Proof

The reader should have no difficulty carrying out the calculations needed to deduce 3.66 (a) and (b) from 3.22 and 3.24 since they are very similar to those carried out in the previous two proofs.

needed to cancel the factor $t^{n(\nu)-n(\lambda)}/t^r$ from 3.63. Moreover, if λ/μ consists of k vertical segments of lengths $r_1, r_2, \ldots, r_k$, then the factor contributed by the denominators from the last product in 3.63 reduces to

$$(-1)^r \prod_{i=1}^{k} \frac{1}{(1-t)(1-t^2)\cdots(1-t^{r_i})} \ .$$

This cancels the $(-1)^r$ in 3.63 and combines with the factor $(1-t)(1-t^2)\cdots(1-t^r)$ in 3.63 to produce the q-multinomial coefficient appearing in 3.60. Carrying out all these cancellations we see that 3.63 reduces to 3.60 as desired. This completes our proof.

Our next task is to rewrite in terms of the $\tilde{H}_\mu$ *'s* the Pieri rule in 2.28.

Theorem 3.11

For $r \geq s$

$$\tilde{H}_s(x;q,t)\tilde{H}_r(x;q,t) = \sum_{i=0}^{s} \begin{bmatrix} s \\ i \end{bmatrix}_q \frac{(q)_r}{(q)_{r-s+i}} \frac{(t-1)\cdots(t-q^{i-1})}{(t-q^{r-s+i})\cdots(t-q^{r+i})}(t-q^{r-s+2i})\tilde{H}_{r-i,s+i}(x;q,t) \ , \qquad 3.65$$

where $\begin{bmatrix} a \\ i \end{bmatrix}_q = (q)_a/(q)_i(q)_{a-i}$ *denotes the q-binomial coefficient and for an integer m*

$$(q)_m \ = \ (1-q)(1-q^2)\cdots(1-q^{m-1}) \ .$$

Proof

The definitions 3.2 and 3.18, specialized to the partitions $\{s\}$,$\{r\}$ and $\{s-i, r+i\}$ give

$$H_r(x;q,t) = (q)_r Q_r[X/(1-t);q,t] \quad , \quad H_s(x;q,t) = (q)_s Q_s[X/(1-t);q,t]$$

and

$$H_{s-i,r+i}(x;q,t) = (q)_{s-i}\,(q)_{r-s+i}\,(1-tq^{r-s+2i+1}t)\cdots(1-tq^{r+i})\,Q_{s-i,r+i}[X/(1-t);q,t)] \ .$$

Multiplying both sides of 2.28 by $(q)_s(q)_r$, making the λ-ring substitution $X \to X/(1-t)$ and using the relations above, after simple cancellations and a reorganization of the factors we get

$$H_s\,H_r \ = \ \sum_{i=0}^{s} \begin{bmatrix} s \\ i \end{bmatrix}_q \frac{(q)_r}{(q)_{r-s+i}} \frac{(1-t)\cdots(1-tq^{i-1})}{(1-tq^{r-s+i})\cdots(1-tq^{r+i})}\,(1-tq^{r-s+2i})\,H_{s-i,r+i}(x;q,t) \ .$$

This given, the definition 3.48 a) immediately yields 3.65.

In the same vein equations 3.22 and 3.34 may be rewritten as

Theorem 3.12

Using the same notation as in 3.22 and 3.24 we have

$$(a) \quad \Gamma_1 \tilde{H}_\mu \ = \ \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\ \tilde{H}_\nu \quad , \quad (b) \quad \tilde{F}_\mu(q,t) \ = \ \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\ \tilde{F}_\nu(q,t) \qquad 3.66$$

with

$$c_{\mu\nu}(q,t) \ = \ \prod_{s\in\tilde{\mathcal{R}}_{\lambda/\mu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in\tilde{\mathcal{C}}_{\lambda/\mu}} \frac{t^{l_\mu(s)+1} - q^{a_\mu(s)}}{t^{l_\nu(s)+1} - q^{a_\nu(s)}} \qquad 3.67$$

Theorem 3.10

$$\tilde{H}_\mu(x;q,t)\;\tilde{H}_{1^r}(x;q,t) \;=\; \sum_{\lambda/\mu\in V_r} v_{\lambda\mu}(q,t)\;\tilde{H}_\lambda(x;q,t)\;, \tag{3.59}$$

where $\lambda/\mu \in V_r$ is to mean that the sum is carried out over Ferrers' diagrams λ such that λ/μ is a vertical r-strip, $r_1, r_2, \ldots, r_k$ are the lengths of the successive vertical segments of λ/μ and

$$v_{\lambda\mu}(q,t) \;=\; \begin{bmatrix} r \\ r_1\; r_2 \cdots r_k \end{bmatrix}_t \prod_{s\in\mathcal{R}_{\lambda/\mu}} \frac{t^{l_\mu(s)+1}-q^{a_\mu(s)}}{t^{l_\lambda(s)+1}-q^{a_\lambda(s)}} \prod_{s\in\tilde{\mathcal{C}}_{\lambda/\mu}} \frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\lambda(s)}-q^{a_\lambda(s)+1}}\;, \tag{3.60}$$

where $\mathcal{R}_{\lambda/\mu}$ is the set of squares of μ that are in a row that intersects λ/μ and $\tilde{\mathcal{C}}_{\lambda/\mu}$ is the set of squares of μ that are in a column that intersects λ/μ but are <u>not</u> in $\mathcal{R}_{\lambda/\mu}$.

Proof

Using 3.1, 3.18, 2.24 a) and the specialization (a) of Theorem 2.5, the Pieri rule 2.27 (ii') is easily converted into

$$H_\mu(x;q,t)\;H_{1^r}(x;q,t) \;=\; \sum_{\lambda/\mu\in V_r} u_{\lambda\mu}(q,t)\;H_\lambda(x;q,t)\;, \tag{3.61}$$

with

$$u_{\lambda\mu}(q,t) \;=\; (1-t)(1-t^2)\cdots(1-t^r) \prod_{s\in\mathcal{R}_{\lambda/\mu}} \frac{h_\mu(s)}{h_\lambda(s)} \prod_{s\in\tilde{\mathcal{C}}_{\lambda/\mu}} \frac{h'_\mu(s)}{h'_\lambda(s)} \prod_{s\in\lambda/\mu} \frac{1}{h_\lambda(s)}\;. \tag{3.62}$$

Replacing t by $1/t$ and multilying both sides of 3.61 by $t^{n(\mu)}t^{n(1^r)}$ from the definition 3.48 a) we immediately get 3.59 with

$$v_{\lambda\mu}(q,t) \;=\; t^{n(\mu)}\,t^{\binom{r}{2}}t^{-n(\lambda)}\,u_{\lambda\mu}(q,1/t)\;.$$

Substituting 3.62 into this and using the formulas in 2.20 gives

$$\begin{aligned} v_{\lambda\mu}(q,t) \;=\;& t^{n(\mu)-n(\lambda)}\,t^{\binom{r}{2}}\frac{(-1)^r}{t^{r+\binom{r}{2}}}\,(1-t)(1-t^2)\cdots(1-t^r) \\ & \prod_{s\in\mathcal{R}_{\lambda/\mu}} \frac{t^{l_\mu(s)+1}-q^{a_\mu(s)}}{t^{l_\lambda(s)+1}-q^{a_\lambda(s)}}\,\frac{t^{l_\lambda(s)}}{t^{l_\mu(s)}} \prod_{s\in\tilde{\mathcal{C}}_{\lambda/\mu}} \frac{t^{l_\mu(s)}-q^{a_\mu(s)+1}}{t^{l_\lambda(s)}-q^{a_\lambda(s)+1}}\,\frac{t^{l_\lambda(s)}}{t^{l_\mu(s)}} \prod_{s\in\lambda/\mu} \frac{-t^{l_\lambda(s)+1}}{1-t^{l_\lambda(s)+1}}\;. \end{aligned} \tag{3.63}$$

Note next that the power of t contributed by the last three products in 3.63 may be rewritten in the form

$$t^r\;t^{\Sigma_{s\in\lambda}\,l_\lambda(s)\;-\;\Sigma_{s\in\mu}\,l_\mu(s)}\;. \tag{3.64}$$

In fact, the t^r is produced by the "+1" in the exponent of the last product in 3.63, and the only extra factors we have included in 3.64 are produced by both sums when s is not in a column or row that intersects λ/μ. However, since for such a square we necessarily have $l_\lambda(s)=l_\mu(s)$, these extra factors cancel each other. Now it is easily seen that the power of t in 3.64 is precisely what is

Since there is no dependence on t here we deduce that $H_n[X,q,t]$ and $\tilde{H}_n[X;q,t]$ are one and the same. Thus 3.56 a) follows from the result stated in Remark III.2.2. An application of the identity in 3.50 then derives $a')$ from $a)$. To establish $b)$ we start from 3.46 which, via the definition 3.1 and 3.18 gives

$$H_\mu[X;1,t] \;=\; h_\mu(1,t)\, e_{\mu'}[X\tfrac{1}{1-t}] \;.$$

Indicating by $\mu'_1, \mu'_2, \ldots, \mu'_h$ the parts of the partition μ' (conjugate of μ), we see from the definition 2.20 that

$$h_\mu(1,t) \;=\; \prod_{i=1}^{h} (1-t)\cdots(1-t^{\mu'_i}) \;.$$

This gives that

$$H_\mu[X;1,t] \;=\; \prod_{i=1}^{h} e_{\mu'_i}[X\tfrac{1}{1-t}]\,(1-t)\cdots(1-t^{\mu'_i}) \;, \qquad 3.57$$

and thus 3.48 a) gives

$$\tilde{H}_\mu[X;1,t] \;=\; \tfrac{(-1)^n}{t^n} \prod_{i=1}^{h} e_{\mu'_i}[X\tfrac{-t}{1-t}]\,(1-t)\cdots(1-t^{\mu'_i}) \;=\; \prod_{i=1}^{h} (1-t)\cdots(1-t^{\mu'_i}) h_{\mu'_i}[X\tfrac{1}{1-t}] \;,$$

where we have made use of III.1.23. Making again use of Remark III.2.2 we see that 3.56 $b)$ holds true as asserted. Equation 3.56 $b')$ follows then by another application of 3.50.

To derive 3.56 $c)$ we need only verify that

$$Q_\mu(x;0,t) \;=\; Q_\mu(x,t) \;. \qquad 3.58$$

Indeed from 2.20 we get that $h'_\mu(0,t) \;=\; 1$ and thus III.2.21 gives

$$H_\mu[X;0,t] \;=\; Q_\mu[X\tfrac{1}{1-t};t] \;=\; H_\mu[X,t]$$

which implies 3.56 $c)$ via the definitions III.2.36 $a)$ and 3.48 $a)$. We clearly see that setting $q=0$ in the Macdonald kernel $\Omega[XY\frac{1-t}{1-q}]$ yields the Hall-Littlewood kernel $\Omega[XY(1-t)]$. So it is quite natural to conclude that $Q_\mu(x;q,t)$ that must tend to $Q_\mu(x;t)$ as $q \rightarrow 0$. Nevertheless, carrying out this passage to the limit rigorously requires a delicate argument which would take us to far out of the present context. We must thus refer the reader to the original source [] for a detailed derivation of 3.58. This given, we are only left to note that formula 3.56 $c')$ then follows by another application of 3.50.

Our next two theorems rewrite in terms of our functions $\tilde{H}_\mu(x;q,t)$ some specializations of the Pieri rules in 2.27 which play a role in the developments that follow.

Now this last equation, combined with 3.48 a), may be rewritten as

$$\tilde{H}_\mu(x,\tfrac{1}{q},t) \;=\; \frac{1}{q^{n(\mu')}}\,\omega\, H_\mu(x;q,t)\ .$$

or better

$$\tilde{H}_\mu(x,q,t) \;=\; q^{n(\mu')}\,\omega\, H_\mu(x;\tfrac{1}{q},t)\ . \tag{3.55}$$

But now 3.54 gives

$$\omega H_\mu(x;\tfrac{1}{q},t) \;=\; H_{\mu'}(x;t,\tfrac{1}{q})\ .$$

Substituting this in 3.55 finally yields

$$\tilde{H}_\mu(x,q,t) \;=\; H_{\mu'}(x;t,\tfrac{1}{q})\, q^{n(\mu')}\ ,$$

which (via 3.48 a)) is precisely 3.50. Q.E.D

Remark 3.2

Here and in the following we shall have to deal with Macdonald polynomials and Hall-Littlewood polynomials all at the same time. Since the latter depend only on the parameter t and the former on both q and t, we only need to use this information to distinguish between them. In this manner we can keep the notation used in Chapter III, without change. For instance $H_\mu(x;t)$ refers to the polynomial defined in III.2.22 and $\tilde{H}_\mu(x;t)$ is as defined by III.2.36 a). By contrast $H_\mu(x;q,t)$ and $\tilde{H}_\mu(x;q,t)$ will still represent the polynomials defined by 3.18 and 3.48 a). Analogous conventions will apply to the coefficients $K_{\lambda\mu}(q,t)$, $\tilde{K}_{\lambda\mu}(q,t)$, $K_{\lambda\mu}(t)$ and $\tilde{K}_{\lambda\mu}(t)$.

This given the following specializations indicate the tight relationships that exist between the material of these last two chapters.

Theorem 3.9

$$\begin{array}{rrcl}
a\,) & \tilde{H}_n(x;q,t) & = & \tilde{H}_{1^n}(x;q)\ , \\
a') & \tilde{H}_{1^n}(x;q,t) & = & \tilde{H}_{1^n}(x;t)\ , \\
b\,) & \tilde{H}_\mu(x;1,t) & = & \tilde{H}_{1^{\mu'_1}}(x;t)\tilde{H}_{1^{\mu'_2}}(x;t)\cdots\tilde{H}_{1^{\mu'_h}}(x;t)\ , \\
b') & \tilde{H}_\mu(x;q,1) & = & \tilde{H}_{1^{\mu_1}}(x;q)\tilde{H}_{1^{\mu_2}}(x;q)\cdots\tilde{H}_{1^{\mu_k}}(x;q)\ , \\
c\,) & \tilde{H}_\mu(x;0,t) & = & \tilde{H}_\mu(x;t)\ , \\
c') & \tilde{H}_\mu(x;q,0) & = & \tilde{H}_{\mu'}(x;q)\ .
\end{array} \tag{3.56}$$

Proof

Equation 3.2 and (b') of Theorem 2.5 give that

$$J_n[X;q,t] \;=\; h'_n(q,t)\, h_n[X\tfrac{1-t}{1-q}]\ .$$

Now 2.22 b) gives that $h'_n(q,t) = (1-q)\cdots(1-q^n)$, and using 3.18 we deduce that

$$H_n[X;q,t] \;=\; (1-q)\cdots(1-q^n)h_n[X\tfrac{1}{1-q}]\ .$$

setting $q = 1$ and using 3.45 we may rewrite this as

$$\sum_{\mu} e_\mu(x)\, m_\mu(y) \;=\; \sum_{\mu} P_\mu(x;1,t)\, m_{\mu'}(y)$$

and 3.46 follows by equating the coefficients of $m_{\mu'}(y)$ on both sides of this identity.

In analogy with what had to be done in section III.2 (see III.2.36) it will be combinatorially more convenient to work with the modified versions $\tilde{H}_\mu[X_n;q,t]$ and $\tilde{K}_\mu(q,t)$ obtained by setting

$$a)\quad \tilde{H}_\mu[X_n;q,t] \;=\; H_\mu[X_n;q,1/t]t^{n(\mu)} \quad \text{and} \quad b)\quad \tilde{K}_{\lambda\mu}(q,t) \;=\; K_{\lambda\mu}(q,1/t)t^{n(\mu)} \;. \qquad 3.48$$

We shall also set

$$\tilde{F}_\mu(q,t) \;=\; \partial_{p_1}^n\, \tilde{H}_\mu(x;q,t) \;=\; F_\mu(q,1/t)\, t^{n(\mu)} \;. \qquad 3.49$$

The rest of this section will be dedicated to restate some of the basic theorems in terms of $\tilde{H}_\mu(x;q,t)$, $\tilde{K}_\mu(q,t)$ and $\tilde{F}(q,t)$.

We start with the following beautiful identity.

Theorem 3.8

For any partition μ we have

$$\tilde{H}_{\mu'}(x;q,t) \;=\; \tilde{H}_\mu(x;t,q) \;, \qquad 3.50$$

in particular

$$\tilde{K}_{\lambda\mu'}(q,t) \;=\; \tilde{K}_{\lambda\mu}(t,q) \;. \qquad 3.51$$

Proof

We start by rewriting quations 2.14 b) and 2.15 a) with λ replaced by μ. Namely,

$$Q_\mu(x;q,t) \;=\; \omega\; P_{\mu'}[X\tfrac{1-q}{1-t};q,t] \;, \qquad 3.52$$

and

$$P_\mu(x;\tfrac{1}{q},\tfrac{1}{t}) \;=\; P_\mu(x;q,t) \;. \qquad 3.53$$

Now, by a straighforward manipulation which uses the identities 3.1 and 3.2, we can rewrite 3.52 in the form

$$J_\mu[\tfrac{X}{1-q};t,q) \;=\; \omega\; J_{\mu'}[\tfrac{X}{1-t};q,t)$$

In view of our definition 3.18 this simply says that

$$H_\mu(x;t,q) \;=\; \omega\; H_{\mu'}(x;q,t) \qquad 3.54$$

On the other hand 3.53, using again 3.1, 3.2 and 3.18, gives

$$H_\mu(x,\tfrac{1}{q},\tfrac{1}{t}) \;=\; \frac{1}{q^{n(\mu')}}\,\frac{1}{t^{n(\mu)}}\;\omega\; H_\mu(x;q,t) \;.$$

Proof.

We give only the basic idea here and refer the reader to [ ] for details. Note that for $t=1$ the operator δ_{qt} reduces to

$$\delta_{qt} \;=\; \sum_{s=1}^{n} T_q^{(s)}$$

and thus

$$\begin{aligned} \delta_{qt}\, m_\lambda(x) \;&=\; \sum_{\lambda(p)=\lambda} x_1^{p_1}\cdots x_s^{p_s} q^{p_s}\cdots x_n^{p_n} \\ &=\; \left(\sum_{s=1}^{n} q^{\lambda_s}\right) m_\lambda(x) \;\; . \end{aligned}$$

In other words the eigenfunctions of δ_{qt} reduce in this case to the basis $\{m_\lambda(x)\}_\lambda$. We should also note that

$$m_\lambda(x) \;=\; S_\lambda(x) + \sum_{\mu\to\lambda} S_\mu(x)\, H_{\lambda\mu}$$

where the $H_{\lambda\mu}$ are the entries in the inverse of the Kostka matrix $K = \|K_{\lambda\mu}\|$. Thus it is reasonable to assume that

$$\lim_{t\to 1} P_\mu(x;q,t) \;=\; m_\mu(x)$$

and this is all that we mean here by 3.45.

Theorem 3.7

$$P_\mu(x;1,t) \;=\; e_{\mu'}(x) \qquad 3.46$$

Proof.

In the course of the proof of the duality theorem 2.1 we have established that if $\{f_n\}$ has the Macdonald property then

$$\Omega(x,y) \;=\; \sum_\mu \omega\, P_\mu(x;f)\, P_{\mu'}(y,\tfrac{1}{f})$$

making the specialization $f_n = \frac{1-t^n}{1-q^n}$ gives

$$\Omega(x,y) \;=\; \sum_\mu \omega\, P_\mu(x;q,t)\, P_{\mu'}(y;t,q) \;\; . \qquad 3.47$$

Since

$$\omega\Omega(x,y) \;=\; \sum_\mu \omega h_\mu(x)\, m_\mu(y) \;=\; \sum_\mu e_\mu(x)\, m_\mu(y)$$

3.47 gives

$$\sum_\mu e_\mu(x)\, m_\mu(y) \;=\; \sum_\mu P_\mu(x;q,t)\, P_{\mu'}(y;t,q)$$

Note that the definitions 2.20 give

$$h_\lambda(t,q) \;=\; h'_{\lambda'}(q,t)\ ; \quad h'_\lambda(t,q) \;=\; h_{\lambda'}(q,t)\ .$$

Thus we may write

$$h'_\lambda(t,q)\,Q_\lambda[\tfrac{X}{1-q};t,q] \;=\; \omega\, h_{\lambda'}(q,t)\,P_{\lambda'}[\tfrac{X}{1-t};q,t]$$

in other words (in view of 3.1 and 3.2) we have

$$J_\lambda[\tfrac{X}{1-q};t,q] \;=\; \omega\, J_{\lambda'}[\tfrac{X}{1-t};q,t]$$

and 3.41 for $\mu=\lambda$ follows then from our definition 3.18. Formula 3.42 is then obtained by equating the coefficients of S_λ in 3.41.

Theorem 3.5

$$H_\mu(x;\tfrac{1}{q},\tfrac{1}{t}) \;=\; \frac{1}{q^{n(\mu')}}\,\frac{1}{t^{n(\mu)}}\,\omega\, H_\mu(x;q,t) \qquad 3.43$$

in particular

$$K_{\lambda'\mu'}(\tfrac{1}{q},\tfrac{1}{t}) \;=\; K_{\lambda'\mu'}(q,t)\,\frac{1}{q^{n(\mu')}t^{n(\mu)}} \qquad 3.44$$

Proof.

Formula 2.15 (b) gives for $|\mu|=n$

$$Q_\mu(x;\tfrac{1}{q},\tfrac{1}{t}) \;=\; \frac{q^n}{t^n}\,Q_\mu(x;q,t)\ .$$

Since

$$h'_\mu(\tfrac{1}{q},\tfrac{1}{t}) \;=\; \frac{1}{q^{n(\mu')}t^{n(\mu)}}\,\frac{(-1)^n}{q^n}\,h'_\mu(q,t)$$

from 3.2 we get

$$J_\mu(x;\tfrac{1}{q},\tfrac{1}{t}) \;=\; \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)}}\,\frac{1}{t^n}\,J_\mu(x;q,t)$$

replacing X by $X/(1-1/t)$ gives

$$\begin{aligned} H_\mu(x;\tfrac{1}{q},\tfrac{1}{t}) \;&=\; \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)}}\,\frac{1}{t^n}\,J_\mu[\tfrac{-tX}{1-t};q,t] \\ &=\; \frac{(-1)^n}{q^{n(\mu')}t^{n(\mu)}}\,J_\mu[-\tfrac{X}{1-t};q,t] \end{aligned}$$

and 3.43 then follows from III.1.23 and 3.18. Equation 3.44 is then obtained by equating coefficients of S_λ in 3.43

Theorem 3.6

$$P_\mu(x;q,1) \;=\; m_\mu(x) \qquad 3.45$$

Proof.

We simply note that for $q=t$, and $|\mu|=n$

$$h_\mu(s) \;=\; 1-t^{a_\mu(s)+l_\mu(s)+1} \;=\; h'_\mu(s) \;.$$

Thus 3.31 reduces to

$$\frac{h_\mu(t,t)}{h_\nu(t,t)} \;=\; \prod_{s\in \tilde{R}_{\mu/\nu}} \frac{h_\mu(s)}{h_\nu(s)} \prod_{s\in \tilde{C}_{\mu/\nu}} \frac{h_\mu(s)}{h_\nu(s)}(1-t) \tag{3.39}$$

substituting in 3.22 and noting that

$$[h_\mu]_t \;=\; \frac{h_\mu(t,t)}{(1-t)^n} \qquad , \qquad [h_\nu]_t \;=\; \frac{h_\nu(t,t)}{(1-t)^{n-1}}$$

gives

$$\Gamma_1\, H_\mu \;=\; \sum_{\nu\to\mu} \frac{h_\mu(t,t)}{h_\nu(t,t)}\,\frac{1}{1-t}\,H_\nu \;=\; \sum_{\nu\to\mu} \frac{h_\mu(t,t)}{h_\nu(t,t)}\,\frac{(1-t)^n}{(1-t)^{n-1}}\,\frac{1}{1-t}\,H_\nu$$

which is 3.37 as desired.

In the same manner 3.34 gives

$$\frac{F_\mu(t,t)}{[h_\mu]_t} \;=\; \sum_{\nu\to\mu} \frac{F_\nu(t,t)}{[h_\nu]_t} \;. \tag{3.40}$$

In other words, the expression $F_\mu(t,t)/[h_\mu]_t$ satisfies the same recursion as the number of standard tableaux of shape μ. Since we clearly have the same initial condition

$$\frac{F_\phi(t,t)}{h_\phi(t,t)} \;=\; 1 \;.$$

Equation 3.38 must necessarily hold true by induction on $|\mu|$.

There are further specializations that are important in our developments. We shall state them as separate theorems.

Theorem 3.4

$$H_\mu(x;t,q) \;=\; \omega\; H_{\mu'}(x;q,t) \tag{3.41}$$

in particular

$$K_{\lambda\mu}(t,q) \;=\; K_{\lambda'\mu'}(q,t) \tag{3.42}$$

Proof.

The duality formula 2.14 (b) gives

$$Q_\lambda(x;q,t) \;=\; \omega\, P_{\lambda'}[X\tfrac{1-q}{1-t};q,t] \;.$$

Theorem 3.2

$$F_\mu(q,t) \;=\; \sum_{\nu\to\mu} \prod_{s\in\tilde{\mathcal{R}}_{\mu/\nu}} \frac{h'_\mu(s)}{h'_\nu(s)} \prod_{s\in\tilde{\mathcal{C}}_{\mu/\nu}} \frac{h_\mu(s)}{h_\nu(s)}\, F_\nu(q,t) \;. \qquad 3.34$$

Proof.

Note that if χ is an S_n-character and H is its Frobenius image then the definition of the Frobenius map gives

$$\chi\Big|_{id} \;=\; \sum_{\sigma\in S_n} \chi(\sigma)\, p_{\lambda(\sigma)}(x)\Big|_{p_1^n} \;=\; n!\, H\big|_{p_1^n} \;. \qquad 3.35$$

Since $\chi\big|_{id} = \chi\big|_{S_{n-1}}\big|_{id}$, this formula for S_{n-1} gives that

$$\chi_{id} \;=\; (n-1)!\Gamma_1 H\big|_{p_1^{n-1}}$$

and thus 3.34 is simply obtained by equating coefficients of p_1^{n-1} of both sides of 3.22.

Remark 3.1

We should note that we do have

$$n!\, H|_{p_1^n} \;=\; \partial_{p_1}^n\, H \;,$$

provided we interpret a symmetric function as a formal power series in the *variables* $p_1, p_2, \ldots p_n$. In the same vein we can easily show that

$$\Gamma_1 \;=\; \partial_{p_1} \;.$$

In particular, this shows that Γ_1 acts as *a differentiation* on symmetric functions. This given, we see that 3.34 is simply obtained by applying $\partial_{p_1}^{n-1}$ to both sides of 3.22.

Formulas 3.22 and 3.34 have remarkable specializations for $q = t$. Indeed, setting

$$[h_\mu]_t \;=\; \prod [1 + a_\mu(s) + a_\nu(s)]_t \qquad 3.36$$

with

$$[m]_t \;=\; 1 + q + \cdots + q^{m-1}$$

we can state

Theorem 3.3

$$\frac{\Gamma_1\, H_\mu(x;t,t)}{[h_\mu]_t} \;=\; \sum_{\nu\to\mu} \frac{H_\nu(x;t,t)}{[h_\nu]_t} \qquad 3.37$$

$$F_\mu(t,t) \;=\; f_\mu\, [h_\mu]_t \qquad 3.38$$

Thus, equating coefficients of $K_\mu(y;q,t)$ we obtain

$$\Gamma_1\, H_\mu(x;q,t) \;=\; \sum_\nu c_{\mu\nu}(q,t)\, H_\nu(x;q,t)\ . \qquad 3.28$$

Recalling 3.25 we see that we may rewrite 3.27 as

$$\frac{1}{1-q}\, P_\nu[(1-q)y;q,t]\, e_1[(1-q)y] \;=\; \sum_\mu \frac{h'_\nu(q,t)}{h'_\mu(q,t)}\, P_\mu[(1-q)y;q,t]\, c_{\mu\nu}(q,t)\ . \qquad 3.29$$

Comparing with the Pieri rule 2.27 (ii'), for $r=1$ and λ,μ respectively replaced by μ and ν gives

$$(1-q)\, c_{\mu\nu}(q,t)\, \frac{h'_\nu(q,t)}{h'_\mu(q,t)} \;=\; \begin{cases} \prod_{s\in\tilde{\mathcal{C}}_{\mu/\nu}} \frac{h_\mu(s)}{h'_\mu(s)}\,\frac{h'_\nu(s)}{h_\nu(s)} & \text{if } \nu\to\mu \\ 0 & \text{otherwise} \end{cases}\ . \qquad 3.30$$

Thus the sum in 3.28 runs only over partitions ν that immediately precede μ. Letting s_0 be the square we must remove from μ to obtain ν, we see from the definition 2.20 that

$$\frac{h'_\nu(q,t)}{h'_\mu(q,t)} \;=\; \prod_{s\in\tilde{\mathcal{R}}_{\mu/\nu}} \frac{h'_\nu(s)}{h'_\mu(s)} \prod_{s\in\tilde{\mathcal{C}}_{\mu/\nu}} \frac{h'_\nu(s)}{h'_\mu(s)}\,\frac{1}{1-q}\ . \qquad 3.31$$

This is because $h'_\mu(s_0)/h'_\nu(s_0)=(1-q)$ and for a square s not in the row or column of s_0 we must have $h'_\mu(s)=h'_\nu(s)$. Substituting this in 3.30 and extracting $c_{\mu\nu}$ we finally get for $\nu\to\mu$

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in\tilde{\mathcal{R}}_{\mu/\nu}} \frac{h'_\mu(s)}{h'_\nu(s)} \prod_{s\in\tilde{\mathcal{C}}_{\mu/\nu}} \frac{h_\mu(s)}{h_\nu(s)}\ ,$$

and our formula 3.22 then follows from 3.28.

This theorem has a very useful corollary. Note that the evaluation of our character $p^\mu(q,t)$ at the identity permutation gives the polynomial

$$F_\mu(q,t) \;=\; \sum_\lambda f_\lambda\, K_{\lambda\mu}(q,t) \;=\; p^\mu(q,t)\Big|_{id}\ . \qquad 3.32$$

Note further that 3.16 gives also

$$F_\mu(q,t) \;=\; \sum_{hk} q^h\, t^k\, p^\mu_{hk}\Big|_{id}$$

and since $p^\mu_{hk}|_{id}$ should give the dimension of the bigraded component $\mathcal{H}_{hk}(M_\mu)$ of M_μ we see that we may also write

$$F_\mu(q,t) \;=\; \sum_{hk} q^h\, t^k\, \dim \mathcal{H}_{hk}(M_\mu)\ . \qquad 3.33$$

In other words, the polynomial $F_\mu(q,t)$ may be viewed as the *bigraded Hilbert series* of M_μ.

This given, Theorem 3.1 immediately yields where Γ_1 is the operator on symmetric polynomials that is dual to multiplication by e_1. In particular, for $\chi = \chi^\lambda$, 3.19 specializes to

$$\Gamma_1 S_\lambda \;=\; s_{\lambda/1} \;=\; \sum_{\lambda^-} S_{\lambda^-}(x)\,\chi(\lambda^- \to \lambda) \qquad 3.20$$

which (section III.3.) may be taken as the definition of Γ_1. We also have observed (see III.2.39) that 3.19 that Γ_1, as an operation acting on symmetric polynomials in the variables $x_1, \ldots, x_n$, satisfies the identity

$$\Gamma_1\,\Omega(x,y) \;=\; \Omega(x,y)\,e_1(y)\ . \qquad 3.21$$

This given, we have the following remarkable formula

Theorem 3.1

$$\Gamma_1\,H_\mu \;=\; \sum_{\nu\to\mu}\ \prod_{s\in\tilde{\mathcal{R}}_{\mu/\nu}} \frac{h'_\mu(s)}{h'_\nu(s)} \prod_{s\in\tilde{\mathcal{C}}_{\mu/\nu}} \frac{h_\mu(s)}{h_\nu(s)}\, H_\nu\ . \qquad 3.22$$

Proof.

The identity

$$\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu Q_\mu(x;q,t)\,P_\mu(y;q,t) \qquad 3.23$$

(using 3.2) may be rewritten in the form

$$\Omega[XY\tfrac{1-t}{1-q}] \;=\; \sum_\mu J_\mu(x;q,t)\,\frac{P_\mu(y;q,t)}{h'_\mu(q,t)}\ .$$

3.18 then gives

$$\Omega(x,y) \;=\; \sum_\mu H_\mu(x;q,t)\,K_\mu(y;q,t) \qquad 3.24$$

where we have set

$$K_\mu(y;q,t) \;=\; P_\mu[Y(1-q);q,t]/h'_\mu(q,t)\ . \qquad 3.25$$

From 3.21 we then get that

$$\sum_\mu \Gamma_1\,H_\mu(x;q,t)\,K_\mu(y;q,t) \;=\; \sum_\nu H_\nu(x;q,t)\,K_\nu(y;q,t)\,e_1(y)\ . \qquad 3.26$$

Setting, for a moment,

$$K_\nu(y;q,t)\,e_1(y) \;=\; \sum_\mu K_\mu(y;q,t)\,c_{\mu\nu}(q,t) \qquad 3.27$$

we then get

$$\sum_\mu \Gamma_1\,H_\mu(x;q,t)\,K_\mu(y;q,t) \;=\; \sum_\mu \Big(\sum_\nu c_{\mu\nu}(q,t)\,H_\nu(x;q,t)\Big)\,K_\mu(y;q,t)\ .$$

The fact that $K_{\lambda\mu}(t,t)$ equals f_λ for all μ simply suggests that a possible approach towards proving the MPK conjecture should be to construct for each μ a bigraded version of the left regular representation of S_n given by a module M_μ whose bigraded character $p^\mu(q,t)$ has precisely the expansion

$$p^\mu(q,t) \;=\; \sum_\lambda \chi^\lambda\, K_{\lambda\mu}(q,t) \;. \qquad 3.14$$

More precisely, if

$$M_\mu \;=\; \bigoplus_{hk} \mathcal{H}_{hk}(M_\mu)$$

is the decomposition of M_μ into its bihomogeneous submodules and

$$p^\mu_{hk} \;=\; \text{char}\; \mathcal{H}_{hk}(M_\mu) \qquad 3.15$$

then 3.14 should hold true with

$$p^\mu(q,t) \;=\; \sum_{h,k} q^h\, t^k\, p^\mu_{hk} \;. \qquad 3.16$$

In particular this would yield not only that $K_{\lambda\mu}(q,t)$ is a polynomial in its arguments but it would result that the coefficient

$$K_{\lambda\mu}(q,t)\Big|_{q^h t^k} \;=\; \langle\, \chi^\lambda, p^\mu_{hk}\, \rangle_{S_n} \qquad 3.17$$

gives the multiplicity of χ^λ in the character of $\mathcal{H}_{hk}(M_\mu)$.

Reversing this reasonning, we see that if $K_{\lambda\mu}(q,t)$ is a polynomial and the coefficients in 3.17 are non negative integers for all choices of λ,h and k, then a bigraded module M_μ whose character decomposes as in 3.14 must necessarily exist. The goal of this chapter is precisely the construction of such a module. However, before we can do this we need to derive the various combinatorial and representation theoretical properties that M_μ and its bigraded character $p^\mu(q,t)$ should possess in order for 3.14 to hold true.

To this end we start by letting

$$H_\mu(x;q,t) \;=\; \sum_\lambda S_\lambda(x)\, K_{\lambda\mu}(q,t) \;,$$

denote the Frobenius image of $p^\mu(q,t)$. From 3.3 and 3.14 we then get that

$$H_\mu(x;q,t) \;=\; J_\mu[X\tfrac{1}{1-t};q,t] \;. \qquad 3.18$$

Now it develops that this relation, combined with the Pieri rules (Theorem 2.6), yields us recursions for the polynomials $H_\mu(x;q,t)$ that will turn out to be crucial for our later developments.

To this end, recall from section III.3. (Proposition III.3.11), that if χ is an S_n-character then the relation between its Frobenius image and the Frobenius image of its restriction to S_{n-1} is given by the formula

$$F\,\chi\big|_{S_{n-1}} \;=\; \Gamma_1\, F\,\chi \qquad 3.19$$

Note that Macdonald states in [ ] that

$$\lim_{q\to t} P_\mu(x;q,t) \;=\; S_\mu(x)\ . \tag{3.4}$$

So, 3.1 gives

$$\lim_{q\to t} J_\mu(x;q,t) \;=\; h_\mu(t)\, S_\mu(x) \tag{3.5}$$

where

$$h_\mu(t) \;=\; \prod_{s\in\mu} (1 - t^{1+a_\lambda(s)+l_\lambda(s)})\ . \tag{3.6}$$

For simplicity of notation we shall symbolically represent all these limit results as *evaluation* results. So, for instance, we shall simply write 3.4 and 3.5 as

$$(a)\quad P_\mu(x;t,t) \;=\; S_\mu(x)\ ; \qquad\qquad (b)\quad J_\mu(x;t,t) \;=\; h_\mu(t)\, S_\mu(x)\ . \tag{3.7}$$

This given, introducing the matrix $K(q,t) = ||K_{\lambda\mu}(q,t)||$ we may write 3.3 in the form

$$\langle\, J_\mu(x;q,t)\,\rangle \;=\; \langle\, S[X(1-t)]\,\rangle\, K(q,t) \tag{3.8}$$

while 3.7 (b) can be written as

$$\langle\, J_\mu(x;t,t)\,\rangle \;=\; \langle S(x)\rangle\, h(t) \tag{3.9}$$

with $h(t) = \text{diag}\ (h_\mu(t))$. Combining 3.8 and 3.9 with III.1.26 we get

$$\langle p(x)\rangle\, \tfrac{1}{z}\, \chi\, h(t) \;=\; \langle S(x)\rangle\, h(t) \;=\; \langle\, S[X(1-t)]\,\rangle\, K(t,t) \;=\; \langle p(x)\rangle\, \tfrac{1}{z}\, p(1-t)\, \chi\, K(t,t)$$

and this gives that

$$\frac{1}{p(1-t)}\frac{1}{z}\,\chi\, h(t) \;=\; \frac{1}{z}\,\chi\, K(t,t)\ .$$

Multiplying this matrix equation on the left by χ^T and using the character relations III.1.29 we finally get

$$K(t,t) \;=\; \chi^T\, \frac{1}{zp(1-t)}\, \chi\, h(t) \tag{3.10}$$

which may be rewritten in the more explicit form

$$K_{\lambda\mu}(t,t) \;=\; \sum_\rho \chi^\lambda_\rho\, \frac{1}{z_\rho p_\rho(1-t)}\ \chi^\mu_\rho\ \ h_\mu(t) \tag{3.11}$$

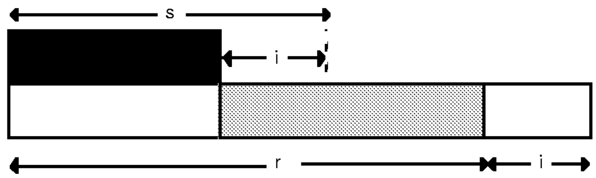
and so

$$K_{\lambda\mu}(1,1) \;=\; \chi^\lambda_{1^n}\, \frac{1}{n!}\, \chi^\mu_{1^n} \prod_s (1 + a_\mu(s) + l_\mu(s)) \;=\; f_\lambda\ . \tag{3.12}$$

the last equality holding true via the hook formula

$$f_\mu \;=\; \frac{n!}{\prod_{s\in\mu}(1+a_\mu(s)+l_\mu(s))}\ . \tag{3.13}$$

where $S(i)$ is the shaded area in the figure below



We see then that

$$\prod_{s\in S(i)} \frac{h_\mu(s)}{h'_\mu(s)} \;=\; \frac{(1-t)\cdots(1-tq^{r-s+i-1})}{(1-q)\cdots(1-q^{r-s+i})}$$

with

$$\prod_{s\in S(i)} \frac{h'_\lambda(s)}{h_\lambda(s)} \;=\; \frac{(1-q^{i+1})\cdots(1-q^{r-s+2i})}{(1-tq^{i})\cdots(1-tq^{r-s+2i-1})} \;.$$

Multiplying these two expressions and carrying out the obvious cancellations we get

$$\prod_{s\in S(i)} \frac{h_\mu(s)}{h'_\mu(s)}\,\frac{h'_\lambda(s)}{h_\lambda(s)} \;=\; \frac{(1-t)\cdots(1-tq^{i-1})}{(1-q)\cdots(1-q^{i})}\,\frac{(1-q^{r-s+i+1})\cdots(1-q^{r-s+2i})}{(1-tq^{r-s+i})\cdots(1-tq^{r-s+2i-1})}$$

as desired.

3. The Macdonald integral forms.

The branch of the theory of Macdonald polynomials which is most closely connected with the theory of orbit harmonics is that which concerns the Macdonald *integral forms.* To see how this comes about we need to review some definitions.

Macdonald in [ ] sets for given μ, q, t:

$$J_\mu(x;q,t) \;=\; h_\mu(q,t)\,P_\mu(x;q,t) \;. \qquad 3.1$$

Combining 2.19 and 2.17 we see that we also have

$$J_\mu(x;q,t) \;=\; h'_\mu(q,t)\,Q_\mu(x;q,t) \;. \qquad 3.2$$

This given, Macdonald writes

$$J_\mu(x;q,t) \;=\; \sum_\lambda S_\lambda[X(1-t)]\,K_{\lambda\mu}(q,t) \qquad 3.3$$

and conjectures that the coefficients $K_{\lambda\mu}(q,t)$ appearing in this expansion are polynomials in q,t with positive integer coefficients. We shall refer to this here and in the following as the *MPK conjecture.*

It is convenient to define for a partition λ and *any* lattice square s:

$$h_\lambda(s) \;=\; \begin{cases} 1 - q^{a_\lambda(s)}\, t^{l_\lambda(s)+1} & \text{if } s \in \lambda \\ 1 & \text{otherwise} \end{cases} . \qquad 2.25$$

and similarly

$$h'_\lambda(s) \;=\; \begin{cases} 1 - q^{a_\lambda(s)+1}\, t^{l_\lambda(s)} & \text{if } s \in \lambda \\ 1 & \text{otherwise} \end{cases} . \qquad 2.26$$

This given, the Pieri rules can be stated as follows

Theorem 2.6

$$\begin{aligned}
(i) \quad P_\mu(x;q,t)\, h_r(X\tfrac{1-t}{1-q}) \;&=\; \sum_{\lambda/\mu \in \mathcal{H}_r} \prod_{s \in \mathcal{C}_{\lambda/\mu}} \frac{h_\lambda(s)}{h'_\lambda(s)} \frac{h'_\mu(s)}{h_\mu(s)} P_\lambda(x;q,t) \\
(ii) \quad Q_\mu(x;q,t)\, h_r(X\tfrac{1-t}{1-q}) \;&=\; \sum_{\lambda/\mu \in \mathcal{H}_r} \prod_{s \in \tilde{\mathcal{R}}_{\lambda/\mu}} \frac{h_\mu(s)}{h'_\mu(s)} \frac{h'_\lambda(s)}{h_\lambda(s)} Q_\lambda(x;q,t) \\
(i') \quad Q_\mu(x;q,t)\, e_r(x) \;&=\; \sum_{\lambda/\mu \in \mathcal{V}_r} \prod_{s \in \mathcal{R}_{\lambda/\mu}} \frac{h_\mu(s)}{h'_\mu(s)} \frac{h'_\lambda(s)}{h_\lambda(s)} Q_\lambda(x;q,t) \\
(ii') \quad P_\mu(x;q,t)\, e_r(x) \;&=\; \sum_{\lambda/\mu \in \mathcal{V}_r} \prod_{s \in \tilde{\mathcal{C}}_{\lambda/\mu}} \frac{h_\lambda(s)}{h'_\lambda(s)} \frac{h'_\mu(s)}{h_\mu(s)} P_\lambda(x;q,t)
\end{aligned} \qquad 2.27$$

where $\lambda/\mu \in \mathcal{H}_r$ means that λ/μ is a horizontal r-strip, $\lambda/\mu \in \mathcal{V}_r$ means that λ/μ is a vertical r-strip and

(i) $\mathcal{C}_{\lambda/\mu}$ denotes the set of lattice squares of λ that are in the columns that intersect λ/μ.
(ii) $\tilde{\mathcal{R}}_{\lambda/\mu}$ denotes the set of lattice squares of λ that are in the rows of λ/μ but <u>not</u> in $\mathcal{C}_{\lambda/\mu}$.
(i') $\mathcal{R}_{\lambda/\mu}$ denotes the set of lattice squares of λ that are in the rows that intersect λ/μ.
(ii') $\tilde{\mathcal{C}}_{\lambda/\mu}$ denotes the set of lattice squares of λ that are in $\mathcal{C}_{\lambda/\mu}$ but <u>not</u> in $\mathcal{R}_{\lambda/\mu}$.

We shall illustrate these identities by working out some examples that will be useful for us in the following. We shall start with

Theorem 2.7

For $r \geq s$

$$Q_s\, Q_r \;=\; \sum_{i=0}^{s} \frac{(1-t)\cdots(1-tq^{i-1})}{(1-q)\cdots(1-q^i)} \cdot \frac{(1-q^{r-s+i+1})\cdots(1-q^{r-s+2i})}{(1-tq^{r-s+i})\cdots(1-tq^{r-s+2i-1})}\, Q_{s-i,r+i} \qquad 2.28$$

Proof.

Formula (ii) of 2.27 gives

$$Q_s\, Q_r \;=\; \sum_{i=0}^{s} \prod_{s \in S(i)} \frac{h_\mu(s)}{h'_\mu(s)} \frac{h'_\lambda(s)}{h_\lambda(s)}\, Q_{s-i,r+i}$$

The proof of this result would take us too far out of the present context. So we shall have to accept it here. For the same reasons we shall also have to refer the reader to [ ] for the proof of the so called Pieri formulas governing the multiplication of Macdonald polynomials. Before we state them here we should note that the Macdonald polynomials P_λ, Q_λ reduce to familiar symmetric functions when λ is specialized to be a simple row or as a simple column. Namely, we have

Theorem 2.5

(a) $P_{1^n}(x;q,t) = e_n(x)$ (b) $Q_{1^n}(x;q,t) = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{(1-q)(1-qt)\cdots(1-qt^{n-1})}\, e_n(x)$

(a′) $P_n(x;q,t) = \frac{(1-q)\cdots(1-q^n)}{(1-t)\cdots(1-tq^{n-1})}\, h_n[X\frac{1-t}{1-q}]$ (b′) $Q_n(x;q,t) = h_n[X\frac{1-t}{1-q}]$

Proof.

The identity in 2.18 (a) is an immediate consequence of the requirement 1.1 (a). Indeed, since there is no partition that is strictly below 1^n we must have that $\forall q,t$

$$P_{1^n}(x;q,t) \;=\; S_{1^n}(x) \;=\; e_n(x)$$

as desired. On the other hand, we can easily see from the duality theorem (formula 2.14 (b)) that

$$Q_n(x;t,q) \;=\; \omega P_{1^n}[X\tfrac{1-q}{1-t};q,t] \;.$$

But this combined with 2.18 (a) immediately gives

$$Q_n(x;t,q) \;=\; h_n[X\tfrac{1-q}{1-t}]$$

as desired.

To derive the remaining identities we use theorem 2.4 which yields

$$d_n(q,t) \;=\; \frac{h_n(q,t)}{h'_n(q,t)} \qquad 2.21$$

with

$$\begin{array}{llll} (a) & h_n(q,t) & = & (1-t)(1-tq)\cdots(1-tq^{n-1}) \;, \\ (b) & h'_n(q,t) & = & (1-q)(1-q^2)\cdots(1-q^n) \;. \end{array} \qquad 2.22$$

and 2.17 gives 2.21 a). Similarly, we get

$$d_{1^n}(q,t) \;=\; h_{1^n}(q,t)/h'_{1^n}(q,t) \qquad 2.23$$

with

$$\begin{array}{llll} (a) & h_{1^n}(q,t) & = & (1-t)(1-t^2)\cdots(1-t^n) \\ (b) & h'_{1^n}(q,t) & = & (1-q)(1-qt)\cdots(1-qt^{n-1}) \;. \end{array} \qquad 2.24$$

and and 2.17 gives 2.21 b).

Q.E.D.

Proof.

Note that we can rewrite III.1.33 for $f_n = \frac{1-t^n}{1-q^n}$ in the form

$$\Omega[XY\tfrac{1-\frac{1}{t}}{1-\frac{1}{q}}] \;=\; \Omega[XY\tfrac{q}{t}\tfrac{1-t}{1-q}] \;=\; \sum_\lambda P[X;q,t]\,Q_\lambda[\tfrac{q}{t}Y;q,t] \;.$$

However, the homogeneity of Q_λ then gives

$$\Omega[XY\tfrac{1-\frac{1}{t}}{1-\frac{1}{q}}] \;=\; \sum_\lambda P(x;q,t)\,(\tfrac{q}{t})^{|\lambda|}\,Q_\lambda(y;q,t) \;. \qquad 2.16$$

Since

$$Q_\lambda(y;q,t) \;=\; d_\lambda(q;t)\,P_\lambda(y;t) \qquad 2.17$$

and $\langle\, P(x;q,t)\,\rangle$ is by definition upper triangularily related to $\langle\, S(x)\,\rangle$ we see that 2.16 implies that $P_\lambda(x;q,t)$ is also the Macdonald basis relative to the scalar product $\langle\ ,\ \rangle_{\frac{1}{q}\frac{1}{t}}$. In other words we must have

$$P_\lambda(x;q,t) \;=\; P_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) \qquad 2.18$$

as asserted. Moreover, 2.16 yields that $\{(\frac{q}{t})^{|\lambda|}Q_\lambda(x;q,t)\}_\lambda$ is dual to $\{P_\lambda(x;\frac{1}{q},\frac{1}{t})\}_\lambda$. Thus

$$(\tfrac{q}{t})^{|\lambda|}\,Q_\lambda(x;q,t) \;=\; Q_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) \;.$$

Using 2.17 we then get

$$(\tfrac{q}{t})^{|\lambda|}\,d_\lambda(q,t)\,P_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) \;=\; Q_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) \;.$$

which finally gives

$$d_\lambda(\tfrac{1}{q},\tfrac{1}{t}) \;=\; (\tfrac{q}{t})^{|\lambda|}\,d_\lambda(q,t) \;,$$

as desired.

For a given partition λ and a given square in the Ferrers diagram of λ it is customary to let $a_\lambda(s)$ and $l_\lambda(s)$ respectively denote the number of squares to the right of s and below s in the Ferrers diagram of λ drawn by the English convention. For this reason, $a_\lambda(s)$ and $l_\lambda(s)$ are usually referred to as the *arm* and *leg* of s in λ.

Macdonald shows in [ ] that the coefficients $d_\lambda(q,t)$ may be computed by the following remarkable formulas

Theorem 2.4

$$d_\lambda(q,t) \;=\; \frac{h_\lambda(q,t)}{h'_\lambda(q,t)} \qquad 2.19$$

where

$$h_\lambda(q,t) \;=\; \prod_{s\in\lambda}(1-q^{a_\lambda(s)}\,t^{l_\lambda(s)+1});\quad h'_\lambda(q,t) \;=\; \prod_{s\in\lambda}(1-q^{a_\lambda(s)+1}\,t^{l_\lambda(s)}) \;. \qquad 2.20$$

Using 2.12 a), 2.3 and 2.13 a) we get

$$\begin{aligned}\Omega(x,y) &= \omega\langle\, S(x)\,\rangle\,\xi\,\eta^T \diamond \langle\, S(y)\,\rangle^T \\ &= \omega\langle\, S(x)\,\rangle\,\xi \diamond \overline{\eta}^T \langle\, S(y)\,\rangle^T \\ &= \omega\langle\, P(x;f)\,\rangle \diamond \langle\, P(y;\tfrac{1}{f})\,\rangle^T \ .\end{aligned}$$

Applying $\Theta_{\frac{1}{f}}$ to both sides of this identity finally gives

$$\Omega_{\frac{1}{f}}(x,y) = \langle\, \Theta_{\frac{1}{f}}\omega P(x;f)\,\rangle \diamond \langle\, P(y;\tfrac{1}{f})\,\rangle^T$$

and this (by proposition III.1.2) yields us that $\{\Theta_{\frac{1}{f}}\omega P(x;f)\}\diamond$ is the dual basis of $\{P(y;\frac{1}{f})\}$. But this is another way of stating 2.2 (b).

To get 2.2 (a) we use 2.12 b) and 2.13 (b) and derive that

$$\begin{aligned}P_\lambda(x;\tfrac{1}{f}) &= \frac{1}{d_\lambda(\frac{1}{f})}\,Q_\lambda(x;\tfrac{1}{f}) = d_{\lambda'}(f)\ \Theta_{\frac{1}{f}}\omega P_{\lambda'}(x;f) \\ &= \Theta_{\frac{1}{f}}\ \omega\ Q_{\lambda'}(x;f)\ ,\end{aligned}$$

as desired.

Specializing theorem 2.1 to the case

$$f_n = \frac{1-t^n}{1-q^n}$$

we immediately obtain the corollary

Theorem 2.2

$$\begin{aligned}&(a)\quad P_\lambda(x;t,q) = \omega Q_{\lambda'}[X\tfrac{1-q}{1-t};q,t] \\ &(b)\quad Q_\lambda(x;t,q) = \omega P_{\lambda'}[X\tfrac{1-q}{1-t};q,t] \\ &(c)\quad d_\lambda(t,q) = \frac{1}{d_{\lambda'}(q,t)}\ .\end{aligned} \qquad 2.14$$

There is another remarkable identity satisfied by the Macdonald polynomials which is an immediate consequence of the definition, namely:

Theorem 2.3

$$\begin{aligned}&(a)\quad P_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) = P_\lambda(x;q,t) \\ &(b)\quad Q_\lambda(x;\tfrac{1}{q},\tfrac{1}{t}) = \frac{q^{|\lambda|}}{t^{|\lambda|}}\,Q_\lambda(x;q,t) \\ &(c)\quad d_\lambda(\tfrac{1}{q},\tfrac{1}{t}) = (\tfrac{q}{t})^{|\lambda|}d_\lambda(q,t)\ .\end{aligned} \qquad 2.15$$

with $\eta = (\xi^T)^{-1}$. If η were dominance upper unitriangular the Macdonality of $\{\frac{1}{f_n}\}$ would be immediate. However, the definition of η does give that it is dominance lower unitriangular. It develops that this is easily fixed. Let ω^x and ω^y denote the operator ω given in III.1.21 as applied to functions of x and y respectively.

Since $\omega p_\rho(x) = (-1)^{n-k(\rho)} p_\rho(x)$ (with $k(\rho) = \#$ parts of ρ) we see that

$$\omega_x \, \omega_y \, \Omega_f(x,y) \;=\; \sum_\rho p_\rho(x) \, \frac{f}{z_\rho} \, p_\rho(y)(-1)^{2n-2k(\rho)} \;=\; \Omega_f(x,y) \; . \tag{2.9}$$

Now the relation $\omega S_\lambda = S_{\lambda'}$ may be expressed in matrix form by setting

$$\omega \langle \, S(x) \, \rangle \;=\; \langle \, S(x) \, \rangle \diamond \tag{2.10}$$

where $\diamond$ denotes the permutation matrix corresponding to conjugation of partitions. We may thus write

$$\begin{aligned} \langle \, S(x) \, \rangle \, \chi^{-1} \tfrac{1}{f} \, \chi \, \langle \, S(y) \, \rangle \;=\; \Omega_{\frac{1}{f}}(x,y) \;=\; \omega_x \, \omega_y \, \Omega_{\frac{1}{f}}(x,y) \;&=\; \langle \, S(x) \, \rangle \diamond \chi^{-1} \tfrac{1}{f} \, \chi \diamond \langle \, S(y) \, \rangle^T \\ &=\; \langle \, S(x) \, \rangle \diamond \eta \, \tfrac{1}{d} \, \eta^T \diamond \langle \, S(x) \, \rangle^T \; . \end{aligned}$$

This gives that (since $\diamond\diamond^T = I$)

$$\begin{aligned} \chi^{-1} \tfrac{1}{f} \, \chi \;&=\; \diamond \, \eta \, \diamond^T \;\; \diamond \tfrac{1}{d} \, \diamond^T \;\; (\diamond \, \eta \, \diamond^T)^T \\ &=\; \overline{\eta} \;\, \overline{d} \;\, \overline{\eta}^T \end{aligned} \tag{2.11}$$

where we have set

$$\begin{aligned} (a) \quad \overline{\eta} \;&=\; \diamond \, \eta \, \diamond^T \\ (b) \quad \overline{d} \;&=\; \diamond \, \tfrac{1}{d} \, \diamond^T \; . \end{aligned} \tag{2.12}$$

Note that the μ, λ entry in $\overline{\eta}$ is simply given by

$$\overline{\eta} \;=\; \sum_{\rho_1, \rho_2} \diamond_{\mu\rho_1} \, \eta_{\rho_1 \rho_2} \, \diamond_{\lambda \rho_2} \;=\; \eta_{\mu' \lambda'} \; .$$

Thus because η is dominance lower unitriangular we must have that $\overline{\eta}_{\mu\lambda} \neq 0$ only if $\mu' >_D \lambda'$ or, equivalently, only if $\mu <_D \lambda$. In other words $\overline{\eta}$ is dominance upper unitriangular. Thus 2.11 gives that $\{\frac{1}{f_n}\}$ has the Macdonald property as desired, and in addition we must have that the bases $\{P_\lambda(x; \frac{1}{f})\}$ and $\{Q_\lambda(x; \frac{1}{f})\}$ must be given by the formulas

$$\begin{aligned} (a) \quad & \langle \, P(x; \tfrac{1}{f}) \, \rangle \;=\; \langle \, S(x) \, \rangle \, \overline{\eta} \\ (b) \quad & Q_\lambda(x, \tfrac{1}{f}) \;=\; \overline{d}_\lambda \, P_\lambda(x, \tfrac{1}{f}) \qquad\qquad (\overline{d}_\lambda = \frac{1}{d_{\lambda'}}) \; . \end{aligned} \tag{2.13}$$

Now observe that the definition of η gives

$$\Omega(x,y) \;=\; \langle \, S(x) \, \rangle \, \xi \, \eta^T \, \langle \, S(y) \, \rangle^T \; .$$

Theorem 2.1

If $\{f_n\}$ has the Macdonald property then so does $\{\frac{1}{f_n}\}$ and

$$\begin{aligned} (a) \quad & P_\lambda(x;\tfrac{1}{f}) &=& \ \Theta_{\frac{1}{f}}\,\omega\, Q_{\lambda'}(x;f) \\ (b) \quad & Q_\lambda(x;\tfrac{1}{f}) &=& \ \Theta_{\frac{1}{f}}\,\omega\, P_{\lambda'}(x;f) \\ (c) \quad & d_\lambda(\tfrac{1}{f}) &=& \ 1/d_\lambda(f) \end{aligned} \tag{2.2}$$

where Θ_f is the operator on symmetric functions defined by setting

$$\Theta_f\, p_\lambda(x) \;=\; p_\lambda(x)\, f_\lambda \ .$$

Proof.

Note that 2.1 (1) may be written as

$$\langle\, P(x;f)\,\rangle \;=\; \langle\, S(x)\,\rangle\, \xi \tag{2.3}$$

where ξ is a matrix which is *dominance* upper unitriangular. By this we mean that $\xi = \|\xi_{\mu\lambda}\|$ with $\xi_{\mu\lambda} \neq 0$ only if $\mu <_D \lambda$ and $\xi_{\mu\mu} = 1$. Clearly 2.1 (2) says that the basis $\{Q_\lambda(x;f)\}$ is given by

$$Q_\lambda(x;f) \;=\; d_\lambda(f)\, P_\lambda(x;f) \tag{2.4}$$

with

$$d_\lambda(f) \;=\; \frac{1}{\langle\, P_\lambda, P_\lambda\,\rangle_f} \ . \tag{2.5}$$

Note further that substituting III.1.28 into III.1.31 (and using III.1.29) we get

$$\Omega_f(x,y) \;=\; \langle\, S_\lambda(x)\,\chi^{-1} f\, \chi\, \langle\, S_\lambda(y)\,\rangle^T \ . \tag{2.6}$$

Now, the duality of $\{P_\lambda(x;f)\}$ and $\{Q_\lambda(x;f)\}$ combined with 2.3 and 2.4 yields

$$\Omega_f(x,y) \;=\; \{P_\lambda(x;f)\}\cdot\{Q_\lambda(x;f)\}^T \;=\; \langle\, S(x)\,\rangle\, \xi\, d\, \xi^T \langle\, S(y)\,\rangle^T \ .$$

Comparing with 2.6 we get

$$\chi^{-1} f\, \chi \;=\; \xi\, d\, \xi^T \ . \tag{2.7}$$

It is easy to see then that the Macdonald property for the sequence $\{f_n\}$ is simply the fact that the matrix on the left of this identity may be factored as in the expression on the right with a matrix ξ which is dominance upper unitriangular.

Now note that inverting both sides of 2.6 we deduce that

$$\chi^{-1} \tfrac{1}{f}\, \chi \;=\; \eta\, \tfrac{1}{d}\, \eta^T \tag{2.8}$$

Proposition 1.2 yields us that the polynomial

$$P_\lambda(x;q,t) \;=\; Z_\lambda(\delta_{q,t})\, S_\lambda(x) \;\;.$$

satisfies 1.1 a). Next, we need to check that 1.1 b) holds true as well. To this end let $\{Q_\lambda(x;q,t)\}$ be the basis dual to $\{P_\lambda(x;q,t)\}$ with respect to the scalar product $\langle\;,\;\rangle_{q,t}$ we see that 1.20 yields that

$$\sum_\lambda \omega_\lambda(q,t) P_\lambda(x;q,t)\, Q_\lambda(y;q,t) \;=\; \sum_\lambda P_\lambda(x;q,t)\, \delta^y_{q,t}\, Q_\lambda(y;q,t) \;\;.$$

We are thus forced to conclude that

$$\delta^y_{q,t}\, Q_\lambda(y;q,t) \;=\; \omega_\lambda(q,t)\, Q_\lambda(y;q,t) \;\;.$$

Since the $\omega_\lambda(q,t)$ are all distinct we deduce that for some coefficients $d_\lambda(q,t)$ we must have

$$Q_\lambda(x;q,t) \;=\; d_\lambda(q,t) P_\lambda(x;q,t)$$

which yields that

$$\Omega\big[XY\tfrac{1-t}{1-q}\big] \;=\; \sum_\lambda d_\lambda(q,t)\, P_\lambda(x;q,t)\, P_\lambda(y;q,t)$$

and thus we must have

$$\langle\, P_\lambda, P_\mu\, \rangle_{q,t} \;=\; 0 \quad \text{if } \lambda \neq \mu \;\;.$$

Finally, we observe that the uniqueness of the family $\{P_\lambda(x;q,t)\}$ is simply due to the fact that 1.1 a) yields that $P_\lambda(x;q,t)$ may be recursively computed by the Gramm-Schmidt orthogonalization process applied to the basis $\{S_\lambda(x)\}$ arranged in the lexicographically increasing order of their partition indexing. This completes our proof.

2. Duality

It will be convenient to say that the sequence $\{f_n\}$ has the Macdonald property if there exists a symmetric function basis $\{P_\lambda(x;f)\}$ with the properties

$$\begin{aligned} &(1) \quad P_\lambda(x;f) \;=\; S_\lambda(x;f) + \sum_{\mu <_D \lambda} S_\mu(x)\, \xi_{\mu\lambda}(f) \\ &(2) \quad \langle\, P_\lambda(x;f), P_\mu(x;f)\, \rangle_f \;=\; 0 \quad \text{if } \lambda \neq \mu \;\;. \end{aligned} \qquad 2.1$$

If such is the case we shall let $\{Q_\lambda(x;f)\}$ denote the basis dual to $\{P_\lambda(x;f)\}$ with respect to the scalar product $\langle\;,\;\rangle_f$ defined in III 1.32.

This given, the basic duality theorem of Macdonald [ ] may be stated as follows:

using this for $m = \sum_{i \notin T} \lambda_i$ and substituting in 1.17 we derive that

$$\begin{aligned} \delta_{q,t} P_\lambda &= \sum_{T \subseteq [1,k]} \prod_{i \notin T} p_{\lambda_i}(x) \prod_{j \in T} \left(-\frac{p_{\lambda_j}(1-q)}{(zt)^{\lambda_j}} \right) \left(\Omega[(t-1)Xz] \frac{t^n}{t-1} - \frac{1}{t-1} \right) \Big|_{z^0} \\ &= p_\lambda \left[X - \frac{1-q}{zt} \right] \left(\Omega[(t-1)Xz] \frac{t^n}{t-1} - \frac{1}{t-1} \right) \Big|_{z^0} . \end{aligned}$$

We can easily see that this reduces to 1.16 when $P = p_\lambda$ since

$$p_\lambda \left[X - \frac{1-q}{zt} \right] \Big|_{z^0} = p_\lambda[X] .$$

Let $\delta^x_{q,t}$ denote our original operator $\delta_{q,t}$ and $\delta^y_{q,t}$ be defined by 1.3 with the alphabet $X = x_1 + x_2 + \cdots + x_n$ replaced by the alphabet $Y = y_1 + y_2 + \cdots + y_n$.

The identity in 1.16 has the following immediate corollary:

Proposition 1.4 (Macdonald [ ])

$$\delta^x_{q,t} \Omega[XY \tfrac{1-t}{1-q}] = \delta^y_{q,t} \Omega\left[XY \tfrac{1-t}{1-q}\right]$$

Proof.

Note that 1.16 gives

$$\frac{\delta^x_{q,t} \Omega[XY(\frac{1-t}{1-q})]}{\Omega[XY(\frac{1-t}{1-q})]} = 1 + \frac{t^n}{t-1} \Omega\left[-\frac{(1-t)}{tz} Y \right] \Omega\left[-(1-t)Xz \right] \Big|_{z^0} . \qquad 1.18$$

Since we can write

$$\begin{aligned} \Omega\left[-\frac{(1-t)}{tz} Y \right] &= \sum_{m \geq 0} h_m[-(1-t)X] \; \frac{1}{t^m z^m} \\ \Omega\left[-(1-t)Yz \right] &= \sum_{m \geq 0} h_m[-(1-t)X] \; z^m . \end{aligned}$$

We see that we may rewrite 1.18 as

$$\frac{\delta^x_{q,t} \Omega[XY \frac{1-t}{1-q}]}{\Omega[XY \frac{1-t}{1-q}]} = 1 + \frac{t^n}{t-1} \sum_{m \geq 0} \frac{h_m[-(1-t)Y] \; h_m[-(1-t)X]}{t^m} . \qquad 1.19$$

Since the right hand side of this equation is invariant under the interchange of X and Y we must conclude that

$$\delta^x_{q,t} \Omega[XY \tfrac{1-t}{1-q}] = \delta^y_{q,t} \Omega[XY \tfrac{1-t}{1-q}] \qquad 1.20$$

as desired.

Proof of Theorem 1.1

Proposition 1.2

The polynomials

$$P_\lambda(x;q,t) \;=\; Z_\lambda(\delta_{q,t})S_\lambda \qquad\qquad \lambda\vdash n$$

form a basis for the space of homogeneous polynomials of degree n. Moreover we have

$$\begin{aligned} &a)\quad P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\lambda\,\xi_{\mu\lambda}(q,t)\\ &b)\quad \delta_{q,t}P_\lambda = \omega_\lambda P_\lambda \end{aligned} \qquad 1.15$$

Proof.

Property a) is an immediate consequence of the stated properties of the matrix $Z_\lambda(D)$. Property b) follows from 1.14. The fact that they are a basis is an immediate consequence of a) which implies that the matrix relating the P_λ *'s* to the S_λ *'s* is upper unitriangular and therefore invertible.

To derive further properties of the basis $\{P_\lambda(x;q,t)\}_{\lambda\vdash n}$ and for later purposes it will be good to express the action of $\delta_{q,t}$ in λ-ring notation.

Proposition 1.3

For any symmetric formal power series $P[X]$, $X = x_1+\cdots+x_n$, we have

$$\delta_{q,t}P \;=\; \frac{P}{1-t}+\frac{t^n}{t-1}\,P\left[X-\frac{1-q}{tz}\right]\Omega[(t-1)Xz]\Big|_{z^0} \qquad 1.16$$

Proof.

Clearly it is sufficient to show 1.16 for the power symmetric function basis. Let then $P = p_{\lambda_1}, p_{\lambda_2},\ldots,p_{\lambda_k}$. From the definition and 1.2 we then get that

$$\begin{aligned} \delta_{q,t}\,p_\lambda \;&=\; \sum_{s=1}^n \prod_{i=1}^{n}{}^{(s)}\,\frac{tx_s-x_i}{x_s-x_i}\,\prod_{j=1}^{k}\left(p_{\lambda_j}(x)+(q^{\lambda_j}-1)x_s^{\lambda_j}\right)\\ &=\; \sum_{T\subseteq[1,k]}\prod_{i\notin T}p_{\lambda_i}(x)\,\prod_{j\in T}\left(-p_{\lambda_j}(1-q)\right)\sum_{s=1}^n\prod_{i=1}^n{}^{(s)}\,\frac{tx_s-x_i}{x_s-x_i}\,x_s^{\sum_{j\in T}\lambda_j}\;. \end{aligned} \qquad 1.17$$

Now, the partial fraction expansion

$$\Omega\big[(t-1)Xz\big] \;=\; \frac{1}{t^n}+\frac{t-1}{t^n}\sum_{s=1}^n\prod_{i=1}^n{}^{(s)}\frac{tx_s-x_i}{x_s-x_i}\,\frac{1}{1-tx_sz}$$

yields us that

$$\frac{1}{t^m}\left(\Omega[(t-1)Xz]\frac{t^n}{t-1}-\frac{1}{t-1}\right)\Big|_{z^m} \;=\; \sum_{s=1}^n\prod_{i=1}^n{}^{(s)}\frac{tx_s-x_i}{x_s-x_i}x_s^m$$

this gives that the coefficient of S_μ in 1.7 vanishes if $\mu \not\leq \lambda$. Moreover, if $\mu = \lambda$, then all the inequalities in 1.8 must be equalities and this forces $\sigma = id$ which gives $p = \mu$. In other words we must have

$$\sum_{\lambda(p)=\lambda} \epsilon_\lambda(p)\, \omega_p(q,t) \;\; = \;\; \omega_\lambda(q,t) \;\; .$$

Thus the identity in 1.5 must hold true as asserted. Setting for a moment

$$\omega_{\mu\lambda} \;\; = \;\; \begin{cases} \sum_{\lambda(p)=\lambda} \epsilon_\mu(p)\, \omega_p(q,t) & \text{for } \mu \leq \lambda \\ 0 & \text{otherwise} \end{cases} \qquad 1.9$$

and letting $\omega = ||\omega_{\mu\lambda}||$ we may rewrite 1.5 in the form

$$\delta_{q,t}\langle\, m(x)\,\rangle \;\; = \;\; \langle\, S(x)\,\rangle\, \omega \qquad 1.10$$

where $\langle\, m\,\rangle$ and $\langle\, S\,\rangle$ denote respectively the monomial and Schur bases written in the lexicographic order of their partition index. Letting $K = ||K_{\lambda\mu}||$ denote the customary Kostka matrix, we may rewrite 1.10 in the form

$$\delta_{q,t}\langle\, S(x)\,\rangle \;\; = \;\; \langle\, S(x)\,\rangle\, \omega\, K^T \;\; .$$

This means that the matrix D of the operator $\delta_{q,t}$ in terms of the Schur basis may be expressed as

$$D \;\; = \;\; \omega\, K^T \;\; . \qquad 1.11$$

Since K^T is upper unitriangular, we see from 1.5 that D is upper triangular with the same diagonal elements as ω. In other words we may write D in the form

$$D \;\; = \;\; \text{diag}\;(\omega_\lambda(q,t)) + U \;\; , \qquad 1.12$$

with U a strictly upper triangular matrix. Note that for generic q, t we can have

$$\omega_\lambda(q,t) \;\; = \;\; \sum_{s=1}^{n} t^{n-s} q^{\lambda_s} \;\; = \;\; \sum_{s=1}^{n} t^{n-s} q^{\mu_s}$$

only if $\lambda = \mu$. Dropping for a moment the dependence in q and t we see that the matrix

$$Z_\lambda(D) \;\; = \;\; \prod_{\mu}^{(\lambda)} \frac{D - \omega_\mu\, I}{(\omega_\lambda - \omega_\mu)} \qquad 1.13$$

(with I the identity matrix) is well defined and upper triangular. Moreover, 1.12 yields that,the diagonal element of $Z_\lambda(D)$ that is in position λ is equal to one and all the other diagonal elements are equal to zero. Finally, a standard argument which is easily put together shows that

$$DZ_\lambda(D) \;\; = \;\; \omega_\lambda Z_\lambda(D) \;\; . \qquad 1.14$$

These properties of $Z_\lambda(D)$ may be translated into the following basic proposition

For a given vector $p = (p_1, p_2, \ldots, p_n)$ let us set

$$\omega_p(q,t) \;=\; \sum_{s=1}^{n} t^{n-s} q^{p_s} \quad .$$

We have the following remarkable identity:

Proposition 1.1 (Macdonald [ ])

$$\delta_{q,t}\, m_\lambda \;=\; \omega_\lambda(q,t) S_\lambda + \sum_{\mu<\lambda} S_\mu \sum_{\lambda(p)=\lambda} \epsilon_\mu(p)\, \omega_p(q,t) \qquad 1.5$$

where

$$\epsilon_\mu(p) \;=\; \begin{cases} \text{sign } (\sigma) & \text{if } p_{\sigma_i} + n - \sigma_i = \mu_i + n - i, \\ 0 & \text{otherwise} \end{cases} \qquad 1.6$$

and $\lambda(p) = \lambda$ means that p may be rearranged to λ.

Proof.

The monomial symmetric function $m_\lambda(x)$ may be written in the form

$$m_\lambda(x) \;=\; \sum_{\lambda(p)=\lambda} x^p \;=\; \sum_{\lambda(p)=\lambda} x_1^{p_{\sigma_1}} x_2^{p_{\sigma_2}} \cdots x_n^{p_{\sigma_n}} \qquad \forall \sigma \in S_n \quad .$$

Thus using 1.4 we get

$$\begin{aligned} \delta_{q,t}\, m_\lambda \;&=\; \sum_{\lambda(p)=\lambda} \frac{1}{\Delta(x)} \sum_{s=1}^{n} \sum_{\sigma\in S_n} \epsilon(\sigma)\, t^{n-\sigma_s} q^{p_{\sigma_s}} x_1^{p_{\sigma_1}+n-\sigma_1} \cdots x_s^{p_{\sigma_s}+n-\sigma_s} \cdots x_n^{p_{\sigma_n}+n-\sigma_n} \\ &=\; \sum_{\lambda(p)=\lambda} \frac{\omega_p(q,t)}{\Delta(x)} \det \|x_i^{p_j+n-j}\| \quad . \end{aligned}$$

Now it is well known [ ] and easy to prove that

$$\frac{\det \|x_i^{p_j+n-j}\|}{\det \|x_i^{n-j}\|} \;=\; \begin{cases} \epsilon(\sigma) S_\mu & \text{if } p_{\sigma_j} + n - \sigma_j = \mu_j + n - j, \\ 0 & \text{otherwise.} \end{cases}$$

This gives that

$$\delta_{q,t}\, m_\lambda \;=\; \sum_{\mu} S_\mu(x) \sum_{\lambda(p)=\lambda} \epsilon_\mu(p)\, \omega_p(q,t) \quad . \qquad 1.7$$

Note that if for some $\sigma \in S_n$, $\quad p_{\sigma_j} + n - \sigma_j = \mu_j + n - j$ then for $i = 1, \ldots, n$ we have

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_i + 1 + 2 + \cdots + i &\geq p_{\sigma_1} + p_{\sigma_2} + \cdots + p_{\sigma_i} + 1 + 2 + \cdots + i \\ &= \mu_1 + \mu_2 + \cdots + \mu_i + \sigma_1 + \sigma_2 + \cdots + \sigma_i \\ &\geq \mu_1 + \mu_2 + \cdots + \mu_i + 1 + 2 + \cdots + i \qquad 1.8 \end{aligned}$$

CHAPTER IV.

Orbits and Kostka-Macdonald coefficients.

1. The Macdonald polynomials.

The scalar product $\langle\ ,\ \rangle_f$ corresponding to the sequence

$$f_n \;=\; \frac{1-t^n}{1-q^n}$$

will here and after be denoted by $\langle\ ,\ \rangle_{q,t}$. In this section we shall review the proof of the following fundamental result of Macdonald.

Theorem 1.1

For each partition λ, and generic q,t there exists a unique family of polynomials $\{P_\lambda(x;q,t)\}_\lambda$ such that

$$\begin{aligned} a)\quad & P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu\, \xi_{\mu\lambda}(q,t) \\ b)\quad & \langle\, P_\lambda\ ,\ P_\mu\,\rangle_{q,t} \;=\; 0 \quad for \quad \lambda\neq\mu \end{aligned} \tag{1.1}$$

We shall derive this from a sequence of auxiliary results which represent the bare minimum necessary to carry through Macdonald's original argument. Since our goal here is to review the background needed for our understanding the material which follows, we shall have to refer the reader to the original manuscript for the more extensive and complete treatment. We shall start by reviewing some notation.

For a given $1\leq s\leq n$ let $T_q^{(s)}$ be the operation on formal power series in $(x_1,\ldots,x_n)$ defined by setting

$$T_q^{(s)}P(x_1,\ldots,x_n) \;=\; P(x_1,\ldots,x_sq,\ldots,x_n)\ .$$

Note that in λ-ring notation

$$T_q^{(s)}P[X] \;=\; P\big[X+(q-1)x_s\big]\ . \tag{1.2}$$

This given, let us set

$$\delta_{q,t}P \;=\; \sum_{s=1}^n \left(\prod_{i=1}^{n}{}^{(s)}\,\frac{tx_s-x_j}{x_s-x_j}\right) T_q^{(s)}P\ . \tag{1.3}$$

Note that, if $\Delta(x)=\prod_{1\leq i<j\leq n}(x_i-x_j)$ is the Vandermonde determinant, we can also write

$$\delta_{q,t}P \;=\; \frac{1}{\Delta(x)}\sum_{s=1}^n \left(T_t^{(s)}\Delta(x)\right) T_q^{(s)}P\ . \tag{1.4}$$