Orbits and Macdonald Polynomials

Lecture notes

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1. Coordinate rings of finite sets

Let $\mathbf{V}_n$ denote the n-dimensional coordinate space $\mathbf{V}_n = \{(x_1, \ldots, x_n) : x_i \in \mathcal{Q}\}$ and let $\mathcal{Q}[x_1, \ldots, x_n] = \mathcal{Q}[x]$ denote the space of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathcal{Q}$. Here $\mathcal{Q}$ is some field which will vary as our develoment proceeds. At the beginning we may assume without loss that $\mathcal{Q}$ is the field of rational numbers. For a finite set $S \subseteq \mathbf{R}_n$ we let $\mathcal{J}_S$ be the ideal of polynomials that vanish on S. In symbols

$$\mathcal{J}_S \;=\; \{\, P \in \mathcal{Q}[x] \;:\; P(a) = 0 \quad \forall \; a \in S \,\}$$

It is easy to see that the quotient

$$\mathbf{R}_S \;=\; \mathcal{Q}[x]/\mathcal{J}_S$$

is a vector space of dimension $|S|$. †

Let us recall that if $J \subseteq \mathcal{Q}[x]$ is an ideal, by $gr\ \mathcal{J}$ we denote the ideal generated by the highest homogeneous components of elements of $\mathcal{J}$. In symbols:

$$gr\ \mathcal{J} \;=\; (\ h(P) \;:\; P \in \mathcal{J}\ )$$

where for a polynomial P here and after $h(P)$ will denote its homogeneous component of highest degree.

We shall use here the *dlex* order of monomials defined by setting

$$x^p <_{DL} x^q \quad \text{if and only if} \quad \begin{cases} deg(x^p) < deg(x^q) & \text{or} \\ \\ deg(x^p) = deg(x^q) & p <_{lex} q \end{cases}$$

This given, we denote by $\lambda(P)$ the dlex-highest monomial in the polynomial P. The collection of leading monomials

$$\mathcal{M}_{\mathcal{J}} \;=\; \{\ \lambda(P) \;:\; P \in \mathcal{J}\ \}$$

will be referred to as the *monomial ideal* of $\mathcal{J}$. It is an upper ideal in the inclusion partial order of monomials "$\prec$" defined by setting

$$x^p \preceq x^q \quad \textit{if and only if } x^p \textit{ divides } x^q$$

It is well known [OH] †† that the complement ${}^c\mathcal{M}_{\mathcal{J}}$ of $\mathcal{M}_{\mathcal{J}}$ gives a basis for the ring $\mathbf{R}_{\mathcal{J}}$. Note that since for any polynomial P the leading monomial in P lies in $h(P)$ we see that we must have

$$\mathcal{M}_{\mathcal{J}} \;=\; \mathcal{M}_{gr\ \mathcal{J}}$$

† In these notes for any set S, the symbol $|S|$ denotes the cardinality of S

†† [OH] refers to the monograph on Orbit Harmonics.

Thus it follows that the quotient

$$gr\, \mathbf{R}_{\mathcal{J}} \;=\; \mathcal{Q}[x]/gr\, \mathcal{J}$$

has the same monomial basis and therefore also the same dimension as $\mathbf{R}_{\mathcal{J}}$.

Let us recall that a theorem of Gordan states that every upper ideal of monomials has a finite number of minimal elements. Gordan also shows that if $m_1, m_2, \ldots, m_N$ are the minimal elements of $\mathcal{M}_{\mathcal{J}}$ and $f_1, f_2, \ldots, f_N$ are elements of $\mathcal{J}$ such that $\lambda(f_i) = m_i$ then $f_1, f_2, \ldots, f_N$ generate $\mathcal{J}$ and every element $P \in \mathcal{J}$ has an expansion of the form

$$P(x) \;=\; \sum_{i=1}^{N} A_i(x)\, f_i(x) \qquad (\; A_i \in \mathcal{Q}[x] \;)$$

with

$$deg\, (A_i f_i) \;\leq\; deg\, P$$

We shall refer to such a set of elements $f_1, f_2, \ldots, f_N$ as a G-basis of $\mathcal{J}$. It is clear from this definition that we have the following fundamental fact

Proposition 1.1

If $f_1, f_2, \ldots, f_N$ are a G-basis of $\mathcal{J}$ then

$$h(f_1)\;,\; h(f_2)\;,\; \ldots\;,\; h(f_N)$$

are a G-basis of $gr\ \mathcal{J}$.

In some particular cases it may be possible to construct generators for $gr\ \mathcal{J}$ by taking the highest homogeneous components of a set of generators for $\mathcal{J}$ without constructing a G-basis. The following is a useful crieterion to this effect.

Proposition 1.2

Let $Q_1, Q_2, \ldots, Q_N$ be elements of an ideal $\mathcal{J}$ and let $P_1, P_2, \ldots, P_M$ be elements of $gr\ (Q_1, Q_2, \ldots, Q_N)$. Suppose that the quotient $\mathcal{Q}[x]/\mathcal{J}$ has dimension d and we are able to show that

$$dim\ \mathcal{Q}[x]/(P_1, P_2, \ldots, P_M) \;\leq\; d \tag{1.1}$$

Then $\mathcal{J}$ is generated by $Q_1, Q_2, \ldots, Q_N$ and $gr\ \mathcal{J}$ is generated by $(P_1, P_2, \ldots, P_M)$.

Proof

Note that the hypotheses imply the two inequalities

$$\begin{array}{ll} a) & d = dim\ \mathcal{Q}[x]/\mathcal{J} \;\leq\; dim\ \mathcal{Q}[x]/(Q_1, Q_2, \ldots, Q_N) \\ b) & dim\ \mathcal{Q}[x]/gr\ (Q_1, Q_2, \ldots, Q_N) \;\leq\; dim\ \mathcal{Q}[x]/(P_1, P_2, \ldots, P_M) \end{array}$$

Since

$$dim\ \mathcal{Q}[x]/(Q_1, Q_2, \ldots, Q_N) \;=\; dim\ \mathcal{Q}[x]/gr\ (Q_1, Q_2, \ldots, Q_N)$$

the estimate in 1.1 converts a) and b) into equalities, and the assertion immediately follows.

Another useful criterion we shall need later in our development may be stated as follows

Proposition 1.2

Let S be a finite set in $\mathbf{V}_n$ and suppose that $Q_1, Q_2, \ldots, Q_N$ generate $\mathcal{J}_S$. Suppose further that a subset $D \subseteq S$ may be decomposed into disjoint union

$$S - D \;=\; E_1 \;+\; E_2 \;+\; \cdots \;+\; E_k$$

and that for each $i = 1, .., k$ we have a polynomial P_i which vanishes in D but doesn't vanish in E_i. Then

$$Q_1, Q_2, \ldots, Q_N \;\; ; \;\; P_1, P_2, \ldots, P_k$$

generate $\mathcal{J}_D$.

Proof

For each element $a \in S$ construct a polynomial $\phi_a(x)$ which has the value **1** on a and vanishes in the rest of S. Then for any polynomial $P \in \mathcal{Q}[x]$ we have the congruence the form

$$P(x) \;\cong\; \sum_{a \in D} P(a)\, \phi_a(x) \;+\; \sum_{i=1}^{k} \sum_{b \in E_i} P(b)\, \phi_b(x) \qquad (\; mod \;\; \mathcal{J}_S \;) \tag{1.2}$$

Note that for each $i = 1..k$ must we have

$$\sum_{b \in E_i} P(b)\, \phi_b(x) \;\cong\; \Big(\sum_{b \in E_i} \frac{P(b)}{P_i(b)} \;\phi_b(x) \Big)\; P_i(x) \qquad (\; mod \;\; \mathcal{J}_S \;) \tag{1.3}$$

Now, if P vanishes in D the first sum in 1.2 will identically vanish and the combination of 1.2 and 1.3 gives that we have

$$P(x) \;=\; \sum_{i=1}^{k} \Big(\sum_{b \in E_i} \frac{P(b)}{P_i(b)} \;\phi_b(x) \Big)\; P_i(x) \;+\; \sum_{i=1}^{N} A_i(x)\, Q_i(x)$$

for suitable polynomials A_i. This proves our assertion.