal{FD}(\lambda)$$

may be referred to by writing

$$\mathrm{GS}\,(\mathrm{P}_{4\text{ parts}}) = \frac{\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\\[-4pt]\square\end{array}}{1-\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\\[-4pt]\square\end{array}}\;\frac{1}{1-\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}}\;\frac{1}{1-\begin{array}{c}\square\\[-4pt]\square\end{array}}\;\frac{1}{1-\square} \tag{1.5}$$

It should be noted that this symbolism goes beyond simply referring to a certain collection of objects, but rather contains information on how these objects may be put together. Indeed, we may infer from 1.5 that, a four-part partition may be constructed by taking

1) a number (≥ 1) of columns of length four,
2) a number ≥ 0 of columns of length three,
3) a number ≥ 0 of columns of length two,
4) and finally a number ≥ 0 of columns of length one

then stick them together **side by side** to obtain a Ferrers diagram.

Analogously, to obtain a partition with four distinct parts, we must take at least one column of length four, at least one column of length three, at least one column of length two, and at least one column of length one. Thus the collection

$$GS(\mathcal{DP}_{4-parts}) \;=\; \sum_{\lambda \;\; has 4-distinct\ parts} \mathcal{FD}(\lambda) \tag{1.6}$$

may be referred to by writing

$$\mathrm{GS}\,(\;\mathrm{P}^{\mathrm{D}}_{4\text{ parts}}) = \frac{\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\\[-4pt]\square\end{array}}{1-\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\\[-4pt]\square\end{array}}\;\frac{\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}}{1-\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}}\;\frac{\begin{array}{c}\square\\[-4pt]\square\end{array}}{1-\begin{array}{c}\square\\[-4pt]\square\end{array}}\;\frac{\square}{1-\square} \tag{1.7}$$

We should call expressions of the kind

$$GS(Family) \;=\; \sum_{object \;\in\; Family} object$$

the "*Generating Series*" of the given family. By contrast a *"Generating Function"* of the family is obtained by summing over all objects of the family, not the objects themselves (as in 1.4), but rather their "*weights*" under a certain assignment of weights. In symbols

$$GF_{weight}(Family) \;=\; \sum_{object \;\in\; Family} weight\,[object]$$

For families of partitions, the most commonly chosen weight is one which takes account of the number of parts and the sum of the parts. For instance, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we shall set here

$$w(\lambda) \;=\; z^k\; q^{\lambda_1+\lambda_2+\cdots+\lambda_k}\;\;.$$

Other weights are also often used. In these notes we shall also use the weights

$$w_1(\lambda) \;=\; q^{\lambda_1+\lambda_2+\cdots+\lambda_k} \;\;\text{and}\;\;\;\; w_{-1}(\lambda) \;=\; (-1)^k\, q^{\lambda_1+\lambda_2+\cdots+\lambda_k}$$

which are respectively obtained by setting $z = 1$ and $z = -1$ in $w(\lambda)$. . The corresponding generating functions will then be denoted GF, GF_1 or GF_{-1} according as we use $w(\lambda)$, $w_1(\lambda)$ or $w_{-1}(\lambda)$ This given, we shall immediately see the power of our symbolism, in that it also offers a convenient mechanism for deriving sums of weights from sums of objects. Infact, whenever the weight function is such that the weight of a product of objects is equal to the product of the weights of the individual factors, we may directly substitute weights for objects either before or after we carry out the products. For instance from the symbol in 1.3 we immediately deduce that

$$GF(\mathcal{P}) \;=\; \frac{1}{(1-zq)}\,\frac{1}{(1-zq^2)}\,\frac{1}{(1-zq^3)}\cdots \;=\; \prod_{n\geq 1}\frac{1}{(1-zq^n)}$$

while 1.4 gives that the generating function of partitions with distict parts is given by the formal series

$$GF(\mathcal{DP}) \;=\; (1+zq)(1+zq^2)(1+zq^3)\cdots \;=\; \prod_{n\geq 1}(1+zq^n) \qquad 1.8$$

Similarly, we get from 1.5 that the generating function of partitions with four parts is given by the expression

$$GF(\mathcal{P}_{4-parts}) \;=\; z^4\,\frac{q^4}{1-q^4}\,\frac{1}{1-q^3}\,\frac{1}{1-q^2}\,\frac{1}{1-q} \qquad 1.9$$

These observations lead us to the following remarkable identities

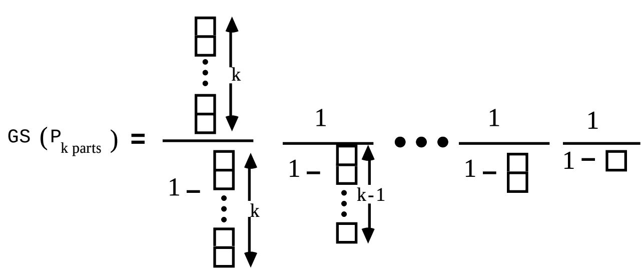
Theorem 1.1 *(Euler)*

$$GF(\mathcal{P}) \;=\; \prod_{n\geq 1}\frac{1}{(1-zq^n)} \;=\; \sum_{k\geq 0}\frac{(zq)^k}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^k)} \qquad 1.10$$

$$GF(\mathcal{DP}) \;=\; \prod_{n\geq 1}(1+zq^n) \;=\; \sum_{k\geq 0}\frac{(zq)^k\,q^{\binom{k}{2}}}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^k)} \qquad 1.11$$

Proof

We can easily see, from the argument that gave 1.5, that partitions with a given number of parts may be constructed one column at the time starting with at least one column of length k and following by any number of columns of lesser lengths. Thus the generating series of k-part partitions is given by the expression



1.12

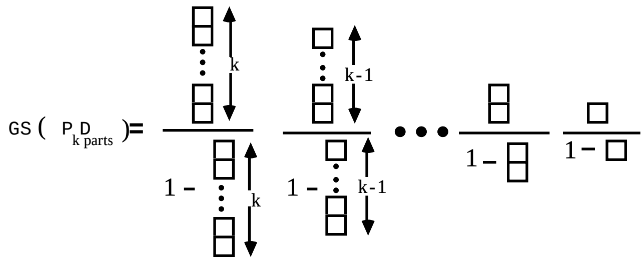
This gives us that

$$GF_1(\mathcal{P}_{k-parts}) \;=\; \frac{q^k}{(1-q^k)(1-q^{k-1})\cdots(1-q^2)(1-q)}\;,$$

and 1.10 immediately follows since, by grouping partitions according to their lengths, we easily deduce that we must have

$$GF(\mathcal{P}) \;=\; 1+\sum_{k\geq 1} z^k\;GF_1(\mathcal{P}_{k-parts})\;.$$

To prove 1.11 we can proceed in an analogous manner an note that we must have (as in the 4-part case)



1.13

and this gives

$$GF_1(\mathcal{DP}_{k-parts}) \;=\; \frac{q^{k+k-1+\cdots+2+1}}{(1-q^k)(1-q^{k-1})\cdots(1-q^2)(1-q)} \;,$$

and then 1.11 follows from the identity

$$GF(\mathcal{DP}) \;=\; 1+\sum_{k\geq 1} z^k \;\, GF_1(\mathcal{DP}_{k-parts}) \;\;.$$

and the fact that

$$k+k-1+\cdots+2+1 \;=\; \binom{k}{2} \;+\; k$$

Remarks 1.1

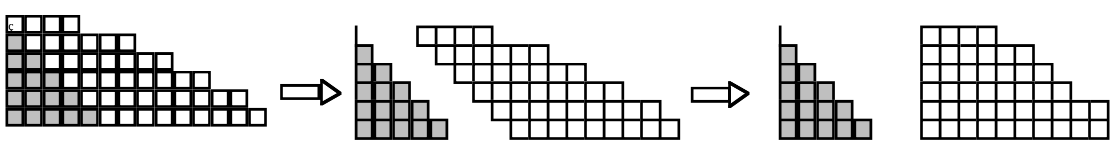
We should note that both equalities in 1.10 and 1.11 are consequences of the simple observation that Ferrers diagrams of partitions may be constructed by rows or by columns. Another fact which emerges from this proof (by comparing 1.12 and 1.13) is that each partition with k distinct parts may be factored into a pair consisting of a k-part partition and a $k-1$-staircase. Analytically, this may be established by noting that if $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ and we set

$$\mu_i \;=\; \lambda_i-(k-i) \qquad \text{for } i=1,\ldots,k$$

Then since $\mu_i-\mu_{i+1}=\lambda_i-\lambda_{i+1}-1$ we see that $\lambda_i>\lambda_{i+1}$ gives that $\mu_i\geq\mu_{i+1}$ assuring that we have the decomposition

$$\lambda=(k-1,k-2,\ldots,1\;,\;0) \;+\; (\mu_1,\mu_2,\ldots,\mu_k)$$

with μ a partition. Pictorially we may represent this decomposition in the form



Here and after we shall use the notation

$$a)\quad e_q(z)=1+\sum_{k\geq 1}\frac{z^k}{(1-q)(1-q^2)\cdots(1-q^k)} \;,$$

$$b)\quad E_q(z)=1+\sum_{k\geq 1}\frac{z^k q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

1.14

These two series are easily seen to be q-analogues of the exponential function. Indeed both tend to

$$\sum_{k\geq 0}\frac{z^k}{1\,2\,\cdots k} \;=\; e^x$$

as we make the substitution $z \to z(1-q)$ and then let q tend to 1.

Note further that because of Euler's Theorem 1.1 we also have the product expansions:

$$a) \quad e_q(z) \;=\; \prod_{n\geq 0} \frac{1}{(1-zq^n)} \;, \qquad b) \quad E_q(z) \;=\; \prod_{n\geq 0} (1+zq^n) \tag{1.15}$$

Remark 1.2

To bring further insight into the results under study, it may be good from time to time to give also manipulatorial proofs of identities we establish in a purely combinatorial manner. In this instance we will give a brief indication how the expansion in 1.14 a) may be obtained if we start with 1.15 a) as the definition of $e_q(x)$. To this end, let us begin by setting

$$e_q(z) \;=\; 1 + \sum_{k\geq 1} \sum_{m\geq 0} c_{k,m} \; z^k q^m \tag{1.16}$$

Now that it is clear from 1.15 a) that we do have

$$c_{k,m} \;=\; \prod_{n=0}^{m} \frac{1}{(1-zq^n)} \;\Big|_{z^k q^m}$$

where the symbol " $\big|_{z^k q^m}$" represents taking the coefficient of $z^k q^m$ in the preceding expression. † The reason for this is that all the omitted terms contain powers of q greater tham m and therefore cannot contribute anything to this coefficient. In the same vein we can see that we must have

$$c_{k,m} \;=\; \prod_{n=0}^{m} \Big(1 + zq^n + (zq^n)^2 + \cdots + (zq^n)^m\Big) \;\Big|_{z^k q^m}$$

because again the omitted terms only contain powers of q greater than m. This shows that to compute the coefficients $c_{k,m}$ we need only a finite number of operations and therefore the expression

$$c_k(q) \;=\; \sum_{m\geq 0} c_{k,m} \; q^m$$

is a well defined formal power series. So we may set

$$e_q(z) \;=\; \sum_{k\geq 0} c_k(q)\, z^k \tag{1.17}$$

with

$$c_o(q) = 1 \tag{1.18}$$

.

Now it is easy to see from 1.15 a) that

$$e_q(z) \;=\; \frac{1}{(1-z)}\, e_q(zq)$$

Using 1.17 we may rewrite this as

$$(1-z) \sum_{k\geq 0} c_k(q) \; z^k \;=\; \sum_{k\geq 0} c_k(q) \; (zq)^k \;.$$

† More generally here and after the symbol " $\big|_m$" (for some monomial m) represents taking the coefficient of m in the expression that just precedes the symbol.

Equating coefficients of z^k in both sides yields the recursion

$$c_k(q) \;-\; c_{k-1}(q) \;=\; c_k(q)\; q^k$$

or better

$$c_k(q) \;=\; \frac{c_{k-1}(q)}{(1-q^k)}$$

which together with 1.17 yields

$$c_k(q) \;=\; \frac{1}{(1-q^k)}\frac{1}{(1-q^{k-1})}\cdots\frac{1}{(1-q)}$$

proving the expansion in 1.14 a). We leave it to the reader to construct a proof of 1.14 b) using a similar argument starting from 1.15 b).

The process of proof we have followed to get 1.14 a from 1.15 a) can be extended to yield a formula which contains both 1.10 and 1.11 as special cases.

Theorem 1.2

$$\prod_{n\geq 0}\frac{(1+xzq^n)}{(1-zq^n)} \;=\; 1+\sum_{n\geq 1}\frac{(1+x)(1+xq)\cdots(1+xq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}\; z^n \qquad 1.19$$

Proof

Proceeding as we did in Remark 1.2 let us set

$$\prod_{n\geq 0}\frac{(1+xzq^n)}{(1-zq^n)} \;=\; \sum_{n\geq 0} c_n(x,q)\; z^n \;\; . \qquad 1.20$$

with

$$c_o(x,q)=1 \qquad 1.21$$

Calling $G(z)$ for a moment the left-hand side of 1.19 we derive from 1.19 that

$$G(z) \;=\; \frac{(1+xz)}{(1-z)}\; G(zq) \;\; .$$

Using 1.20 we may rewrite this as

$$(1-z)\sum_{n\geq 0} c_n(x,q)\; z^n \;=\; (1+xz)\sum_{n\geq 0} c_n(x,q)\;(zq)^n \;\; .$$

Equating coefficients of z^n on both sides yields that we must have

$$c_n(x,q)-c_{n-1}(x,q) \;=\; c_n(x,q)\; q^n + xc_{n-1}(x,q)q^{n-1}$$

or better

$$c_n(x,q) \;=\; \frac{1+xq^{n-1}}{1-q^n}\; c_{n-1}(x,q) \;\; .$$

Iterating this recursion and using 1.21 we are led to

$$c_n(x,q) \;=\; \frac{(1+x)(1+xq)\cdots(1+xq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}$$

proving 1.19.

As we shall see, the identity in 1.19 has a wide variety of applications, for the moment we shall have to be contented with two important corollaries:

Theorem 1.3 (The "q-Binomial" identity)

$$(1+x)(1+xq)\cdots(1+xq^{n-1}) \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\, x^k \qquad 1.22$$

where we have set

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \;=\; \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)(1-q)(1-q^2)\cdots(1-q^{n-k})} \qquad 1.23$$

Proof

Since the left-hand side of 1.19 is none other than $E_q(xz)\,e_q(z)$ we may use the expansions in 1.14 and rewrite 1.19 in the form

$$\Big(\sum_{k\geq 0} \frac{x^k z^k q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big)\Big(\sum_{h\geq 0} \frac{z^h}{(1-q)(1-q^2)\cdots(1-q^h)}\Big) \;=$$

$$= \; 1 + \sum_{n\geq 1} \frac{(1+x)(1+xq)\cdots(1+xq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}\, z^n$$

Equating coefficients of z^n on both sides gives

$$\sum_{k\geq 0}^{n} \frac{x^k\, q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)\;\,(1-q)(1-q^2)\cdots(1-q^{n-k})} \;=\; \frac{(1+x)(1+xq)\cdots(1+xq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}$$

and 1.23 immediately follows upon multiplying both sides by

$$(1-q)(1-q^2)\cdots(1-q^n)$$

Theorem 1.4

$$\frac{1}{(1-z)(1-zq)\cdots(1-zq^{k-1})} \;=\; 1+\sum_{n\geq 1} \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix} z^n \qquad 1.24$$

Proof

Note that we may write

$$\frac{1}{(1-z)(1-zq)\cdots(1-zq^{k-1})} \;=\; \frac{(1-zq^k)(1-zq^{k+1})(1-zq^{k+2})(1-zq^{k+3})\cdots}{(1-z)(1-zq)(1-zq^2)(1-xq^3)\cdots} = E_q(-zq^k)e_q(z)\;\;.$$

But the latter is none other than the left-hand side of 1.19 for $x=-q^k$. Thus making this substitution in 1.19 we get

$$\frac{1}{(1-z)(1-zq)\cdots(1-zq^{k-1})} \;=\; 1 \;+\; \sum_{n\geq 1} \frac{(1-q^k)(1-q^{k+1})\cdots(1-q^{k+n-1})}{(1-q)(1-q^2)\cdots(1+q^n)}\;z^n$$

which, in view of the definition 1.23 is another way of writing 1.24.

Remark 1.3

Note that setting $q=1$ in 1.23 reduces it to the binomial coefficient $\binom{n}{k}$. This is because dividing the numerator and denominator by $(1-q)^n$ it may be rewritten in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} \;=\; \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad 1.25$$

where for any integer m we set

$$[m]_q! \;=\; [m]_q[m-1]_q\cdots[2]_q[1]_q \qquad 1.26$$

with

$$[m]_q = \frac{1-q^k}{1-q} \;=\; 1+q+q^2+\cdots+q^{m-1} \qquad 1.27$$

We shall refer to 1.26 as the "*q-analogue of n!*", to 1.27 as the "*q-analogue of n*" and to 1.23 as the "*q-binomial coefficients*". We should infer from the few examples we have seen that a q-analogue of a classical identity is none other than an identity which contains powers of q everywhere which, nevertheless reduces to the classical one when $q \to 1$. In particular we see that 1.22 and 1.24 are respectively q-analogues of

$$(1+x)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \text{and} \qquad \frac{1}{(1-x)^k} \;=\; \sum_{k=0}^{n} \binom{n+k-1}{k-1} x^n$$

A beautiful corollary of Theorem 1.4 is the following combinatorial result

Theorem 1.5

The q-binomial coefficient

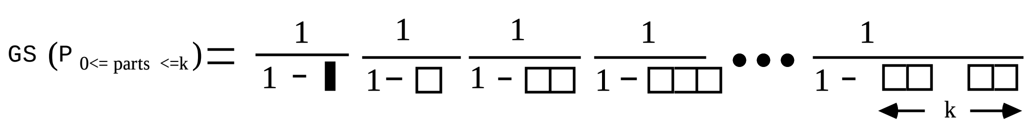
$$\begin{bmatrix} n+k \\ k \end{bmatrix}$$

gives w_1-generating function of partitions contained in a $n \times k$ *rectangle. More precisely*

$$\sum_{\lambda \text{ has } \leq n \text{ parts } \leq k} q^{|\lambda|} \;=\; \begin{bmatrix} n+k \\ k \end{bmatrix} \qquad 1.28$$

Proof

We can view a partition with $\leq n$ parts as a partition with n-parts some of which may be of zero length. If all the parts must be $\leq k$ then the generating series for these partitions may be depicted as



where the symbol " ▮ " is to represent a zero part. Thus we see that

$$GF\big(\mathcal{P}_{0\leq parts\leq k}\big) \;=\; \frac{1}{(1-z)(1-zq)\cdots(1-zq^k)} \;. \qquad 1.29$$

Using 1.24 with k replaced by $k+1$ gives

$$GF\big(\mathcal{P}_{0\leq parts\leq k}\big) \;=\; 1+\sum_{n\geq 1} \begin{bmatrix} n+k \\ k \end{bmatrix} z^n \;. \qquad 1.30$$

Since

$$GF\big(\{\lambda\in\mathcal{P} \text{ with } \leq n \text{ parts } \leq k\}\big) \;=\; GF\big(\mathcal{P}_{0\leq parts\leq k}\big)\Big|_{z^n}$$

1.28 immediately follows by taking the coefficient of z^n in 1.30.

We terminate this section with the famed and most useful "*Jacobi Triple Product Identity*".

Theorem 1.5

$$\prod_{n\geq 1} (1-q^{2n})\ (1+q^{2n-1}z)\ (1+q^{2n-1}/z) \;=\; \sum_{n=-\infty}^{+\infty} q^{n^2}\ z^n \tag{1.31}$$

Proof

It is not difficult to predict 1.30 by computer experimentation, once one is capricious enough to concoct the left-hand side. Analytically, we may discover it as follows. Let us set

$$G(z) \;=\; \prod_{n\geq 1}(1+q^{2n-1}z) \;=\; (1+qz)(1+q^3z)(1+q^5z)\cdots$$

and

$$\Gamma(z) \;=\; G(z)\,G(1/z)$$

Now clearly we have

$$G(z) \;=\; (1+qz)\,G(q^2z) \qquad \text{and} \qquad G(1/q^2z) \;=\; (1+1/qz)\,G(1/z)\ .$$

Thus it follows that

$$\Gamma(z) \;=\; (1+qz)G(q^2z)\,\frac{1}{(1+1/qz)}G(1/q^2z) \;=\; qz\ \Gamma(q^2z)$$

Now, if (as it must be) we have

$$\Gamma(z) \;=\; \sum_{n=-\infty}^{+\infty} c_n(q)\,z^n$$

then the coefficients $c_n(q)$ must satisfy the recursion

$$c_n(q) \;=\; q^{2n-1}\,c_{n-1}(q)$$

So it remains to construct one of these coefficients to recursively compute them all. In particular we see that for $n\geq 0$ we must have

$$c_n(q) \;=\; c_o(q)\ q^{2n-1\,+\,2n-2\,+\,\cdots\,+\,3\,+\,1} \;=\; c_o(q)\ q^{n^2}\ .$$

We leave it to the reader to show that

$$c_o(q) \;=\; \prod_{n\geq 1}\frac{1}{(1-q^{2n})}\ , \tag{1.32}$$

for we prefer to give a completely combinatorial proof of this remarkable identity. To this end it will be convenient to rewrite it in the form

$$\prod_{n\geq 1}(1+q^{n-1}z)(1+q^n/z) \;=\; \sum_{n=-\infty}^{+\infty}\frac{q^{\binom{n}{2}}}{\prod_{r\geq 1}(1-q^r)}\ z^n \tag{1.33}$$

which is easily seen to be what 1.31 becomes if we make the substitutions $q\to q^{1/2}$ and $z\to z/q^{1/2}$.

Now rewriting the left-hand side of 1.33 by means of the expansions

$$\prod_{n\geq 1}(1+q^{n-1}z) = \sum_{k\geq 0}\frac{q^{\binom{k}{2}}z^k}{(1-q)(1-q^2)\cdots(1-q^k)}\ , \qquad \prod_{n\geq 1}(1+q^n/z) = \sum_{h\geq 0}\frac{q^{\binom{h}{2}+h}z^{-h}}{(1-q)(1-q^2)\cdots(1-q^h)}$$

formula 1.33 becomes

$$\Big(\sum_{k\geq 0} \frac{q^{\binom{k}{2}} z^k}{(1-q)(1-q^2)\cdots(1-q^k)}\Big)\Big(\sum_{h\geq 0} \frac{q^{\binom{h+1}{2}} z^{-h}}{(1-q)(1-q^2)\cdots(1-q^h)}\Big) = \sum_{n=-\infty}^{+\infty} \frac{q^{\binom{n}{2}}}{\prod_{r\geq 1}(1-q^r)} \; z^n \; .$$

Equating coefficients of z^n we see that we must have

a) For $n \geq 0$:

$$\sum_{k-h=n} \Big(\frac{q^{\binom{k}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \quad \frac{q^{\binom{h+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^h)} = \frac{q^{\binom{n}{2}}}{\prod_{r\geq 1}(1-q^r)} \Big) \tag{1.34}$$

b) For $n \to -n < 0$:

$$\sum_{k-h=-n} \Big(\frac{q^{\binom{k}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \quad \frac{q^{\binom{h+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^h)} = \frac{q^{\binom{n+1}{2}}}{\prod_{r\geq 1}(1-q^r)} \Big) \tag{1.35}$$

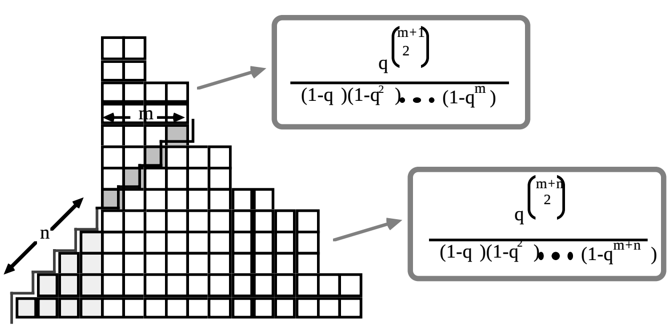
Now making the replacements $h \to m$ and $k \to m+n$ 1.34 becomes

$$\sum_{m\geq 0} \Big(\frac{q^{\binom{m+n}{2}}}{(1-q)(1-q^2)\cdots(1-q^{m+n})} \quad \frac{q^{\binom{m+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)} \Big) = \frac{q^{\binom{n}{2}}}{\prod_{r\geq 1}(1-q^r)} \; . \tag{1.36}$$

Similarly, with $k \to m$ and $h \to m+n$ 1.35 becomes

$$\sum_{m\geq 0} \Big(\frac{q^{\binom{m}{2}}}{(1-q)(1-q^2)\cdots(1-q^m)} \quad \frac{q^{\binom{m+n+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^{m+n})} \Big) = \frac{q^{\binom{n+1}{2}}}{\prod_{r\geq 1}(1-q^r)} \; . \tag{1.37}$$

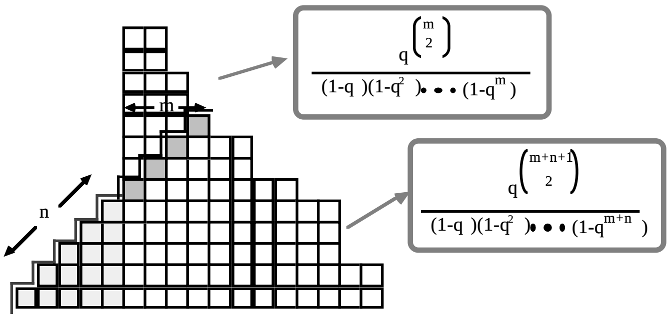
Thus to prove 1.33 we need only verify that 1.36 and 1.37 hold true for every $n \geq 0$ and every $n \geq 1$ respectively. We shall do this by giving a combinatorial inerpretation of the left hand sides and show that the objects they represent can be reassembled into the objects represented by the right-hand side. We shall begin with the left hand side of 1.36. Note that it may be interpreted as q-enumerating by area (of the corresponding Ferrers diagram) the collection of pairs consisting of a partition with $m+n$ distinct "≥ 0 " parts and a partition with m distinct parts. On the other hand the right hand-side may be interpreted as q-enumerating by area the collection of pairs consisting of an $n-1$ staircase and an arbitrary partition. Thus the equality may be shown by simply verifying that the former pairs may be combined to produce the latter pairs. We can immediately see how this can be done from the following self-explanatory picture which illustrate the process in a particular case.



Here we have prepended to the Ferrers diagram of the partition $(12, 12, 10, 10, 10, 8, 6, 6, 4, 4, 4, 2, 2)$ the 4-staircase (in lightly shaded squares). We then indicate how the composite figure is to be cut to produce a partition α with $m = 4$ distinct parts ≥ 1 and a partition β with $m+n$ distinct parts ≥ 0. We must read vertically the parts of α and horizontally the parts of β obtaining

$$\alpha = (8, 7, 4, 3) \quad , \qquad\qquad \beta = (16, 15, 12, 11, 10, 7, 4, 3, 0)$$

The identity in 1.37 is established in a similar manner as illustrated in the figure below

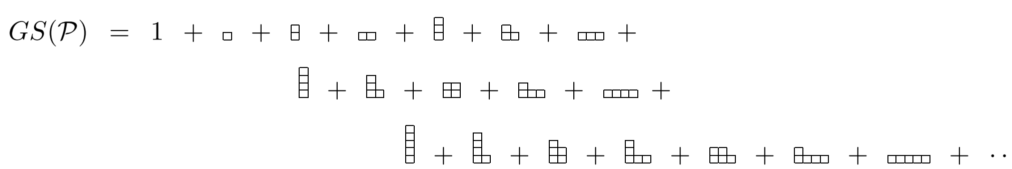


as we see, in this case we prepend an arbitrary partition with the n-staircase but now after we perform the cut, the partition α whose distorted diagram is above the cut has $m \geq 0$ parts a and the partition β obtained by left justifying the remaining squares has necessarily $m + n$ parts ≥ 1, which is precisely as required for 1.37 to hold true.

This completes our proof.

2. Two Theorems of Euler

We may recursively construct all Ferrers diagrams with n cells by constructing for each $k \geq 1$ all Ferrers diagrams with $n - k$ cells and no more than k rows then prepending each of them with a column of length k. Proceeding in this manner we obtain the first 19 terms of the generating series of $\mathcal{P}$ in the following order



Where again the symbol "1" is to represent the empty partition.

Passing to the generating function with weight $w_1(\lambda) = q^{|\lambda|}$ we deduce from this that

$$GF_1(\mathcal{P}) \;=\; \prod_{n\geq 1} \frac{1}{(1-q^n)} \;=\; 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots \tag{2.1}$$

More generally, we shall have

$$\prod_{n\geq 1} \frac{1}{(1-q^n)} \;=\; \sum_{n\geq 0} p(n)\, q^n \tag{2.2}$$

with $p(n)$ denoting the number of partitions of n. We have seen that

$$p(0) = 1\ ,\ p(1) = 1\ ,\ p(2) = 2\ ,\ p(3) = 3\ ,\ p(4) = 5\ ,\ p(5) = 7\ . \tag{2.3}$$

If we do a little more work and put together all diagrams with 6 and 7 cells we obtain that

$$p(6) = 11 \quad \text{and} \quad p(7) = 15\ . \tag{2.4}$$

There is a beautiful algorithm discovered by Euler which enables us to successively compute the coefficients $p(n)$ without constructing the diagrams. It will be good to see how we may be led to discover the Euler result before stating it. The idea is to note that if we set

$$\prod_{n\geq 1}(1-q^n) \;=\; 1-c_1q-c_2q^2-c_3q^3-c_4q^4-\cdots \;=\; 1 \;-\; \sum_{k\geq 1} c_k\, q^k \;, \tag{2.5}$$

then equating coefficients of q^n in both sides of the equation

$$1=\Big(\prod_{n\geq 1}(1-q^n)\Big)\Big(\prod_{n\geq 1}\frac{1}{(1-q^n)}\Big) \;=\; \Big(1-\sum_{k\geq 1} c_k\, q^k\Big)\Big(\sum_{n\geq 0} p(n)\, q^n\Big)$$

we obtain the recursion

$$p(n) \;=\; c_1\, p(n-1) \;+\; c_2\, p(n-2) \;+\; c_3\, p(n-3) \;+\; \cdots \;+\; c_{n-1}\, p(1) \;+\; c_n\, p(0)\;. \tag{2.6}$$

In particular, we derive from 2.6 that

$$\begin{aligned}
p(1) &= c_1\\
p(2) &= c_1\, p(1)+c_2\\
p(3) &= c_1\, p(2)+c_2\, p(1)+c_3\\
p(4) &= c_1\, p(3)+c_2\, p(2)+c_3\, p(1)+c_4\\
p(5) &= c_1\, p(4)+c_2\, p(3)+c_3\, p(2)+c_4\, p(1)+c_5\\
p(6) &= c_1\, p(5)+c_2\, p(4)+c_3\, p(3)+c_4\, p(2)+c_5\, p(1)+c_6\\
p(7) &= c_1\, p(6)+c_2\, p(5)+c_3\, p(4)+c_4\, p(3)+c_5\, p(2)+c_6\, p(1)+c_7
\end{aligned}$$

and substituting our findings in 2.3 and 2.4 we get

$$\begin{aligned}
1 &= c_1\\
2 &= c_1+c_2 \qquad\Longrightarrow\qquad c_2=1\\
3 &= 2\,c_1+c_2+c_3 \qquad\Longrightarrow\qquad c_3=0\\
5 &= 3\,c_1+2\,c_2+c_3+c_4 \qquad\Longrightarrow\qquad c_4=0\\
7 &= 5\,c_1+3\,c_2+2\,c_3+c_4+c_5 \qquad\Longrightarrow\qquad c_5=-1\\
11 &= 7\,c_1+5\,c_2+3\,c_3+2\,c_4+c_5+c_6 \qquad\Longrightarrow\qquad c_6=0\\
15 &= 11\,c_1+7\,c_2+5\,c_3+3\,c_4+2\,c_5+c_6+c_7 \qquad\Longrightarrow\qquad c_7=-1
\end{aligned}$$

Thus using these values in 2.5 gives

$$\prod_{n\geq 1}(1-q^n) \;=\; 1-q-q^2+q^5+q^7 \;+\; \cdots$$

Alternatively, we may use MATHEMATICA and ask for the first 40 terms of the Taylor expansion of this product and obtain that

$$\prod_{n\geq 1}(1-q^n) \;=\; 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+\cdots \tag{2.7}$$

A close inspection of the powers of q appearing in this sum might reveal that they are all of the form

$$1+2+\cdots+n+n^2 \;=\; \frac{3}{2}n^2+\frac{1}{2}n \qquad\text{or}\qquad 1+2+\cdots+n-1+n^2 \;=\; \frac{3}{2}n^2-\frac{1}{2}n$$

with the corresponding sign being $(-1)^n$. Having made all these observations we would be led to the statement of the "*Euler Pentagonal Number Theorem*", namely

Theorem 2.1

$$\prod_{n\geq 1}(1-q^n) \;=\; \sum_{n=-\infty}^{+\infty}(-1)^n\, q^{\frac{3}{2}n^2+\frac{1}{2}n} \tag{2.8}$$

Proof

Now it develops that this is an immediate consequence of the Jacobi triple product identity. In fact, making the replacements $q \to q^{3/2}$ and $z \to -q^{1/2}$ in 1.31 we obtain

$$\prod_{n\geq 1}(1-q^{3n})(1-q^{3n-3/2}q^{1/2})(1-q^{3n-3/2}q^{-1/2}) \;=\; \sum_{n=-\infty}^{+\infty}(-1)^n\, q^{3n^2/2}q^{n/2}$$

or better

$$\prod_{n\geq 1}(1-q^{3n})(1-q^{3n-1})(1-q^{3n-2}) \;=\; \sum_{n=-\infty}^{+\infty}(-1)^n\, q^{3n^2/2+n/2}$$

which is easily seen to be just another way of writing 2.8.

Although this proof of 2.8 may be combinatorialized by modifying the manipulations that led to a combinatorial proof of the Jacobi Triple Product identity, this approach does not reveal the combinatorial mechanism that produces such a beautiful identity. Now there is a remarkable combinatorial argument, given by F. Franklin in 1881, which makes 2.8 almost self evident reducing it to two pictures that are difficult to forget. Before we present this argument it may be good to make a few preliminary observations.

Note first that we may write

$$GS\,(DP_{\text{signed by \# parts}}) = (1-\square)(1-\square\square)(1-\square\square\square)(1-\square\square\square\square)\bullet\bullet\bullet$$

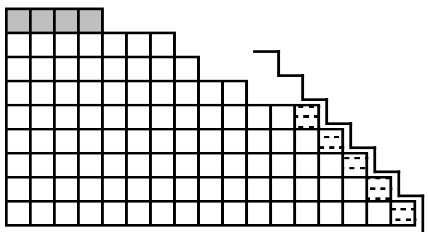
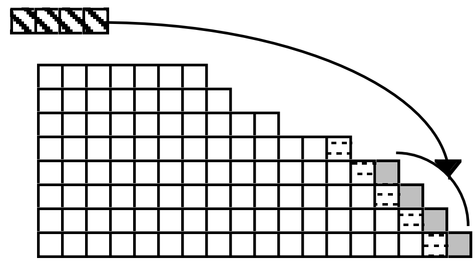
where by $\mathcal{DP}_{\text{signed by \#parts}}$ we mean the collection of partitions with distinct parts preceded by the sign "+" or "−" according as the number of parts is even or odd. Thus, combinatorially, 2.7 may be viewed as the result of a massive cancellation process where each power q^n missing in 2.7 is caused by the fact that for that particular n the number of partition of n with an even number of distinct parts is equal to the number of of partition of n with an odd number of distinct parts. Infact note that by means of MATHEMATICA we find that

$$\begin{aligned}\prod_{n\geq 1}(1+q^n) \;=\;& 1+q+q^2+2\,q^3+2\,q^4+3\,q^5+4\,q^6+5\,q^7+\\ & 6\,q^8+8\,q^9+10\,q^{10}+12\,q^{11}+15\,q^{12}+18\,q^{13}+\\ & 22\,q^{14}+27\,q^{15}+32\,q^{16}+38\,q^{17}+46\,q^{18}+54\,q^{19}+\\ & 64\,q^{20}+76\,q^{21}+89\,q^{22}+104\,q^{23}+122\,q^{24}+142\,q^{25}+\\ & 165\,q^{26}+192\,q^{27}+222\,q^{28}+256\,q^{29}+296\,q^{30}+\\ & 340\,q^{31}+390\,q^{32}+448\,q^{33}+512\,q^{34}+585\,q^{35}+\\ & 668\,q^{36}+760\,q^{37}+864\,q^{38}+982\,q^{39}+1113\,q^{40}+\cdots\end{aligned} \tag{2.9}$$

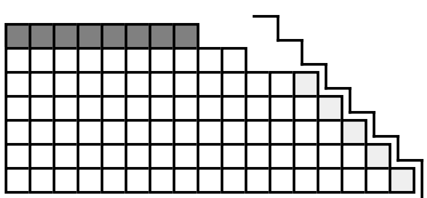
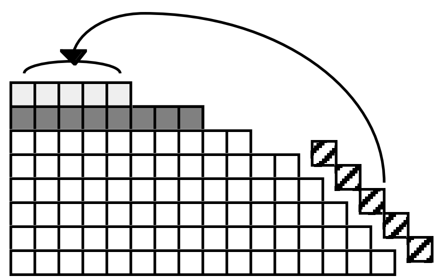
and we see that, since there is no term with q^{39} in 2.7, we must conclude from 2.9 that there must be 491 partitions of 39 with an even number of distinct parts and an equal number having an odd number of parts. In the same vein comparing 2.7 and 2.9 we see that there must be 556 partitions of 40 with an even number of distinct parts versus 557 with an odd number of parts, yielding a difference of -1. Denoting respectively by $\mathcal{EDP}_n$ and $\mathcal{ODP}_n$ the collections of partitions of n with an even and with an odd number of distinct parts, we may then write

$$\prod_{n\geq 1}(1-q^n) \;=\; 1+\sum_{n\geq 1}\big(\,\#\mathcal{EDP}_n \;-\; \#\mathcal{ODP}_n\,\big)\,q^n$$

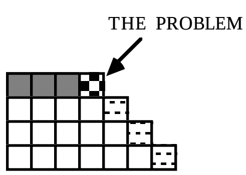
Franklin's idea consists in constructing for each $m \geq 1$ a pairing of each element of $\mathcal{EDP}_m$ with an element of $\mathcal{ODP}_m$ in the cases where the term in q^m is missing in 2.8 and a similar pairing which misses only one element of $\mathcal{EDP}_n$ when q^m is present and misses only one element of $\mathcal{ODP}_m$ when $-q^m$ is present. We shall illustrate the construction in two particular cases. In the first step we attach a "*staircase*" to the right of the Ferrers diagram of the given partition π, as indicated below and lightly shade the cells that are in contact with the staircase. Denote the number of these cells by $s(\pi)$. Then darkly shade the cells of the top row, and denote the number of them by $t(\pi)$. Now if $s(\pi) \leq t(p)$, as is the case here, we simply move the darkly shaded cells from the top row to the ends of the $t(p)$ bottom rows, thus obtaining a new partition π^* with one less row but the same total number of cells..

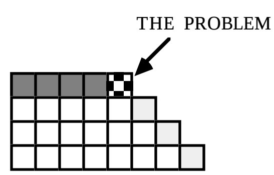
$\prod =$  $\Longrightarrow$ $\prod^* =$ 

On the other hand, if $s(\pi) < t(\pi)$ as in our next case, then we move the lightly shaded cells to form a new top row of length $s(\pi)$ and thereby obtain a new partition π^* with the same number of cells but one more row.

$\prod =$  $\Longrightarrow$ $\prod^* =$ 

We can easily see that the map $\pi \rightarrow \pi^*$ is well defined except in the two cases illustrated below.

$\prod_1 =$ 

$\prod_2 =$ 

2.10

It may happen that the highest lightly shaded cell falls right on top of the rightmost cell of the first row and we have $s(\pi) = t(\pi)$ as for the partition π_1 or $s(\pi) = t(\pi) - 1$ as for the partition π_2. In first case we would like to move the darkly shaded cells but once we do that we lose one of the rows we were supposed to lengthen by a cell. In the second case we would like to move the lightly shaded cells but once we do that the first row becomes of length $s(\pi_2)$ and the $s(\pi_2)$ lightly shaded cells we would place on top of it would create a partition with two equal rows, thereby placing us out of the collection of objects we are working with. It will be convenient to extend the definition of the Franklin bijection also to these problem cases by simply setting $\pi^* = \pi$ in either case. This given we see that the map $\pi \rightarrow \pi^*$ has the following three basic properties:

(1) It is an involution of $\mathcal{DP}$ onto itself (that is $(\pi^*)^* = \pi$)

(2) It preserves the number of cells

(3) It changes the parity of the number of parts.

The conclusion is that in passing from 2.8 to 2.7 the pairs (π, π^*) with $\pi \neq \pi^*$ cancel each other out and the only surviving terms are those corresponding to the fixed points of the involution. However, in the case of π_1 in 2.10 the total number of cells is of the form

$$n^2 + n - 1 + n - 2 + \cdots + 2 + 1 = n^2 + \binom{n}{2}$$

and in the case of π_2 the total number of cells is of the form

$$n^2 + n + n - 1 + n - 2 + \cdots + 2 + 1 = n^2 + \binom{n+1}{2}$$

where n gives in either case the number of parts. Thus the contribution to 2.7 of such a pair is the term

$$(-1)^n \left(q^{(3n^2-n)/2} \ + \ q^{(3n^2+n)/2} \right)$$

completing Franklin's proof of 2.8.

The next result of Euler we shall present here may be considered as the starting point of the study of partitions with "*Restricted Parts*". Although this is a very extensive chapter of the Theory of Partitions, we shall have to limit ourselves here to the definitions and some of the most salient results. To begin with, for any given finite or infinite subset S of the natural numbers we let $\mathcal{P}_S$ denote the collection of partitions all of whose parts are in S. In symbols:

$$\mathcal{P}_S \ = \ \{ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in \mathcal{P} \ : \ \lambda_i \in S \ \forall \ i \} \ .$$

Clearly we have

$$GF_1(\mathcal{P}_S) \ = \ \prod_{i \in S} \frac{1}{(1-q^i)}$$

Denoting by $\mathcal{O}$ the set of all odd integers we have the following beautiful result of Euler.

Theorem 2.2 (Euler 1748)

$$GF_1(\mathcal{DP}) \ = \ GF_1(\mathcal{P}_{\mathcal{O}}) \qquad 2.11$$

In words, there are as many partitions of n with distinct parts as there are with odd parts.

Proof

The manipulatorial proof of 2.11 is immediate. In fact we have

$$GF_1(\mathcal{DP}) \ = \ \prod_{n\geq 1} (1+q^n) \ = \ \prod_{n \geq 1} \frac{(1-q^{2n})}{(1-q^n)} \ = \ \prod_{n\geq 1} \frac{1}{(1-q^{2n-1})} \ = \ GF_1(\mathcal{P}_{\mathcal{O}}) \ . \qquad 2.12$$

QED

This slick argument may prove the result, nevertheless it leaves us wondering if there is a better way of seeing what is going on. Combinatorially the natural thing to do in proving that two sets of objects are equinumerous is to give a bijection between the two sets. Ideally we should provide a recipe for disassembling the object of one set into components that may be reassembled into an object of the other set. I.E. what we may call a "*transformer toy*" proof. Well this is precisely what was provided in 1883 by J.W. L. Glaisher.

He gave the following three step recipe for transforming a partition π with odd parts into partition π^* with distinct parts:

(1) Start by writing π in the form:

$$\pi \;=\; 1^{m_1}3^{m_3}5^{m_5}7^{m_7}\cdots$$

where m_{2n-1} denotes the multiplicity of $2n-1$ in π

(2) Construct the binary expansion of each of these multiplicities: That is set, for each $n \geq 1$

$$m_{2n-1} \;=\; \epsilon_o^{(n)} \;+\; \epsilon_1^{(n)}2 \;+\; \epsilon_2^{(n)}2^2 \;+\; \epsilon_3^{(n)}2^3 \;+\; \epsilon_4^{(n)}2^4 \;+\; \cdots \qquad (\text{with } \epsilon_i^{(n)} = 0,1\ )$$

(3) Finally, for each i with $\epsilon_i^{(n)} = 1$ line-up 2^i rows of length $2n-1$ of the diagram of π into a single row of length $2^i\ (2n-1)$ and place the resulting row in π^*.

Note that

(a) there is no loss of cells in going from π to π^* and

(b) π^* will necessarily have distinct parts.

We should mention that part b) follows from the fact that each integer m has a unique factorization of the form $m = 2^k(2n-1)$ (a power of two times an odd number). This fact makes it also clear that this construction may be inverted to give a map of $\mathcal{P}_{\mathcal{O}}$ into $\mathcal{DP}$, proving that the Glaisher recipe provides a w_1-preserving bijection between these two families of partitions, thereby establishing Euler's result in a purely combinatorial manner.

Having seen the argument we carried out in 2.12, we should be tempted to emulated it by noting that the expression

$$\prod_{n\geq 1} \frac{(1-q^{3n})}{(1-q^n)}$$

may be rewritten in the two forms given below

$$\prod_{n\geq 1} \frac{(1-q^{3n})}{(1-q^n)} \;=\; \prod_{n\geq 1}(1+q^n+q^{2n}) \qquad \text{and} \qquad \prod_{n\geq 1} \frac{(1-q^{3n})}{(1-q^n)} \;=\; \prod_{n\geq 1} \frac{1}{(1-q^{3n-1})(1-q^{3n-2})} \qquad 2.13$$

leading us to the discovery of the following general result of Glaisher:

Theorem 2.3

There are as many partitions of n with parts repeated at most $p-1$ times as there are with parts not divisible by p.

Proof

Just imitate what we have done in 2.13 (which yields the case $p = 3$) . Alternatively the reader may find it challenging to extend to this more general case Glaisher's "tranformer toy" proof of the Euler Theorem.

There are a number of similar results showing that q-series giving the w_1-generating functions of certain families of partitions may be re-expressed as infinite products giving the w_1-generating functions of partitions with restricted parts. These are usually referred to as "series-product" identities. The reader may find it amusing to give manipulatorial or combinatorial proof of the statements given below:

A) The number of partitions of n in which only odd parts may be repeated is equal to the number of partitions of n in which no part appears more than three times.

B) The number of self-conjugate partitions of n is equal to the number of partitions of n with distinct odd parts.

C) There are as many partition of n with parts greater than 1 and no consecutive parts as there are with no part appearing exactly once.

D) The absolute value of the difference between the number of partitions of n with an odd number of parts and the number of partitions of n with an even number of parts equals the number of partitions of n

with distinct odd parts.

Having seen a number of series-product identities yieding surprising results such as the ones stated above we may be tempted, each times we have a q-series

$$F(q) = 1 + c_1 q + c_2 q^2 + \cdots + c_k q^k + \cdots$$

to try and find out if there is a subset S of the positive integers giving

$$1 + c_1 q + c_2 q^2 + \cdots + c_k q^k + \cdots \;=\; \prod_{i \in S} \frac{1}{(1-q^i)} \;. \qquad 2.14$$

This given, we should be pleased to know that the knowledge of the coefficients $c_1, c_2, \ldots, c_k, \ldots$ completely determines the set S. In fact, we can formulate an algorithm which recursively constructs S from the sequence c_k. More precisely we have

Proposition 2.1

For a given subset S of the natural numbers and an integer N let $S^{<N}$ denote the subset consisting of the elements of S that are less than N. Furthermore, set

$$GF_1(\mathcal{P}_S) \;=\; 1 + \sum_{n \geq 1} p(S,n)\, q^n \qquad \text{and} \qquad GF_1(\mathcal{P}_{S^{<N}}) \;=\; 1 + \sum_{n \geq 1} p(S^{<N},n)\, q^n$$

then

$$\chi(N \in S) \;=\; p(S,N) \;-\; p(S^{<N},N) \qquad 2.15$$

In words, N is a member of S if and only if there is one more way to get a partition of N with parts in S than there is using parts of S that are strictly less than N.

Proof

This result is immediate. For the factors in 2.14 that contain powers of q greater than N contribute nothing to $p(S,n)$ for any $n \leq N$. Thus

$$\begin{aligned} p(S,N) \;&=\; (1 + q^N \chi(N \in S)) \prod_{i \in S\,;\, i<N} \frac{1}{(1-q^i)} \;\Big|_{q^N} \;= \\ &=\; \prod_{i \in S\,;\, i<N} \frac{1}{(1-q^i)} \;\Big|_{q^N} \;+\; \chi(N \in S) \prod_{i \in S\,;\, i<N} \frac{1}{(1-q^i)} \;\Big|_{q^0} \;= \\ &=\; p(S^{<N},N) \;+\; \chi(N \in S) \end{aligned}$$

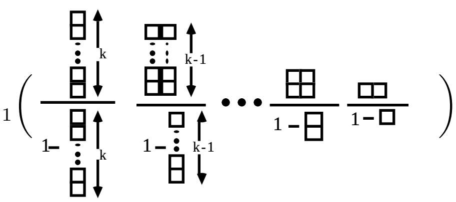
QED

The following two MATHEMATICA procedures are an implementation of the algorithm resulting from the identity in 2.15:

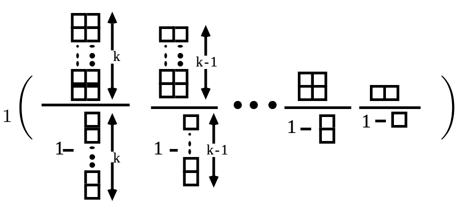
$$parsS[\,S\_\,] := Product[1/(1-q\text{^}S[[i]]), \{i, 1, Length[S]\,\}\,]$$

```
findfs[ EX_, N_ ] :=Block[{coes, tay, te, S, m, cte, ctay},
        (tay = Series[EX, q, 0, N];
        If[Coefficient[tay, 1] == 1, S = {1}, S = {}];
        Do[
            (te = Series[parsS[S], q, 0, m];
            cte = coe[te, m];
            ctay = coe[tay, m];
            If[ctay − cte == 1,  (S = Join[S, {m}]; Print[m good] )  ]
                    ) ,     {m, 1, N}];
        Return[S])]
```

The first question we might pose, after all this, is whether the generating function of partitions with distinct parts differing by two at least may be given a product form. To be precise there are two cases possible depending on whether or not we allow 1 to be a part. In the first case (part 1 allowed) the generating series may be represented by the symbol

$GS(\mathcal{DD}_1) \;=\; 1 \;+\; \sum_{k\geq 1}$ 

and in the second case (part 1 not allowed) we may set

$GS(\mathcal{DD}_2) \;=\; 1 \;+\; \sum_{k\geq 1}$ 

From which we easily derive that

$$GF(\mathcal{DD}_1) \;=\; 1 \;+\; \sum_{k\geq 1} \frac{q^{k+2\binom{k}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \;=\; 1 \;+\; \sum_{k\geq 1} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

and

$$GF(\mathcal{DD}_2) \;=\; 1 \;+\; \sum_{k\geq 1} \frac{q^{2\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)} \;=\; 1 \;+\; \sum_{k\geq 1} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

Now using MATHEMATICA we find that the first few terms of these series are

$$GF(\mathcal{DD}_1) = 1 + q + q^2 + q^3 + 2\,q^4 + 2\,q^5 + 3\,q^6 + 3\,q^7 + 4\,q^8 + 5\,q^9 + 6\,q^{10} + \\ + 10\,q^{13} + 12\,q^{14} 7\,q^{11} + 9\,q^{12} + +14\,q^{15} + 17\,q^{16} + 19\,q^{17} + 23\,q^{18} + 26\,q^{19} + 31\,q^{20} + O(q)^{21}$$

$$GF(\mathcal{DD}_2) = 1 + q^2 + q^3 + q^4 + q^5 + 2\,q^6 + 2\,q^7 + 3\,q^8 + 3\,q^9 + 4\,q^{10} + 4\,q^{11} + 6\,q^{12} + \\ + 6\,q^{13} + 8\,q^{14} + 9\,q^{15} + 11\,q^{16} + 12\,q^{17} + 15\,q^{18} + 16\,q^{19} + 20\,q^{20} + 22\,q^{21} + 26\,q^{22} + O(q)^{23}$$

Processing this information by means of the MATHEMATICA procedures given above produces the following initial elements for the sets S_1 and S_2 which might give infinite product expressions for the series $GF(\mathcal{DD}_1)$

and $GF(\mathcal{DD}_2)$, namely:

$$S_1 \;=\; \{1,4,6,9,11,14,16,19,\ldots\} \qquad \text{and} \qquad S_2 \;=\; \{2,3,7,8,12,13,17,18,22,,\ldots\}$$

We might thus be led to conjecture that

A) The number of partitions of n with distinct parts differing at least by two is equal to the number of partitions of n all of whose parts are congruent to 1 or 4 modulo 5.

B) The number of partitions of n with distinct parts differing at least by two and smallest part > 1 is equal to the number of partitions of n all of whose parts congruent to 2 or 3 modulo 5.

As you may suspect this is in fact true and it amounts to the so-called "*Rogers-Ramanujan identities*":
Theorem 2.5 (Rogers 1893, Schur 1916)

$$1 \;+\; \sum_{k\geq 1} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \;=\; \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{2.16}$$

$$1 \;+\; \sum_{k\geq 1} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \;=\; \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{2.17}$$

We are not yet ready to prove these identities since they are of several orders of magnitude deeper than anything we have shown so far. However it should be clear, from what we have shown, how somebody familiar with the XIX century results in partition theory, and skilful with the manipulations necessary to compute coefficients in Taylor expansions, might be led to conjecture them. So it should not come as a surprise that a researcher with the talent of Ramanujan could come up with them. To prove them is another matter. In the next section we shall give a proof which is a slight modification of one of Rogers proofs. Both Rogers and Schur make use of the Jacobi Triple Product identity in one of the steps. To see how this comes about, we should take a close look at our manipulatorial proof of Euler's pentagonal number theorem. There, we were able to get from 1.31 the three congruence classes (mod 3) of factors $(1-q^{3n})$, $(1-q^{3n-1})$, $(1-q^{3n-2})$ by making the substitutions $q \to q^{3/2}$ and $z \to -q^{1/2}$. It should not be difficult to guess, after examining the mechanism that produced these factors, that we could obtain in the same manner the power series expansion of the two products:

$$A = \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3}) \quad \text{and} \qquad B = \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4})$$

In fact, we can obtain A from the left hand side of 1.31 by the replacements $q \to q^{5/2}$ and $z \to -q^{1/2}$ and B by the replacements $q \to q^{5/2}$ and $z \to -q^{3/2}$, obtaining the expansions

$$\begin{aligned} &a) \qquad \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3}) \;=\; \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{(5n^2-n)/2}\;, \\ &b) \qquad \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4}) \;=\; \sum_{n=-\infty}^{+\infty} (-1)^n\, q^{(5n^2-3n)/2}\;. \end{aligned} \tag{2.18}$$

This given, the basic step consists in multiplying both sides of 2.16 and 2.17 by $\prod_{n\geq 1}(1-q^n)$ obtaining A and B as right hand sides:

$$\Big(\prod_{n\geq 1}(1-q^n)\Big)\Big(1 \;+\; \sum_{k\geq 1}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \;=\; \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3})$$

$$\Big(\prod_{n\geq 1}(1-q^n)\Big)\Big(1 \ + \ \sum_{k\geq 1} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \ = \ \prod_{n\geq 1}(1-q^{5n})(1-q^{5n-1})(1-q^{5n-4})$$

and then using 2.18 a) and b) to rewrite 2.16 and 2.17 in the form

$$a) \qquad \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{(5n^2-n)/2} \ = \ \Big(\prod_{n\geq 1}(1-q^n)\Big)\Big(1+\sum_{k\geq 1} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \ ,$$

2.19

$$b) \qquad \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{(5n^2-3n)/2} \ = \ \Big(\prod_{n\geq 1}(1-q^n)\Big)\Big(1+\sum_{k\geq 1} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \ .$$

In summary, by means of the triple product identity we have reduced Theorem 2.5 to to proving these two identities. We shall see in the next section that both 2.19 a) and b) are immediate consequences of a single 1893 identity of Rogers.

3. The Rogers identity

This section is dedicated to the proof of the following identity

Theorem 3.1 (Rogers 1893)

$$\sum_{n\geq 0} \frac{(-1)^n \, z^{2n} \, q^{(5n-1)n/2}(1-zq^{2n})(1-z)(1-zq)\cdots(1-zq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \ = \qquad 3.1$$

$$= \ \Big(\prod_{m\geq 0}(1-zq^m)\Big)\Big(\sum_{k\geq 0} \frac{z^k q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big)$$

Before we proceed with our arguments it will be good to see that

Proposition 3.1

The identities in 2.19 a) and b) are essentially special evaluations of 3.1. More precisely, 2.19 b) is simply 3.1 for $z=q$ while 2.19 a) is obtained by dividing both sides of 3.1 by $(1-z)$ and letting $z\to 1$.

Proof

Note that the left-hand side of 3.1 can be written in the form

$$(1-z) \ + \ \sum_{n\geq 1} \frac{(-1)^n \, z^{2n} \, q^{(5n-1)n/2}(1-zq^{2n})(1-z)(1-zq)\cdots(1-zq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}$$

Thus we may divide both sides of 3.1 by $(1-z)$ and obtain

$$1 \ + \ \sum_{n\geq 1} \frac{(-1)^n \, z^{2n} \, q^{(5n-1)n/2}(1-zq^{2n})(1-zq)\cdots(1-zq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \ =$$

$$= \ \Big(\prod_{m\geq 1}(1-zq^m)\Big)\Big(\sum_{k\geq 0} \frac{z^k q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \ .$$

Interpreting this as an identity relating two formal power series in q with coefficients polynomials in z we can let $z=1$ and get

$$1 \ + \ \sum_{n\geq 1} (-1)^n \ q^{(5n-1)n/2} \frac{(1-q^{2n})}{(1-q^n)} \ = \ \Big(\prod_{m\geq 1}(1-q^m)\Big)\Big(\sum_{k\geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}\Big) \ . \qquad 3.2$$

Now the left-hand side of 3.2 may be rewritten as

$$
\begin{aligned}
1 \ + \ \sum_{n\geq 1} (-1)^n \ q^{(5n-1)n/2} \frac{(1-q^{2n})}{(1-q^n)} \ &= \ \sum_{n\geq 0} (-1)^n \ q^{(5n-1)n/2} \ + \ \sum_{n\geq 1} (-1)^n \ q^{(5n-1)n/2+n} \\
&= \ \sum_{n\geq 0} (-1)^n \ q^{(5n-1)n/2} \ + \ \sum_{n\geq 1} (-1)^n \ q^{(5n+1)n/2} \\
&= \ \sum_{n=\infty}^{+\infty} (-1)^n \ q^{(5n-1)n/2} \ .
\end{aligned}
$$

Substituting this in 3.2 gives 2.19 a) as desired.

Note next that setting $z = q$ in 3.1 gives

$$
\sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2}(1-q^{2n+1}) \ = \ \Big(\prod_{m\geq 1} (1-q^m) \Big) \Big(\sum_{k\geq 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \Big) \ . \qquad 3.3
$$

Expanding the left-hand side of 3.3 gives

$$
\sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2}(1-q^{2n+1}) \ = \ \sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2} \ - \ \sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2+2n+1} \qquad 3.4
$$

However the simple identity

$$
(5n+3)n/2 + 2n + 1 \ = \ (5(n+1)-3)(n+1)/2
$$

allows us to rewrite 3.4 as

$$
\begin{aligned}
\sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2}(1-q^{2n+1}) \ &= \ \sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2} \ + \ \sum_{n\geq 0} (-1)^{n+1} \, q^{(5(n+1)-3)(n+1)/2} \\
&= \ \sum_{n\geq 0} (-1)^n \, q^{(5n+3)n/2} \ + \ \sum_{n\geq 1} (-1)^n \, q^{(5n-3)n/2} \\
&= \ \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{(5n+3)n/2} \ = \ \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{(5n-3)n/2}
\end{aligned}
$$

Substituting this in 3.3 gives 2.19 b) completing our proof.

The Rogers identity may be proved by a relatively short sequence of ad hoc manipulations see for instance Andrews [] (1976). We prefer to derive it here by means of a package of tools that may also be used to unravel a wide variety of q-series identities. We shall therefore postpone the proof of 3.1 until after we have developed a bit of theory.

Our main ingredients will be formal power series

$$
A(z) \ = \ \sum_{n\geq 0} A_n(q) \, z^n
$$

whose coefficients $A_n(q)$ are rational functions of q. It will be convenient to denote the vector space of all such series by $\mathcal{RAT}(q)$. Note that the operation

$$
\Downarrow A(z) \ = \ \sum_{n\geq 0} A_n(1/q) \, z^n
$$

is well defined in $\mathcal{RAT}(q)$. We shall refer to it as the *basic involution*.

We next introduce two operations acting on $\mathcal{RAT}(q)$ which we call *roofing* and *unroofing* by setting

$$^{\wedge}A \;=\; \sum_{n\geq 0} A_n(q)\, q^{-\binom{n}{2}}\, z^n \qquad \text{and} \qquad ^{\vee}A \;=\; \sum_{n\geq 0} A_n(q)\, q^{\binom{n}{2}}\, z^n \;.$$

It often happens in the theory of q-series that we have expressions of the form

$$\sum_{n\geq 0} A_n(q)\, z^n\, B(zq^n)$$

with $A(z)$ and $B(z)$ in $\mathcal{RAT}(q)$. We shall call such an expression the "*q-tangle product* " of A and B and denote it by $A \odot_q B$. Note this operation is completely symmetric in A and B. In fact if

$$B(z) = \sum_{m\geq 0} B_m(q)\, z^m$$

then we see that

$$\big(A \odot_q B\big)(z) \;=\; \sum_{n\geq 0}\sum_{m\geq 0} A_n(q) B_m(q)\, q^{nm}\, z^{n+m} \;=\; \big(B \odot_q A\big)(z) \;. \tag{3.5}$$

It is also convenient to set

$$\big(A \odot_{q^{-1}} B\big)(z) = \sum_{n\geq 0} A_n(q)\, z^n\, B(zq^{-n}) = \sum_{m\geq 0} A_n(q) B_m(q)\, q^{-nm}\, z^{n+m} \;=\big(B \odot_{q^{-1}} A\big)(z) \tag{3.6}$$

and call it the "*1/q-tangle product* ". Now it develops that roofing and unroofing "*untangle* " these two products into ordinary products. More precisely we have

Proposition 3.2

$$^{\wedge}\big(A \odot_q B\big) \;=\; {}^{\wedge}A \;\times\; {}^{\wedge}B \tag{3.7}$$

$$^{\vee}\big(A \odot_{q^{-1}} B\big) \;=\; {}^{\vee}A \;\times\; {}^{\vee}B \tag{3.8}$$

Proof

From 3.5 we get that

$$^{\wedge}\big(A \odot_q B\big) \;=\; \sum_{n\geq 0}\sum_{m\geq 0} A_n(q) B_m(q)\, q^{nm-\binom{n+m}{2}} z^{n+m} \;. \tag{3.9}$$

However the simple identity

$$\binom{n+m}{2} \;=\; \binom{n}{2} \;+\; \binom{m}{2} \;+\; nm$$

reduces 3.9 to

$$^{\wedge}\big(A \odot_q B\big) \;=\; \sum_{n\geq 0}\sum_{m\geq 0} A_n(q)\, B_m(q)\, q^{-\binom{n}{2}} q^{-\binom{m}{2}} z^{n+m} \;.$$

which is another way of writing 3.7. The identity in 3.8 is established in a similar way. Alternatively we may use the basic involution to deduce 3.8 from 3.7.

Note next that, when $A(0) = 1$, we can write

$$A(z) \;=\; 1 + \sum_{n\geq 1} A_n(q)\, z^n \;=\; exp\Big(\sum_{k\geq 1} p_k(A)\, z^k/k\Big) \tag{3.10}$$

with

$$p_k(A) \;=\; (-1)^{n-1} det \begin{pmatrix} A_1 & 1 & 0 & \cdots & \cdots & 0 \\ 2A_2 & A_1 & 1 & \cdots & \cdots & 0 \\ 3A_3 & A_2 & A_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & A_1 & 1 \\ kA_k & A_{k-1} & \cdots & \cdots & A_2 & A_1 \end{pmatrix} . \qquad 3.11$$

Conversely, given any sequence $\{p_k\}_{k\geq 1}$, the coefficient A_n in the expansion

$$exp\Big(\sum_{k\geq 1} p_k\, z^k/k\Big) = A(z) = 1 + \sum_{n\geq 1} A_n\, z^n$$

may be obtained from the formula

$$A_n \;=\; \frac{1}{n!}\, det \begin{pmatrix} p_1 & -1 & 0 & \cdots & \cdots & 0 \\ p_2 & p_1 & -2 & \cdots & \cdots & 0 \\ p_3 & p_2 & p_1 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & p_1 & -n+1 \\ p_n & p_{k-1} & \cdots & \cdots & p_2 & p_1 \end{pmatrix} \qquad 3.12$$

These formulas, originally due to Newton, can be easily established by applying Cramer's rule to the system of linear equations obtained by differentiating 3.10 and equating coefficients of equal powers of z.

Given a series $A \in \mathcal{RAT}(q)$ with $A(0) = 1$ we would like to be able to construct another series $\theta(z) \in \mathcal{RAT}(q)$ with the property that

$$A(z) \;=\; \frac{\theta(z)}{\theta(zq)} \;. \qquad 3.13$$

Note that this would imply that for all $k \geq 0$

$$\theta(z) \;=\; A(z)A(zq)\cdots A(zq^{k-1})\,\theta(zq^k) \;. \qquad 3.14$$

We may thus be tempted to set

$$\theta(z) \;=\; A(z)A(zq)A(zq^2)\cdots \;=\; \prod_{k\geq 0} A(zq^k)$$

However, from this definition it is not obvious that $\theta(z) \in \mathcal{RAT}(q)$ or even that the coefficients of $\theta(z)$ may be obtained by a finite number of operations. It turns out that, using Newton's relations, we can achieve the same purpose by setting instead

$$\theta(z) \;=\; A^*(z)$$

where

$$A^*(z) \;=\; exp\Big(\sum_{k\geq 1} p_k(A)\, \frac{1}{1-q^k}\, z^k/k\Big) \;. \qquad 3.15$$

Infact, if we also set

$$^*A(z) \;=\; exp\Big(\sum_{k\geq 1} p_k(A)\, \frac{1}{1-(1/q)^k}\, z^k/k\Big) \;. \qquad 3.16$$

we have the following basic result

Proposition 3.3

For all $A(z) \in \mathcal{RAT}(q)$ with $A(0)=1$ both $A^*(z)$ and ${}^*A(z)$ are formal series in $\mathcal{RAT}(q)(z)$. Moreover

$$A(z) \;=\; \frac{\theta(z)}{\theta(zq)} \quad\Longleftrightarrow\quad \theta(z)=A^*(z) \tag{3.17}$$

and

$$A(z) \;=\; \frac{\theta(z)}{\theta(z/q)} \quad\Longleftrightarrow\quad \theta(z)={}^*A(z)\ . \tag{3.18}$$

Remarkably, we also have that

$$A^*(z)\ {}^*A(z/q) \;=\; 1 \tag{3.19}$$

Proof

The first assertion is an immediate consequence of Newton's relations. In fact, 3.11 and 3.12 give that the A_n's are polynomials in the p_k's and viceversa. Thus the rationality of the coefficients of $A^*(z)$ and ${}^*A(z)$ follows from the rationality of the coefficients of $A(z)$. Note next that if we use Newton's relations and set

$$\theta(z) \;=\; exp\Big(\sum_{k\geq 1} p_k(\theta)\, z^k/k\Big)$$

then from the definition in 3.15 we derive that the first equality in 3.17 is equivalent to

$$exp\Big(\sum_{k\geq 1} p_k(A)\, z^k/k\Big) \;=\; exp\Big(\sum_{k\geq 1} p_k(\theta)\,(1-q^k)\, z^k/k\Big)\ .$$

and this in turn is equivalent to

$$p_k(A) \;=\; (1-q^k)\, p_k(\theta)$$

or better

$$p_k(\theta) \;=\; \frac{p_k(A)}{1-q^k}$$

proving 3.17. A similar argument gives 3.18. To prove 3.19 we need to show that

$$exp\Big(\sum_{k\geq 1} p_k(A)\,\frac{1}{1-q^k}\, z^k/k\Big)\ exp\Big(\sum_{k\geq 1} p_k(A)\,\frac{(1/q)^k}{1-(1/q)^k}\, z^k/k\Big) \;=\; 1$$

but this is immediate since

$$\frac{1}{1-q^k} \;+\; \frac{(1/q)^k}{1-(1/q)^k} \;=\; 0\ .$$

This completes our proof.

Remark 3.1

It might be good to mention that the Euler result that the two series

$$e_q(z)=\sum_{n\geq 0}\frac{z^n}{(1-q)(1-q^2)\cdots(1-q^n)} \qquad\text{and}\qquad E_q(-z)=\sum_{n\geq 0}\frac{(-1)^n q^{\binom{n}{2}} z^n}{(1-q)(1-q^2)\cdots(1-q^n)}$$

are inverse of each other is an almost immediate consequence of 3.19. Indeed, after we notice that

$$e_q(z) \;=\; e_q(zq) \;+\; z e_q(z)$$

or better

$$\frac{1}{1-z} \;=\; \frac{e_q(z)}{e_q(zq)}$$

we must conclude that

$$e_q(z) \;=\; \left(\frac{1}{1-z}\right)^* .$$

Now, from 3.19 we get

$$\left(\frac{1}{1-z}\right)^* \quad {}^*\left(\frac{1}{1-z/q}\right) \;=\; 1 \;. \qquad 3.20$$

However, since

$$\begin{aligned} {}^*\left(\frac{1}{1-z/q}\right) \;=\; \Downarrow \left(\frac{1}{1-z\,q}\right)^* \;&=\; \Downarrow \sum_{n\geq 0} \frac{q^n z^n}{(1-q)\cdots(1-q^n)} \\ &=\; \sum_{n\geq 0} \frac{q^{-n} z^n}{(1-1/q)\cdots(1-1/q^n)} \\ &=\; \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(1-q)\cdots(1-q^n)} \;=\; E_q(-z) \;, \end{aligned}$$

we see that 3.20 is none other than

$$e_q(z)\, E_q(-z) \;=\; 1 \;.$$

To terminate our presentation of this mini-theory of *roofing* and *starring* operations we need one additional simple but powerful result, namely

Proposition 3.4

For any two series $A(z), B(z) \in \mathcal{RAT}(q)$ we have

$$^\wedge A(z) \;\; ^\vee B(zq) \;\Big|_{z^n} \;=\; q^{-\binom{n}{2}}\, A(z)\, B(zq^n) \;\Big|_{z^n} \qquad 3.21$$

Proof

Note that

$$q^{-\binom{n}{2}}\, A(z)\, B(zq^n) \;\Big|_{z^n} \;=\; \sum_{h+k=n} A_h\, B_k\, q^{hk-\binom{n}{2}} \;. \qquad 3.22$$

Now we can easily verify for $n = h + k$ we have

$$hk - \binom{n}{2} \;=\; (h+k)\,k \;-\; \binom{h+k}{2} \;=\; \binom{k}{2} + k - \binom{h}{2} \;.$$

Thus we may rewrite 3.22 in the form

$$q^{-\binom{n}{2}}\, A(z)\, B(zq^n) \;\Big|_{z^n} \;=\; \sum_{h+k=n} A_h\, q^{-\binom{h}{2}}\;\; B_k\, q^{\binom{k}{2}}\, q^k$$

and this is 3.21.

We now have all the ingredients needed to prove the Rogers identity. However as a warm-up we shall begin with a closely related identity of Sylvester.

Theorem 3.2

$$\sum_{n\geq 0} (-z)^n\, q^{n(3n-1)/2}\, (1-zq^{2n})\; \frac{(1-z)\cdots(1-zq^{n-1})}{(1-q)\cdots(1-q^n)} \;=\; \prod_{m\geq 0} (1-zq^m) \qquad 3.23$$

Proof

Note that the left-hand side of 3.23 divided by $(1-z)^*$ becomes

$$\phi(z) \;=\; \sum_{n\geq 0} \frac{(-z)^n\, q^{n(3n-1)/2}}{(1-q)\cdots(1-q^n)}\,(1-zq^{2n})\;\frac{1}{(1-zq^n)^*}\;. \tag{3.24}$$

and 3.23 reduces to

$$\phi(z)\equiv 1\;. \tag{3.25}$$

This given, since $\phi(0)=1$, to prove 3.23, we only need to verify that

$$\phi(z)\;\Big|_{z^n} \;=\; 0 \qquad\quad (\;\forall\; n\geq 1\;).$$

The crucial point is that we have

$$\phi(z) \;=\; \sum_{n\geq 0} \frac{(-z)^n\, q^{n(3n-1)/2}}{(1-q)\cdots(1-q^n)}\,\frac{1}{(1-zq^n)^*} \;-\; \sum_{n\geq 0} \frac{(-z)^n\, q^{n(3n-1)/2+n}}{(1-q)\cdots(1-q^n)}\;\frac{zq^n}{(1-zq^n)^*}\;.$$

revealing that $\phi(z)$ is the difference of two q-tangle products. This makes 3.24 an almost immediate consequence of 3.21. In fact, noting that

$$n(3n-1)/2 \;=\; 3\binom{n}{2}+n$$

we may write

$$\phi(z) \;=\; \sum_{n\geq 0} \frac{(-z)^n\, q^{3\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}\,\frac{1}{(1-zq^n)^*} \;-\; \sum_{n\geq 0} \frac{(-z)^n\, q^{3\binom{n}{2}+2n}}{(1-q)\cdots(1-q^n)}\;\frac{zq^n}{(1-zq^n)^*}\;.$$

so roofing gives

$$^\wedge\phi(z) \;=\; \Bigg[\sum_{n\geq 0} \frac{(-z)^n\, q^{2\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}\Bigg]\;\; ^\wedge\Big[\frac{1}{(1-z)^*}\Big] \;-\; \Bigg[\sum_{n\geq 0} \frac{(-z)^n\, q^{2\binom{n}{2}+2n}}{(1-q)\cdots(1-q^n)}\Bigg]\;\; ^\wedge\Big[\frac{z}{(1-z)^*}\Big]\;.$$

or better yet

$$^\wedge\phi(z) \;=\; ^\vee\Bigg[\sum_{n\geq 0} \frac{(-z)^n\, q^{\binom{n}{2}+n}}{(1-q)\cdots(1-q^n)}\Bigg]\;\; ^\wedge\Big[\frac{1}{(1-z)^*}\Big] \;-\; ^\vee\Bigg[\sum_{n\geq 0} \frac{(-z)^n\, q^{\binom{n}{2}+2n}}{(1-q)\cdots(1-q^n)}\Bigg]\;\; ^\wedge\Big[\frac{z}{(1-z)^*}\Big]\;.$$

Now reversing the order of factors and using Euler's Theorem (formula 1.11), we get

$$^\wedge\phi(z) \;=\; ^\wedge\Big[\frac{1}{(1-z)^*}\Big]\; ^\vee\Big[(1-qz)^*\Big] \;-\; ^\wedge\Big[\frac{z}{(1-z)^*}\Big]\; ^\vee\Big[(1-q^2z)^*\Big]\;.$$

This puts each of the summands in the form needed to apply 3.21 obtaining

$$\begin{aligned}\phi(z)\,\Big|_{z^n} = q^{-\binom{n}{2}}\;\; ^\wedge\phi(z)\,\Big|_{z^n} = &\quad \frac{1}{(1-z)^*}\;(1-zq^n)^*\,\Big|_{z^n} \;-\; \frac{z}{(1-z)^*}\;(1-zq^{n+1})^*\,\Big|_{z^n}\\ = &\quad \frac{1}{(1-z)\cdots(1-zq^{n-1})}\,\Big|_{z^n} - \;\frac{1}{(1-z)\cdots(1-zq^n)}\,\Big|_{z^{n-1}}\end{aligned}$$

and 1.24 gives that for all $n\geq 1$ we must have

$$\phi(z)\,\Big|_{z^n} = \;\begin{bmatrix} n+n-1\\ n-1\end{bmatrix} \;-\; \begin{bmatrix} n-1+n\\ n\end{bmatrix} \;=\; 0$$

this completes our proof.

We shall give two proofs of 3.1: one follows closely the above argument and the other reduces it to 3.23.

1st Proof of Theorem 3.1

Proceeding as before, we divide both sides of 3.1 by $\prod_{m\geq 0}(1-zq^m)$ and rewrite it in the form

$$\sum_{n\geq 0} \frac{(-1)^n\, z^{2n}\, q^{(5n-1)n/2}}{(1-q)\cdots(1-q^n)}\;(1-zq^{2n})\,\frac{1}{(1-zq^n)^*} \;=\; \sum_{k\geq 0}\frac{z^k q^{k^2}}{(1-q)\cdots(1-q^k)}\;. \qquad 3.26$$

Using the identity

$$(5n-1)n/2 \;=\; \binom{2\,n}{2}+\binom{n+1}{2}$$

and splitting the left-hand side into the difference of two sums, 3.26 becomes

$$\sum_{n\geq 0} \frac{(-1)^n\, z^{2n}\, q^{\binom{2\,n}{2}+\binom{n+1}{2}}}{(1-q)\cdots(1-q^n)}\;\frac{1}{(1-zq^n)^*}\;- \\ -\sum_{n\geq 0} \frac{(-1)^n\, z^{2n}\, q^{\binom{2\,n}{2}+\binom{n+1}{2}+n}}{(1-q)\cdots(1-q^n)}\;\frac{zq^n}{(1-zq^n)^*} \;=\; \sum_{k\geq 0}\frac{z^k q^{k^2}}{(1-q)\cdots(1-q^k)}\;.$$

Now the left-hand side would be the difference of two q-tangle products if we had z^n instead of z^{2n} in these sums. However the situation is easily fixed: we simply replace q by q^2 and obtain

$$\sum_{n\geq 0} \frac{(-1)^n\, z^{2n}\, q^{2\binom{2\,n}{2}+2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})}\;\frac{1}{\prod_{m\geq 0}(1-zq^{2n+2m})}\;- \\ -\sum_{n\geq 0} \frac{(-1)^n\, z^{2n}\, q^{2\binom{2\,n}{2}+2\binom{n+1}{2}+2n}}{(1-q^2)\cdots(1-q^{2n})}\;\frac{zq^{2n}}{\prod_{m\geq 0}(1-zq^{2n+2m})} \;= \qquad 3.27 \\ =\sum_{k\geq 0}\frac{z^k q^{2k^2}}{(1-q^2)\cdots(1-q^{2k})}\;.$$

Calling $\psi(z)$ the left-hand side of this equation, we see that we may write as before

$${}^\wedge\psi(z) = {}^\vee\!\left[\sum_{n\geq 0}\frac{(-1)^n\, z^{2n}\, q^{2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})}\right]{}^\wedge\!\left[\frac{1}{\prod_{m\geq 0}(1-zq^{2m})}\right] - \\ -\;{}^\vee\!\left[\sum_{n\geq 0}\frac{(-1)^n\, z^{2n}\, q^{2\binom{n+1}{2}+2n}}{(1-q^2)\cdots(1-q^{2n})}\right]{}^\wedge\!\left[\frac{z}{\prod_{m\geq 0}(1-zq^{2m})}\right]$$

This may be rewritten as

$${}^\wedge\psi(z) = {}^\wedge\Delta(z)\;\;{}^\vee\Gamma(z)\;-\;{}^\wedge\big(z\Delta(z)\big)\;\;{}^\vee\Gamma(zq)$$

with

$$\Gamma(z) \;=\; \sum_{n\geq 0}\frac{(-1)^n\, z^{2n}\, q^{2\binom{n+1}{2}}}{(1-q^2)\cdots(1-q^{2n})} \;=\; \prod_{m\geq 1}(1-z^2q^{2m})$$

and

$$\Delta(z) \;=\; \prod_{m\geq 0}\frac{1}{(1-zq^{2m})}$$

This given we may use 3.21 again obtaining

$$\psi(z)\,\big|_{z^n} \;=\; q^{-\binom{n}{2}}\;{}^\wedge\psi(z)\,\big|_{z^n} \;=\; \Delta(z)\;\Gamma(zq^{n-1})\,\big|_{z^n} \;-\; z\,\Delta(z)\;\Gamma(zq^n)\,\big|_{z^n}\;. \qquad 3.28$$

Now $\Gamma(z)$ and $\Delta(z)$ are easily seen to satisfy the following two identities

$$a)\quad \Gamma(z) \;=\; (1-z^2q^2)\Gamma(zq) \qquad \text{and} \qquad b)\quad (1-z)\Delta(z) \;=\; \Delta(zq^2) \tag{3.29}$$

So using 3.29 a) (with $z \to zq^{n-1}$), and later 3.29 b), 3.28 becomes

$$\begin{aligned}
\psi(z)\big|_{z^n} &= \Delta(z)\ \big(1-z^2q^{2n-2+2}\big)\Gamma(zq^n)\big|_{z^n} \;-\; z\,\Delta(z)\ \Gamma(zq^n)\big|_{z^n}\\
&= (1-z)\Delta(z)\ \Gamma(zq^n)\big|_{z^n} \;-\; \Delta(z)\ z^2q^{2n}\Gamma(zq^n)\big|_{z^n}\\
&= \Delta(zq^2)\ \Gamma(zq^n)\big|_{z^n} \;-\; \Delta(z)\ z^2q^{2n}\Gamma(zq^n)\big|_{z^n}\\
&= q^{2n}\ \Big(\Delta(z)\ \Gamma(zq^{n-2})\big|_{z^n} \;-\; \Delta(z)\ z^2\Gamma(zq^n)\big|_{z^n}\Big)
\end{aligned} \tag{3.30}$$

Using 3.29 a) and b) once again gives

$$\begin{aligned}
q^{-2n}\,\psi(z)\big|_{z^n} &= \Delta(z)\ \Gamma(zq^{n-2})\big|_{z^n} \;-\; \Delta(z)\ z^2\Gamma(zq^n)\big|_{z^n}\\
&= \Delta(z)\ (1-z^2q^{2n-2})\Gamma(zq^{n-1})\big|_{z^n} \;+\; z\big(\Delta(zq^2)-\Delta(z)\big)\ \Gamma(zq^n)\big|_{z^n}\\
&= \big(\,\Delta(z)\,\Gamma(zq^{n-1}) - z\Delta(z)\Gamma(zq^n)\,\big)\big|_{z^n} \;+\; \big(\,\Delta(zq^2)\Gamma(zq^n) - zq^{2n-2}\Gamma(zq^{n-1})\big)\big|_{z^{n-1}}\\
&= \big(\,\Delta(z)\,\Gamma(zq^{n-1}) - z\Delta(z)\Gamma(zq^n)\,\big)\big|_{z^n} \;+\; q^{2n-2}\big(\,\Delta(z)\Gamma(zq^{n-2}) - z\Gamma(zq^{n-1})\big)\big|_{z^{n-1}}
\end{aligned}$$

Comparing with 3.28 we see that we have established the identity

$$q^{-2n}\,\psi(z)\big|_{z^n} \;=\; \psi(z)\big|_{z^n} \;+\; q^{2n-2}\,\psi(z)\big|_{z^{n-1}}$$

from which we easily deduce the recursion

$$\psi(z)\big|_{z^n} \;=\; \frac{q^{4n-2}}{1-q^{2n}}\,\psi(z)\big|_{z^{n-1}}\ .$$

Since $\psi(0)=0$, this gives

$$\psi(z)\big|_{z^n} \;=\; \frac{q^{2(2n-1+2n-3+\cdots+3+1)}}{(1-q^2)\cdots(1-q^{2n})} \;=\; \frac{q^{2n^2}}{(1-q^2)\cdots(1-q^{2n})}$$

proving that

$$\psi(z) \;=\; \sum_{n\ge 0}\frac{z^n\,q^{2n^2}}{(1-q^2)\cdots(1-q^{2n})}\ ,$$

Now this is 3.27 since $\psi(z)$ was originally defined as the left-hand side of that equation. Our proof is thus complete.

2^{nd} Proof of Theorem 3.1

We shall use here a reduction process introduced by S. Milne † in the proof of a more general family of identities. The idea is to show that the left-hand side of 3.1 divided by $\prod_{m\ge 0}(1-zq^m)$, namely the formal series

$$\theta(z) \;=\; \sum_{n\ge 0}\frac{(-1)^n\,z^{2n}\,q^{(5n-1)n/2}}{(1-q)\cdots(1-q^n)}\ \frac{(1-zq^{2n})}{\prod_{m\ge 0}(1-zq^{n+m})} \tag{3.31}$$

may be written in the form

$$\theta(z) \;=\; \sum_{k\ge 0}\frac{q^{k^2}\,z^k}{(1-q)\cdots(1-q^k)}\ \phi(zq^{2k}) \tag{3.32}$$

† (Amer. J. of Math. V.104 #3(1982) pp. 635-643.

with $\phi(z)$ given by 3.24. That is

$$\phi(z) \;=\; \sum_{n\geq 0} \frac{(-z)^n\, q^{n(3n-1)/2}}{(1-q)\cdots(1-q^n)}\,(1-zq^{2n})\;\frac{1}{(1-zq^n)^*}\;. \qquad 3.33$$

Of course the establishment of 3.32 would immediately yield

$$\theta(z) \;=\; \sum_{k\geq 0} \frac{q^{k^2}\, z^k}{(1-q)\cdots(1-q^k)} \qquad 3.34$$

since we proved the Sylvester identity precisely by showing that

$$\phi(z) \;\equiv\; 1$$

Now it develops that to prove 3.32 we need only make a clever use of the following simple but interesting identity

Proposition 3.5

$$z^n \;=\; \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}}}{q^{nk}}\;(1-z)(1-zq)\cdots(1-zq^{k-1}) \qquad 3.35$$

Proof

For any polynomial $P(z)$ set

$$D_q\, P(z) \;=\; \frac{P(z)-P(z/q)}{z}\;.$$

The operator D_q may be viewed as a q-analogue of the difference operator, in that it acts on the polynomials

$$(z)_n \;=\; (1-z)(1-zq)\cdots(1-zq^{n-1}) \qquad 3.36$$

precisely as the difference operator acts on the lower factorial polynomials. More precisely, we can easily verify that for all $n\geq 0$ we have

$$D_q\, z^n \;=\; \frac{(1-q^n)}{q}\,(z)_{n-1}$$

From this it immediately follows that we have

$$\frac{q^k}{(1-q)\cdots(1-q^k)}\;D_q^k\,(z)_n\,\Big|_{z=1} \;=\; \begin{cases} 1 & \text{if } k=n \\ 0 & \text{if } k\neq n \end{cases}$$

This given, the well known argument that establishes the Taylor expansion theorem proves that every polynomial $P(z)$ of degree n may be expanded in the form

$$P(z) \;=\; \sum_{k=0}^{n} \frac{q^k\;D_q P(z)\,\big|_{z=1}}{(1-q)\cdots(1-q^k)}\quad (1-z)(1-zq)\cdots(1-zq^{k-1}) \qquad 3.37$$

Now it develops that 3.35 is none other than 3.37 for $P(z)=z^n$. We leave this easy verification to the reader.

To proceed with the proof of 3.32 we use the identity

$$(5n-1)n/2 \;=\; 2n^2+\binom{n}{2}$$

to write a portion of the numerator of 3.31 in the form

$$z^{2n}\, q^{(5n-1)n/2} \;=\; z^n\, q^{n^2+\binom{n}{2}}(zq^n)^n$$

then use 3.5 with $z \to zq^n$ to get

$$z^{2n}\, q^{(5n-1)n/2} \;=\; z^n\, q^{n^2+\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}}}{q^{nk}}\; (zq^n)_k$$

Substituting this in 3.31 gives

$$\begin{aligned}
\theta(z) \;&=\; \sum_{n\geq 0} \frac{(-1)^n\, z^n\, q^{n^2+\binom{n}{2}}}{(1-q)\cdots(1-q^n)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}}}{q^{nk}}\; \frac{(1-zq^{2n})\;(zq^n)_k}{\prod_{m\geq 0}(1-zq^{n+m})} \\
&=\; \sum_{n\geq 0} \frac{(-1)^n\, z^n\, q^{n^2+\binom{n}{2}}}{(1-q)\cdots(1-q^n)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}}}{q^{nk}}\; \frac{(1-zq^{2n})}{\prod_{m\geq 0}(1-zq^{n+k+m})} \qquad 3.31 \\
&=\; \sum_{n\geq 0} (-1)^n\, z^n\, q^{n^2+\binom{n}{2}} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q)_k\,(q)_{n-k}\; q^{nk}}\; \frac{(1-zq^{2n})}{(1-zq^{n+k})^*}
\end{aligned}$$

where for convenience we have set

$$(1-q)\cdots(1-q^k) = (q)_k \qquad \text{and} \qquad (1-q)\cdots(1-q^{n-k}) = (q)_{n-k}\;.$$

Interchanging order of summation we then get

$$\theta(z) \;=\; \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}\, z^k}{(q)_k} \sum_{n\geq k} \frac{(-1)^{n-k} z^{n-k}}{(q)_{n-k}}\; \frac{q^{n^2+\binom{n}{2}}}{q^{nk}}\quad \frac{(1-zq^{2n})}{(1-zq^{n+k})^*}\;.$$

Making the change of variables $n-k=m$ gives

$$\begin{aligned}
\theta(z) \;&=\; \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}\, z^k}{(q)_k} \sum_{m\geq 0} \frac{(-1)^m z^m}{(q)_m}\; \frac{q^{(m+k)^2+\binom{m+k}{2}}}{q^{mk+k^2}}\quad \frac{(1-zq^{2k+2m})}{(1-zq^{m+2k})^*} \\
&=\; \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}\, z^k}{(q)_k} \sum_{m\geq 0} \frac{(-1)^m z^m}{(q)_m}\; \frac{q^{3m^2/2+3k^2/2+2km+km-m/2-k/2}}{q^{mk+k^2}}\quad \frac{(1-zq^{2k+2m})}{(1-zq^{2k+m})^*} \\
&=\; \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}\, z^k}{(q)_k} \sum_{m\geq 0} \frac{(-1)^m (zq^{2k})^m}{(q)_m}\; q^{m(3m-1)/2+k^2/2-k/2}\quad \frac{(1-zq^{2k+2m})}{(1-zq^{2k+m})^*} \\
&=\; \sum_{k\geq 0} \frac{q^{k^2}\, z^k}{(q)_k} \sum_{m\geq 0} \frac{(-1)^m (zq^{2k})^m}{(q)_m}\; q^{m(3m-1)/2}\quad \frac{(1-zq^{2k+2m})}{(1-zq^{2k+m})^*}
\end{aligned}$$

This proves 3.32 and completes our 2^{nd} proof of Theorem 3.1