CHAPTER III.

Orbits and Kostka-Foulkes polynomials

1. Review of λ-ring notation.

In the next two sections we shall have to review a number of identities for symmetric functions.* Our presentation will be greatly simplified if we make use of a formalism which has come to be referred to as "λ-ring" notation. To make our treatment self-contained, we shall give here a brief informal review of the basic constructs. The reader is referred to [ ] for a more formal and rigorous approach.

Our main need is to be able to represent in a convenient and helpful way the operation of **substitution** of a symmetric function Q **into** a symmetric function P. We shall denote here the result of this operation by the symbol $P[Q]$ to make sure that we remember what goes in and what stays out. This operation will be restricted here to formal series Q with integer coefficients. Clearly we can represent such a Q in the form

$$Q \;=\; Q^+ - Q^-$$

where

$$Q^{\pm} \;=\; \sum_{x^p \in M^{\pm}} x^p$$

where M^+ and M^- are two multisets sets of monomials. For instance, if

$$Q \;=\; 3x_1^2y_2 + 2x_1x_2x_3 - 2x_2y_3^2 - 3y_3^2$$

then

$$\begin{aligned} M^+ &= \{x_1^2y_2, x_1^2y_2, x_1^2y_2, x_1x_2x_3, x_1x_2x_3\} \\ M^- &= \{x_2y_2^2, x_2y_3^2, y_3^2, y_3^2, y_3^2\} \ . \end{aligned}$$

When $Q^- = 0$ then $P[Q]$ has a very natural definition. Indeed, if

$$M^+ \;=\; \{m_1, m_2, \ldots, m_N\}$$

then $P[Q]$ is the polynomial obtained by substituting the monomials of M for the variables of P. More precisely we first write P as a polynomial in N variables

$$P \;=\; P[y_1, y_2, \ldots, y_N]$$

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* Here and in the following *a function* means *a formal power series* in particular it also means a *polynomial* in the given variables

then let

$$P[Q] \;=\; P(y_1, y_2, \ldots, y_N)\Big|_{y_i=m_i} \tag{1.1}$$

We can easily see from this that if P and Q are both symmetric in the variables $x_1, x_2, \ldots, x_n$, then $P[Q]$ will also be symmetric, moreover $P[Q]$ will be homogeneous of degree $p \cdot q$ if P and Q themselves are homogeneous of degree p and q respectively.

Note that when P is the power symmetric function p_s then 1.1 reduces to

$$p_s[Q] \;=\; \sum_{i=1}^{N} (m_i)^s \;. \tag{1.2}$$

From this we easily deduce the two basic properties

$$\begin{aligned} (i) \quad p_s[Q_1+Q_2] \;&=\; p_s[Q_1]+p_s[Q_2] \\ (ii) \quad p_s[Q_1Q_2] \;&=\; p_s[Q_1]\, p_s[Q_2] \;. \end{aligned} \tag{1.3}$$

The idea is then to use these properties as the starting point for extending the definition of $P[Q]$ to the general case.

In other words we shall simply let our definition be a consequence of the requirement that 1.3 (i) and (ii) be valid in full generability. In particular, (i) forces

$$p_s[0]=0 \quad \textit{and} \quad p_s[-Q^-]=-p_s[Q^-] \;.$$

Thus we must take

$$p_s[Q^+-Q^-] \;=\; p_s[Q^+]-p_s[Q^-] \tag{1.4}$$

This given, the evaluation of $P[Q]$ can be routinely carried out by expanding P in terms of the power symmetric function basis and then using 1.4.

It is best to illustrate all this with a few examples we shall need in the sequel. For instance, let us see what we get if $P=h_n$ and $Q=(1-t)X$ with $X=x_1+x_2+\cdots+x_m$. Now the expansion of the homogeneous symmetric function h_n in terms of the power basis may be written as

$$h_n \;=\; \sum_{\rho\vdash n} p_\rho/z_\rho \tag{1.5}$$

with

$$z_\rho \;=\; 1^{m_1}2^{m_2}3^{m_3}\ldots m_1!m_2!m_3!\ldots \qquad\qquad (\text{if } \rho=1^{m_1}2^{m_2}3^{m_3}\ldots) \;.$$

Thus, 1.4 gives

$$p_s\big[(1-t)X\big] \;=\; (1-t^s)\sum_{i=1}^{m} x_i^s \;,$$

and we must have

$$h_n\big[(1-t)X\big] \;=\; \sum_{\rho\vdash n} \frac{p_\rho(x_1,\ldots,x_m)}{z_\rho} \prod_i (1-t^{\rho_i}) \;. \tag{1.6}$$

Interpreting $X/(1-q)$ as the formal power series

$$\sum_{i=0}^{\infty} q^i X$$

we are, in the same manner, led to the expansion

$$h_n\big[\tfrac{X}{1-q}\big] \;=\; \sum_{\rho\vdash n} \frac{p_\rho(x_1,\ldots,x_n)}{z_\rho} \prod_i \frac{1}{1-q^{\rho_i}} \;. \tag{1.7}$$

In our operation we need not restrict P itself to be a polynomial. For instance it is customary to let Ω denote the basic symmetric function kernel

$$\Omega(x_1,\ldots,x_m) \;=\; \prod_{i=1}^{m} \frac{1}{1-x_i} \;. \tag{1.8}$$

Note that since we may write

$$\Omega \;=\; \sum_{n\geq 0} h_n \;=\; \sum_{\rho} \frac{p_\rho}{z_\rho} \;=\; \exp\Big\{\sum_{s\geq 1} \frac{1}{s}\, p_s\Big\}$$

then a simple calculation based on 1.4 leads to the direct formula

$$\Omega[Q^+ - Q^-] \;=\; \prod_{m\in M^+} \frac{1}{1-m} \prod_{m\in M^-} (1-m) \;. \tag{1.9}$$

Thus we can easily see that we must have for $X = x_1 + x_2 + \cdots + x_M$, $Y = y_1 + y_2 + \cdots + y_N$

$$\Omega\big[(1-t)X\,Y\big] \;=\; \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{1-tx_iy_j}{1-x_iy_j} \;. \tag{1.10}$$

The later is usually referred to as the Hall-Littlewood kernel.

Allowing both P and Q to be formal power series, the Macdonald kernel [ ]

$$\Omega_{qt}(x,y) \;=\; \prod_{ij}\prod_{k\geq 0} \frac{1-tx_iy_jq^k}{1-x_iy_jq^k} \tag{1.11}$$

may then simply be written in the form

$$\Omega_{qt} \;=\; \Omega\big[\tfrac{1-t}{1-q} X\,Y\big] \;. \tag{1.12}$$

We should also note that 1.3 (ii) allows us to extend the identity

$$\Omega(x,y) \;=\; \prod \frac{1}{1-x_iy_j} \;=\; \sum_{\lambda} S_\lambda(x) S_\lambda(y) \;\;,$$

to

$$\Omega[PQ] \;=\; \sum_{\lambda} S_\lambda[P]\, S_\lambda[Q] \qquad 1.13$$

which we shall here and after refer to as the *general Cauchy Identity*.

In particular, we also have

$$h_n[PQ] \;=\; \sum_{\lambda\vdash n} S_\lambda[P]\, S_\lambda[Q] \;\; .$$

Thus setting $P=X$ and $Q=\frac{1}{1-q}$ we deduce that

$$h_n\big[\tfrac{X}{1-q}\big] \;=\; \sum_{\lambda\vdash n} S_\lambda(x)\, S_\lambda\big[\tfrac{1}{1-q}\big] \;\; .$$

To complete our calculation we use the identity I.3.12

$$S_\lambda\big[\tfrac{1}{1-q}\big] \;=\; S_\lambda[1,q,q^2,\ldots] \;=\; \sum_{T\in ST(\lambda)} \frac{q^{c(T)}}{(1-q)(1-q^2)\cdots(1-q^n)} \;\; .$$

which gives

$$h_n\big[\tfrac{X}{1-q}\big] \;=\; \sum_{\lambda\vdash n} S_\lambda(x) \sum_{T\in ST(\lambda)} \frac{q^{c(T)}}{(1-q)(1-q^2)\cdots(1-q^n)} \;\; . \qquad 1.14$$

Let us recall that the Frobenius isomorphism F between the center of the algebra of S_n and the space $SY(n)$ of homogeneous symmetric polynomials of degree n is given by the formula

$$F\,\chi = \frac{1}{n!}\sum_{\sigma\in S_n} \chi(\sigma)\, p_{\lambda(\sigma)}(x) \qquad 1.15$$

with

$$\lambda(\sigma) \;=\; 1^{m_1(\sigma)} 2^{m_2(\sigma)} \cdots n^{m_n(\sigma)} \qquad 1.16$$

where $m_i(\sigma)$ denotes the number of cycles of length i in the cycle factorization of σ. Denoting by χ_ρ the value of χ at the permutations with cycle structure ρ, 1.15 may also be written as

$$F\,\chi \;=\; \sum_{\rho\vdash n} \chi_\rho\, \frac{p_\rho(x)}{z_\rho} \;\; . \qquad 1.17$$

It is well known [ ] that F maps the irreducible character χ^λ into the Schur function S_λ. That is, we have

$$S_\lambda \;=\; \frac{1}{n!}\sum_{\sigma\in S_n} \chi^\lambda(\sigma)\, p_{\lambda(\sigma)} \;=\; \sum_{\rho} \chi^\lambda_\rho\, p_\rho/z_\rho \;\; . \qquad 1.18$$

Thus the Frobenius image of the q-character in I.3.10 has the Schur function expansion

$$F\chi^R \;=\; \sum_{\lambda} S_\lambda(x)\, S_\lambda\big(\tfrac{1}{1-q}\big) \;\; .$$

Comparing with 1.14 we see that we may now write

$$F\chi^{R} \;=\; h_n\big[\tfrac{X}{1-q}\big] \;. \tag{1.19}$$

In the same vein we can rewrite formula II.5.8 as

$$F\chi^{H_{S_n}} \;=\; (1-q)(1-q^2)\cdots(1-q^n)\;\; h_n\big(\tfrac{X}{1-q}\big) \;. \tag{1.20}$$

Let us recall that the space SY of symmetric formal power series in an infinite number of variables has an involution ω given by setting

$$\omega\, h_n \;=\; e_n \tag{1.21}$$

and it is well known that

$$\omega\, S_\lambda \;=\; S_{\lambda'} \;. \tag{1.22}$$

This involution has a remarkably simple expression in λ-ring notation. Namely, we have

Proposition 1.1

If P is a homogeneous symmetric function of degree n then for all Q we have

$$(\omega\, P)[Q] \;=\; (-1)^n P[-Q] \;. \tag{1.23}$$

Proof.

It is sufficient to verify this for the power symmetric function basis. Now since 1.21 implies that

$$\omega\, \Omega[X] \;=\; \prod_{x\in X} (1+x) \;,$$

we see that we must have

$$\omega \exp\Big(\sum_{s\ge 1} \frac{1}{s}\, p_s[X]\Big) \;=\; \exp\Big(\sum_{s\ge 1} \frac{(-1)^{s-1}}{s}\, p_s[X]\Big) \;.$$

Thus we must necessarily have

$$\omega p_s[X] \;=\; (-1)^{s-1} p_s(x) \;=\; (-1)^s p_s[-X] \;.$$

This shows 1.23 for $P=p_s$. The result for

$$P \;=\; p_\rho \;=\; \prod_i p_{\rho_i} \qquad\qquad \rho=(\rho_1,\rho_2,\ldots,\rho_k)\vdash n$$

then follows from 1.4 (ii).

Q. E. D.

For simplicity it is best to carry out some of our computations in matrix notation. To this end all our matrices are assumed to have entries $a_{\lambda\mu}$ indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ arranged in lexicographic order. This given the relation 1.18

$$S_\lambda \;=\; \sum_\mu \chi^\lambda_\mu \, p_\mu / z_\mu \tag{1.25}$$

yielding the connection between the Schur and power symmetric function bases, will be written in the form

$$\langle\, S(x) \,\rangle \;=\; \langle\, p(x) \,\rangle \, \tfrac{1}{z} \, \chi \tag{1.26}$$

where $\langle\, S(x) \,\rangle$ and $\langle\, p(x) \,\rangle$ respectively denote the bases $\{S_\lambda(x)\}$ and $\{p_\lambda(x)\}$ written as row vectors in the lexicographic order of their partition index, z denotes the diagonal matrix with entries z_μ and $\chi = \|\chi^\lambda_\mu\|$ is the matrix which at the intersection of row μ with column λ has the value of the irreducible character χ^λ at permutations with cycle structure μ. By contrast the reverse expansion

$$p_\mu \;=\; \sum_\lambda S_\lambda \, \chi^\lambda_\mu \tag{1.27}$$

will then have to be written in the form

$$\langle\, p(x) \,\rangle \;=\; \langle\, S(x) \,\rangle \, \chi^T \;\; . \tag{1.28}$$

Here the superscript "T" denotes transposition. In particular by combining 1.26 with 1.28 we obtain the character relations

$$\chi^T \tfrac{1}{z} \, \chi \;=\; \chi \, \chi^T \tfrac{1}{z} \;=\; I \;\; . \tag{1.29}$$

For a given sequence $\{f_n\}_{n \geq 0}$ (with $f_n \neq 0 \;\forall n$) set

$$f_\rho \;=\; f_{\rho_1} f_{\rho_2} \cdots f_{\rho_k}$$

and let f denote the diagonal matrix whose entries are the f_ρ's. In this vein, the kernel

$$\Omega_f(x,y) \;=\; \sum_\rho p_\rho(x) \, p_\rho(y) \, \frac{f_\rho}{z_\rho} \tag{1.30}$$

may be written in the form

$$\Omega_f(x,y) \;=\; \langle\, p(x) \,\rangle \, \frac{f}{z} \, \langle\, p(y) \,\rangle^T \;\; . \tag{1.31}$$

The kernel defined by 1.30 is associated with the scalar product $\langle \;,\; \rangle_f$ defined by setting

$$\langle\, p_{\rho_1}, p_{\rho_2} \,\rangle_f \;=\; \begin{cases} \frac{z_\rho}{f_\rho} & \text{if } \rho_1 = \rho_2 = \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{1.32}$$

The following fact is basic in the present developments

Proposition 1.2

The bases $\{P_\lambda(x;f)\}$, $\{Q_\lambda(x;f)\}$ *are dual with respect to the scalar product* $\langle\ ,\ \rangle_f$ *if and only if*

$$\Omega_f(x,y) \;=\; \sum_\lambda P_\lambda(x;f)\, Q_\lambda(y;f) \qquad 1.33$$

Proof.

The defining relation in 1.32 may be written in matrix notation in the form

$$\langle\, p(x)\,\rangle^T \cdot_f \langle\, p(x)\,\rangle \;=\; \frac{z}{f} \qquad 1.34$$

where the symbol "$\cdot_f$" denotes the operation of taking the scalar product $\langle\ ,\ \rangle_f$. This given, if

$$\langle\, P(x;f)\,\rangle \;=\; \langle\, p(x)\,\rangle\, A \quad \text{ and } \quad \langle\, Q(x;f)\,\rangle = \langle\, p(x)\,\rangle\, B \qquad 1.35$$

give the expansions of the bases $\{P_\lambda(x;f)\}$ and $\{Q_\lambda(x;f)\}$ in terms of the power symmetric function basis $\{p_\lambda(x)\}$, then the duality of $\{P_\lambda(x;f)\}$ and $\{Q_\lambda(x;f)\}$ can be expressed in the form

$$\langle\, P(x;f)\,\rangle^T \cdot_f \langle\, Q(x;f)\,\rangle \;=\; A^T \frac{z}{f} B \;=\; I\ . \qquad 1.36$$

On the other hand the identity in 1.33 simply says that

$$\langle\, p(x)\,\rangle \frac{f}{z} \langle\, p(y)\,\rangle^T \;=\; \langle\, P(x;f)\,\rangle \langle\, Q(y;f)\,\rangle^T \qquad 1.37$$

or, which is the same, (using 1.35)

$$\langle\, p(x)\,\rangle \frac{f}{z} \langle\, p(y)\,\rangle^T \;=\; \langle\, p(x)\,\rangle\, A\, B^T \langle\, p(y)\,\rangle^T\ .$$

But this holds true if and only if

$$\frac{f}{z} \;=\; A\, B^T$$

which is easily seen to be equivalent to 1.36.

2. Review of Hall-Littlewood polynomials.

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Z_k = \{z_1, z_2, \ldots, z_k\}$ be two alphabets. Our starting point is the formula

$$Q_\mu[X_n;t] \;=\; \prod_{1\le i<j\le k} \frac{1 - z_j/z_i}{1 - t z_j/z_i}\, \Omega[(1-t)X_n Z_k]\, \big|_{z_1^{\mu_1}\cdots z_k^{\mu_k}} \qquad 2.1$$

which we take as definition of the Hall-Littlewood polynomial $Q_\mu[X_n;t]$. Otherwise we shall use the same notation as in the previous section. In particular our next ingredient is the Hall-Littlewood scalar product $\langle\ ,\ \rangle_t$. This is the special case $f_\rho = 1/p_\rho[1-t]$ of 1.32. That is we set for the power symmetric function basis:

$$\langle\, p_\mu, p_\nu\,\rangle_t \;=\; \begin{cases} 0 & \text{if } \mu \neq \nu\ , \\ z_\mu / p_\mu[1-t] & \text{if } \mu = \nu \end{cases} \qquad 2.2$$

Our goal in this section is to show that the family of polynomials $\{Q_\mu[X_n;t]\}_\mu$ is orthogonal with respect to the scalar product $\langle\ ,\ \rangle_t$. We shall also determine the norm $\langle\, Q_\mu, Q_\mu \,\rangle_t$ and derive a number of identities that will be of crucial use in the developments of the next section.

All of this can be carried out quite easily by means of the following basic recursion.

Theorem 2.1

For any partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$

$$Q_{\mu_1,\ldots,\mu_k}[X_n;t] \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\,\frac{x_s - tx_i}{x_s - x_i}\; x_s^{\mu_1}\; Q_{\mu_2,\ldots,\mu_k}[X_n - x_s;t] \tag{2.3}$$

where the superscript (s) *is to indicate that, in the product,* i *is not to take the value* s

Proof

The defining formula may be rewritten as

$$Q_{\mu_1,\ldots,\mu_k}[X_n;t] \;=\; \prod_{1<j\leq k}\frac{1 - z_j/z_1}{1 - tz_j/z_1}\,\Omega[(1-t)X_n z_1]\,\Big|_{z_1^{\mu_1}}\;\; REST\,\Big|_{z_2^{\mu_2}\cdots z_k^{\mu_k}} \tag{2.4}$$

where

$$REST \;=\; \prod_{2\leq i<j\leq k}\frac{1 - z_j/z_i}{1 - tz_j/z_i}\,\Omega[(1-t)X_n(Z_k - z_1)]\;\;. \tag{2.5}$$

Setting for a moment

$$\prod_{1<j\leq k}\frac{1 - z_j x}{1 - tz_j x} \;=\; \sum_{m\geq 0} c_m(t)\,x^m \tag{2.6}$$

and using the partial fraction expansion

$$\Omega[(1-t)X_n z] \;=\; t^n \;+\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\,\frac{x_s - tx_i}{x_s - x_i}\;\frac{1}{1 - x_i z}$$

formula 2.4 becomes

$$Q_\mu[X_n;t] \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\,\frac{x_s - tx_i}{x_s - x_i}\Big(\sum_{m\geq 0}\frac{c_m(t)}{z_1^m}\Big)\frac{1}{1 - x_s z_1}\;\;\Big|_{z_1^{\mu_1}}\;\; REST\,\Big|_{z_2^{\mu_2}\cdots z_k^{\mu_k}}$$

and this is equivalent to

$$Q_\mu[X_n;t] \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\,\frac{x_s - tx_i}{x_s - x_i}\Big(\sum_{m\geq 0} c_m(t)x_s^{m+\mu_1}\Big)\;REST\,\Big|_{z_2^{\mu_2}\cdots z_k^{\mu_k}}\;\;.$$

Making the substitution $x = x_s$ in 2.6 gives (using 2.5)

$$\begin{aligned}Q_\mu[X_n;t] \;=\; (1-t)\sum_{s=1}^{n}\prod_{i=1}^{n}{}^{(s)}\,\frac{x_s - tx_i}{x_s - x_i}\,x_s^{\mu_1}\prod_{1<j\leq k}\frac{1 - z_j x_s}{1 - tz_j x_s}&\\ \prod_{2\leq i<j\leq k}\frac{1 - z_j/z_i}{1 - tz_j/z_i}\,\Omega[(1-t)X_n(Z_k - z_1)]\,\Big|_{z_2^{\mu_2}\cdots z_k^{\mu_k}}&\;.\end{aligned}$$

However we can carry out the cancellations

$$\prod_{1<j\leq k} \frac{1-z_j x_s}{1-tz_j x_s}\ \Omega[(1-t)X_n(Z_k-z_1)] \;=\; \Omega[(1-t)(X_n-x_s)(Z_k-z_1)]$$

and obtain

$$\begin{aligned} Q_\mu[X_n;t] \;=\; (1-t) \sum_{s=1}^{n} \prod_{i=1}^{n}{}^{(s)} \frac{x_s-tx_i}{x_s-x_i}\, x_s^{\mu_1} \\ \times \prod_{2\leq i<j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i}\, \Omega[(1-t)(X_n-x_s)(Z_k-z_1)]\,\big|_{z_2^{\mu_2}\cdots z_k^{\mu_k}} \end{aligned} \qquad 2.7$$

Using 2.1 with μ specialized to the partition $\mu^* = (\mu_2,\ldots,\mu_k)$, we can easily see that 2.7 is but another way of writing 2.4.

Another result interesting in its own right may be stated as follows.

Theorem 2.2

For any symmetric polynomial $R[X_n]$ and any partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ we have

$$\langle\, R\,,\, Q_\mu \,\rangle_t \;=\; R[Z_k] \prod_{1\leq i<j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i}\,\big|_{z_1^{\mu_1}\cdots z_k^{\mu_k}} \qquad 2.8$$

Proof

From the definition 2.1 we get (using the linerity of the scalar product)

$$\langle\, R\,,\, Q_\mu \,\rangle_t \;=\; \langle\, R[X_n]\,,\, \Omega[(1-t)X_nZ_k] \,\rangle_t \prod_{1\leq i<j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i}\,\big|_{z_1^{\mu_1}\cdots z_k^{\mu_k}}\ . \qquad 2.9$$

On the other hand the definition of $\langle\ ,\ \rangle_t$ yields that for any power symmetric function $p_\mu = p_{\mu_1}\cdots p_{\mu_k}$ we have

$$\langle\, p_\mu[X_n]\,,\, \Omega[(1-t)X_nZ_k] \,\rangle_t \;=\; \sum_\rho \frac{p_\rho[Z_k]p_\rho[1-t]}{z_\rho}\, \langle\, p_\mu\,,\, p_\rho \,\rangle_t \;=\; p_\mu[Z_k]\ .$$

Making use of this in 2.9 (for $R = p_\mu$) yields

$$\langle\, p_\mu\,,\, Q_\mu \,\rangle_t \;=\; p_\mu[Z_k] \prod_{1\leq i<j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i}\,\big|_{z_1^{\mu_1}\cdots z_k^{\mu_k}}\ .$$

This verifies the identity 2.8 for the power basis. Thus it must hold true for all symmetric functions as desired.

We are now in a position to prove the orthogonality result.

Theorem 2.3

For any two partitions λ and μ we have

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \prod_{i=1}^n (t)_{m_i} & \text{if } \lambda = \mu = 1^{m_1} 2^{m_2} \cdots n^{m_n} \end{cases} \quad , \tag{2.10}$$

where for any integer m we set $(t)_m = (1-t)(1-t^2)\cdots(1-t^m)$.

Proof

It is not difficult to see that only monomials of degree $\mu_1 + \mu_2 + \cdots + \mu_k$ can come out of the the right hand side of 2.1. That means that $Q_\mu[X_n;t]$ will necessarily be a homogeneous symmetric polynomial of that degree. Thus the orthogonality of Q_λ and Q_μ when λ and μ are partitions of different integers is trivial. We may assume that λ and μ are both partitions of n. Clearly, formula 2.1 is not affected if we add a number of zero parts to a partition. Thus we shall set

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0) \quad , \quad \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0) \ . \tag{2.11}$$

It is also convenient to let

$$\lambda^* = (\lambda_2, \lambda_3, \ldots, \lambda_n) \ , \ \mu^* = (\mu_2, \mu_3, \ldots, \mu_n) \ .$$

Clearly by the symmetry of the scalar product we can also assume that

$$\lambda_1 \geq \mu_1 \tag{2.12}$$

Under these conventions we shall show that

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; \begin{cases} 0 & \text{if } \lambda_1 > \mu_1 \\ (t)_m \langle\, Q_{\lambda^*} \,,\, Q_{\mu^*} \,\rangle_t & \text{if } \lambda_1 = \mu_1 \end{cases} \tag{2.13}$$

where m denotes the multiplicity of λ_1 in λ. This given, 2.10 will immediately follow by induction on n, (the theorem is trivially true for $n = 0$).

Now, it develops that 2.13 is a simple consequence of the identities 2.3 and 2.8. In fact, using 2.8 we can write

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; Q_\lambda[Z_n] \prod_{1 \leq i < j \leq n} \frac{1 - z_j/z_i}{1 - t z_j/z_i} \Big|_{z_1^{\mu_1} \cdots z_n^{\mu_n}} \ , \tag{2.14}$$

on the other hand 2.3 with μ replaced by λ and the $x_i's$ by the $z_i's$ gives

$$\langle\, Q_\lambda, Q_\mu \,\rangle_t = (1-t) \sum_{s=1}^n \prod_{i=1}^n {}^{(s)} \frac{z_s - t z_i}{z_s - z_i} \, z_s^{\lambda_1} \, Q_{\lambda^*}[Z_n - z_s; t] \prod_{1 \leq i < j \leq n} \frac{1 - z_j/z_i}{1 - t z_j/z_i} \Big|_{z_1^{\mu_1} \cdots z_n^{\mu_n}} \ . \tag{2.15}$$

Note that we can write

$$\begin{aligned} \prod_{i=1}^n {}^{(s)} \frac{z_s - t z_i}{z_s - z_i} \prod_{1 \leq i < j \leq n} \frac{z_i - z_j}{z_i - t z_j} \;&=\; \frac{z_s - t z_1}{z_s - z_1} \cdots \frac{z_s - t z_{s-1}}{z_s - z_{s-1}} \\ &\quad \times \ \frac{z_1 - z_s}{z_1 - t z_s} \cdots \frac{z_{s-1} - z_s}{z_{s-1} - t z_s} \prod_{1 \leq i < j \leq n} {}^{(s)} \frac{z_i - z_j}{z_i - t z_j} \end{aligned}$$

where the superscript (s) in the last product is to indicate that neither i nor j are to take the value s. Carrying out the cancellations we get

$$\frac{tz_1 - z_s}{z_1 - tz_s} \cdots \frac{tz_{s-1} - z_s}{z_{s-1} - tz_s} \prod_{1\le i<j\le n}^{(s)} \frac{z_i - z_j}{z_i - tz_j}$$

and substituting in 2.15 gives

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t) \sum_{s=1}^{n} \prod_{i=1}^{s-1} \frac{tz_i - z_s}{z_i - tz_s}\, z_s^{\lambda_1}\, Q_{\lambda^*}[Z_n - z_s; t] \prod_{1\le i<j\le n}^{(s)} \frac{z_i - z_j}{z_i - tz_j} \Big|_{z^\mu} \;.$$

Taking first the coefficient of $z_s^{\mu_s}$ in the s^{th} summand reduces this to

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t) \sum_{s=1}^{n} \prod_{i=1}^{s-1} \frac{tz_i - z_s}{z_i - tz_s} \Big|_{z_s^{\mu_s-\lambda_1}} Q_{\lambda^*}[Z_n - z_s; t] \prod_{1\le i<j\le n}^{(s)} \frac{z_i - z_j}{z_i - tz_j} \Big|_{z^\mu / z_s^{\mu_s}} \;.$$

This clearly evaluates to zero if $\lambda_1 > \mu_1$. On the other hand, when $\lambda_1 = \mu_1$ we see that the first factor in the s^{th} summand evaluates to t^{s-1} if $\mu_s = \lambda_1$ and yields zero if $\mu_s < \lambda_1$. This implies that

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t) \sum_{s=1}^{m} t^{s-1}\, Q_{\lambda^*}[Z_n - z_s; t] \prod_{1\le i<j\le n}^{(s)} \frac{z_i - z_j}{z_i - tz_j} \Big|_{z^\mu / z_s^{\mu_s}} \;.$$

Making a further use of 2.9, we get that when $\lambda_1 = \mu_1$

$$\langle\, Q_\lambda \,,\, Q_\mu \,\rangle_t \;=\; (1-t) \Big(\sum_{s=1}^{m} t^{s-1}\Big)\; \langle\, Q_{\lambda^*} \,,\, Q_{\mu^*} \,\rangle_t \;.$$

This establishes 2.13 as desired.

This theorem has the following immediate corollary:

Theorem 2.4

For $\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n}$ *set*

$$d_\mu(t) \;=\; \prod_{i=1}^{n} (t)_{m_i} \qquad 2.16$$

and let

$$P_\mu[X_n; t] \;=\; \frac{1}{d_\mu(t)}\, Q_\mu[X_n; t] \;. \qquad 2.17$$

Then

$$\Omega[(1-t)X_n Y_n] \;=\; \sum_{\mu} Q_\mu[X_n; t] P_\mu[Y_n; t] \qquad 2.18$$

Proof

Formula 2.10 simply says that the family $\{P_\mu[X_n;t]\}_\mu$ defined by 2.16 and 2.17 is dual to $\{Q_\mu[X_n;t]\}_\mu$ with respect to the scalar product $\langle\ ,\ \rangle_t$. Thus the identity in 2.18 is an immediate consequence of proposition 1.2 for the case $f_\rho = 1/p_\rho[1-t]$.

Note that the family of polynomials $\{S_\lambda[(1-t)X_n]\}_\lambda$ is also a basis for the symmetric functions in n variables. In fact, from 1.18 we derive the expansion

$$S_\lambda[(1-t)X_n] \;=\; \sum_\rho \chi^\lambda_\rho \, p_\rho[X_n]p_\rho[1-t]/z_\rho \ , \tag{2.19}$$

and the assertion follows from the invertibility of the character matrix $\chi = \|\chi^\lambda_\rho\|$.

Remark 2.1

More generally we can easily see that if $\{Q_\lambda[X_n]\}_\lambda$ is a symmetric function basis so must be the family $\{Q_\lambda[(1-t)X_n]\}_\lambda$. This is due to the simple reason that the matrix expressing $\{Q_\lambda[(1-t)X_n]\}_\lambda$ in terms of the power basis differs from that expressing $\{Q_\lambda[X_n]\}_\lambda$ only by pre-multiplication by the invertible diagonal matrix with elements $p_\rho[1-t]$.

We must therefore have an expansion of the form

$$Q_\mu[X_n;t] \;=\; \sum_\lambda S_\lambda[(1-t)X_n]\, K_{\lambda\mu}(t) \tag{2.20}$$

In view of the definitions 2.1 and 2.19 all we can say about the coefficients $K_{\lambda\mu}(t)$ is that they are rational functions of t. However, it is well known that they are indeed polynomials with non negative integer coefficients. This result was originally a consequence of deep developments in algebraic geometry (see []). It was also proved by Lascoux-Schützenberger ([],[] and []) who showed that the coefficients of $K_{\lambda\mu}(t)$ count certain column strict tableau of shape λ and content μ. Just the same, their proof, though elementary, is considerably intricate and only accessible to a handful of experts in the combinatorics of the Robinson-Schensted correspondence. An elementary and reasonably accessible representation theoretical proof was recently given in []. In the next section we shall show how the latter proof is related to the theory of orbit harmonics. But before we can do this we need to establish some further identities.

To this end it is convenient to set

$$H_\mu[X_n;t] \;=\; Q_\mu[X_x/(1-t);t] \tag{2.21}$$

In view of the defining formula 2.1 we see that this is equivalent to setting

$$H_\mu[X_n;t] \;=\; \prod_{1\le i<j\le k} \frac{1-z_j/z_i}{1-tz_j/z_i}\, \Omega[X_nZ_k]\,\big|_{z_1^{\mu_1}\cdots z_k^{\mu_k}} \ . \tag{2.22}$$

This yields us the following remarkable fact

Theorem 2.5

The coefficients $K_{\lambda\mu}(t)$ are polynomials which are different from zero if and only if $\lambda \geq \mu$. Moreover we have

$$H_\mu[X_n;t] \;=\; \sum_{\lambda\geq\mu} S_\lambda[X_n]\, K_{\lambda\mu}(t) \qquad 2.23$$

with

$$a) \quad K_{\lambda\mu}(t)\,|_{t=1} = K_{\lambda\mu} \qquad\qquad b) \quad K_{\mu\mu}(t) \equiv 1 \qquad 2.24$$

the latter denoting the classical Kostka coefficients yielding the expansion

$$h_\mu[X_n] \;=\; \sum_{\lambda\geq\mu} S_\lambda[X_n]\, K_{\lambda\mu} \qquad 2.25$$

of the homogeneous basis in terms of the Schur basis

Proof

Note that we may write

$$\prod_{1\leq i<j\leq k} \frac{1-z_j/z_i}{1-tz_j/z_i} \;=\; \sum_p \frac{c_p(t)}{z^p} \qquad 2.26$$

where the coefficients $c_p(t)$ are polynomials in t and the exponent vectors p are all of the form

$$p \;=\; \sum_{1\leq i<j\leq k} p_{i,j}(e^{(i)}-e^{(j)})$$

where $e^{(1)}, e^{(2)}, \ldots, e^{(k)}$ are the unit coordinate vectors of k-dimensional space and the $p_{i,j}$ are non-negative integers. Using this fact, we see that 2.22 breaks up into a finite sum of terms of the form

$$c_p(t)\;\Omega[X_nZ_k]\,|_{z^{\mu+p}} \qquad 2.27$$

Since for a given integer vector $q=(q_1,q_2,\ldots,q_k)$ we have

$$\Omega[X_nZ_k]\,|_{z^q} \;=\; \begin{cases} h_{q_1}[X_n]h_{q_2}[X_n]\cdots h_{q_k}[X_n] & \text{if all } q_i\geq 0 \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

we see that the coefficient in 2.27 evaluates either to zero or to a homogeneous basis element h_ν indexed by a partition ν which dominates μ. From the classical expansion 2.25 we then easily deduce that $H_\mu[X_n;t]$ must be linear combinations with polynomial coefficients of Schur functions $S_\lambda[X_n]$ indexed by partitions λ which dominate μ. This shows the first assertion of the theorem and 2.23 is then given by 2.20. Moreover, we see that the right hand side of defining formula 2.22, for $t=1$, reduces to

$$\Omega[X_nZ_k]\,|_{z_1^{\mu_1}\cdots z_k^{\mu_k}} \;=\; h_\mu[X_n]$$

and thus 2.24 a) is an immediate consequence of 2.25. This also shows that $K_{\lambda\mu}(t)$ cannot be vanishing identically when $\lambda\geq\mu$. Finally we see from 2.25 that the only summand 2.27 that contains a term in $S_\mu[X_n]$ is the one which produces h_μ. But this can only be gotten for $p=0$. This yields 2.24 b) and our proof is now complete.

It will be good to keep in mind also the following identities.

Theorem 2.6

The bases $\{H_\mu[X_n;t]\}_\mu$ and $\{P_\mu[X_n;t]\}_\mu$ are dual with respect to the customary Hall scalar product, that is

$$\Omega[X_nY_n] \;=\; \sum_{\mu} H_\mu[X_n;t]P_\mu[Y_n;t] \tag{2.28}$$

in particular we must have

$$S_\lambda[X_n] \;=\; \sum_{\mu\le\lambda} K_{\lambda\mu}(t)\, P_\mu[X_n;t] \;\; . \tag{2.29}$$

Proof

The first of these identities is simply obtained by replacing X_n by $X_n/(1-t)$ in 2.18 and using the definition 2.21. The second is obtained by eliminating $H_\mu[X_n;t]$ from 2.28 by means of 2.23 obtaining

$$\Omega[X_nY_n] \;=\; \sum_{\lambda} S_\lambda[X_n]\Big(\sum_{\mu\le\lambda} K_{\lambda\mu}(t)\, P_\mu[Y_n;t]\Big)$$

then equating coefficients of $S_\lambda[X_n]$ on the right hand sides of this identity and the classical Cauchy formula

$$\Omega[X_nY_n] \;=\; \sum_{\lambda\ge\mu} S_\lambda[X_n]S_\lambda[Y_n] \;\; .$$

Formulas 2.23 and 2.28 yield us two important special cases.

Theorem 2.7

$$\begin{aligned} a)\quad & H_n[X_n;t] \;=\; h_n[X_n] \\ b)\quad & H_{1^n}[X_n;t] \;=\; (1-t)(1-t^2)\cdots(1-t^n)e_n[X_n/(1-t)] \end{aligned} \tag{2.30}$$

Proof

The first of these identities is an immediate consequence of 2.23. Indeed, since there is no partition that stricly dominates the "one row" partition (n), 2.23 reduces to

$$H_n[X_n;t] \;=\; S_n[X_n]K_{nn}(t)$$

and 2.30 a) follows then from 2.24 b) and the equality of $S_n[X_n]$ and $h_n[X_n]$. For the second identity, we use 2.29 for $\lambda=1^n$. Here, since there is no partition that is strictly below 1^n in dominance, we end up with

$$S_{1^n}[X_n] \;=\; K_{1^n1^n}(t)P_{1^n}[X_n;t] \;\; .$$

Using 2.16, 2.17 and 2.24 b) we then get that

$$Q_{1^n}[X_n;t] \;=\; \prod_{i=1}^{n}(1-t^i)\, P_{1^n}[X_n;t] \;=\; \prod_{i=1}^{n}(1-t^i)\, S_{1^n}[X_n] \;.$$

and and 2.30 b) then follows from the equality of $S_{1^n}[X_n]$ and the elementary symmetric function $e_n[X_n]$.

To proceed further, we need a λ-ring reformulation of the identity in 2.22. This may be stated as follows

Proposition 2.1

For a given variable z and symmetric polynomial $P[X_n]$ set

$$\mathbf{H}(z)\, P[X_n] \;=\; \Omega[X_n z]\, P\big[X_n - \tfrac{1-t}{z}\big] \;, \qquad 2.31$$

and let $\mathbf{H}_m$ be the linear operator on symmetric polynomials defined by setting

$$\mathbf{H}_m\, P[X_n] \;=\; \mathbf{H}(z)\, P[X_n]\,\big|_{z^m} \;. \qquad 2.32$$

Then, for $\mu=(\mu_1,\mu_2,\ldots,\mu_k)$

$$H_\mu[X_n;t] \;=\; \mathbf{H}_{\mu_1}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_k}\; 1 \qquad 2.33$$

Proof

Note that applying $\mathbf{H}(z_2)$ and $\mathbf{H}(z_1)$ in succession we get

$$\begin{aligned}\mathbf{H}(z_1)\mathbf{H}(z_2)\, P[X_n] \;&=\; \mathbf{H}(z_1)\,\Omega[X_n z_2]\, P\big[X_n - \tfrac{1-t}{z_2}\big] \\ &=\; \Omega[X_n z_1]\Omega\big[(X_n - \tfrac{1-t}{z_1})z_2\big]\, P\big[X_n - \tfrac{1-t}{z_1} - \tfrac{1-t}{z_2}\big] \end{aligned} \;.$$

Using the multiplicativity property of the kernel Ω we can rewrite this as

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\, P[X_n] \;=\; \Omega[X_n Z_2]\, \tfrac{1-z_2/z_1}{1-tz_2/z_1}\, P\big[X_n - \tfrac{1-t}{z_1} - \tfrac{1-t}{z_2}\big] \;. \qquad 2.34$$

An easy induction argument then yields that

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\cdots\mathbf{H}(z_k)\, P[X_n] = \Omega[X_n Z_k] \prod_{1\le i<j\le k} \tfrac{1-z_j/z_i}{1-tz_j/z_i}\, P\big[X_n - \tfrac{1-t}{z_1} - \tfrac{1-t}{z_2} - \cdots - \tfrac{1-t}{z_k}\big] \;.$$

In particular, for $P=1$ we get

$$\mathbf{H}(z_1)\mathbf{H}(z_2)\cdots\mathbf{H}(z_k)\; 1 \;=\; \Omega[X_n Z_k] \prod_{1\le i<j\le k} \tfrac{1-z_j/z_i}{1-tz_j/z_i} \;.$$

Equating the coefficients of z^μ on both sides of this identity, we see that formula 2.33 is just another way of writing 2.22.

The operators $\mathbf{H}_m$ satisfy remarkable commutativity relations, for our purposes we need only the followin basic ones.

Proposition 2.2

For all $m \geq 1$ *we have*

$$\mathbf{H}_{m-1}\mathbf{H}_m \;=\; t\,\mathbf{H}_m\mathbf{H}_{m-1} \tag{2.35}$$

Proof

From 2.34 we easily derive that

$$(z_2 - tz_1)\mathbf{H}(z_2)\mathbf{H}(z_1) \;+\; (z_1 - tz_2)\mathbf{H}(z_1)\mathbf{H}(z_2) \;=\; 0\ ,$$

and formula 2.35 follows immediately by taking the coefficient of $z_1^m z_2^m$.

Remark 2.2

We should note that formula 2.33 enables us to extend the definition of $H_\mu[X;t]$ to the case in which μ is only a composition. That is, when its parts are not necessarily weakly decreasing.

For technical and combinatorial reasons it is more convenient to work with the modified versions $\tilde{H}_\mu[X_n;t]$ and $\tilde{K}_\mu(t)$ obtained by setting

$$a)\quad \tilde{H}_\mu[X_n;t] \;=\; H_\mu[X_n;1/t]t^{n(\mu)} \quad \text{and} \quad b)\quad \tilde{K}_{\lambda\mu}(t) \;=\; K_{\lambda\mu}(1/t)t^{n(\mu)}\ , \tag{2.36}$$

where for a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ we set

$$n(\mu) \;=\; \sum_{i=1}^{k}(i-1)\mu_i \tag{2.37}$$

Our final goal in this section is to derive a number of basic recursions which completely characterize the coefficients $\tilde{K}_{\lambda\mu}(t)$. These recursions are in a sense *dual* to the classical Pieri rules that can be found in []. They are best expressed in terms of the operator Γ_{1^k} which is the Hall inner product adjoint of multiplication by $e_k[X_n]$. To be precise, we can define Γ_{1^k} in terms of its action on the Schur function basis by setting

$$\Gamma_{1^k}\ S_\lambda[X_n] \;=\; S_{\lambda/1^k}[X_n]\ . \tag{2.38}$$

Using the self-duality of the Schur basis with respect to the Hall inner product, it easy to derive that the definition 2.38 is equivalent to the identity

$$\Gamma^x_{1^k}\ \Omega[X_nY_n] \;=\; \Omega[X_nY_n]\, e_k[Y_n] \tag{2.39}$$

where for a moment we let $\Gamma^x_{1^k}$ represent Γ_{1^k} when it acts on polynomials in the variables $x_1, x_2, \ldots, x_n$.

Let us still use the english convention for the partition μ but change the last index to the height of μ. More precisely,

$$\mu \;=\; (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{h(\mu)} > 0)\ .$$

For a given set of indices

$$I \;=\; \big\{1 \le i_1 < i_2 < \cdots < i_k \le h(\mu)\big\} \;,$$

let $p(\mu, I)$ denote the composition obtained from μ upon decreasing by one each of the parts $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k}$ and let $\mu^{(I)}$ denote the partition obtained by rearranging $p(\mu, I)$ back again into a partition. Under these conventions, we have the following remarkable identity.

Theorem 2.7

For any $1 \le k \le h(\mu)$ we have

$$\Gamma_{1^k}\, \tilde{H}_\mu[X_n;t] \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le h(\mu)} t^{i_1-1+i_2-1+\cdots+i_k-1}\, \tilde{H}_{\mu^{(I)}}[X_n;t] \;. \qquad 2.40$$

Proof

Our identity is an immediate consequence of the following beautiful commuting properties of the operators Γ_{1^k} and $\mathbf{H}_m$. Namely,

$$\Gamma_{1^k}\mathbf{H}_m \;=\; \mathbf{H}_m\Gamma_{1^k} \;+\; \mathbf{H}_{m-1}\Gamma_{1^{k-1}} \;. \qquad 2.41$$

To prove this we resort to a generating function argument. We set $\Gamma(x) = \sum_{k=0}^{n}(-x)^k\Gamma^x_{1^k}$ and note that 2.39 gives

$$\Gamma(x)\,\Omega[X_nY_n] \;=\; \Omega[X_nY_n]\sum_k(-x)^k\, e_k[Y_n] \;=\; \Omega[X_nY_n]\Omega[-xY_n] \;=\; \Omega\big[(X_n-x)Y_n\big] \;.$$

This implies that for any symmetric polynomial $P[X_n]$ we must have

$$\Gamma(x)\; P[X_n] \;=\; P[X_n - x] \;. \qquad 2.42$$

Thus, using the defining relation 2.31, we deduce that

$$\mathbf{H}(z)\Gamma(x)\; P[X_n] \;=\; \Omega[X_nY_n]P[X_n - x - \tfrac{1-t}{z}]$$

as well as

$$\Gamma(x)\mathbf{H}(z)\; P[X_n] \;=\; \Omega\big[(X_n-x)Y_n\big]P[X_n - x - \tfrac{1-t}{z}] \;=\; (1-xz)\Omega[X_nY_n]P[X_n - x - \tfrac{1-t}{z}] \;.$$

In other words

$$\Gamma(x)\mathbf{H}(z) \;=\; (1-xz)\mathbf{H}(z)\Gamma(x) \;,$$

and 2.41 follows by equating coefficients of the monomial t^kz^m. This given, we see that we must have

$$\Gamma_{1^k}\mathbf{H}_{\mu_1}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_h}\,1 \;=\; \mathbf{H}_{\mu_1}\Gamma_{1^k}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_h}\,1 \;+\; \mathbf{H}_{\mu_1-1}\Gamma_{1^{k-1}}\mathbf{H}_{\mu_2}\cdots\mathbf{H}_{\mu_h}\,1 \;,$$

and an obvious induction argument, based on the fact that $\Gamma_{1^s}1 = 0$, (when $s \ge 1$) immediately yields that

$$\Gamma_{1^k}H_\mu[X_n;t] \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le h(\mu)} H_{p(\mu,I)}[X_n;t] \;. \qquad 2.43$$

where the meaning of $H_{p(\mu,I)}[X_n;t]$ is as indicated in Remark 2.2. To obtain 2.40 we first rearrange $p(\mu,I)$ to a partition using the commutativity property in 2.35 then get the relation involving the $\tilde{H}_\mu\,'s$, by means of the identities

$$\tilde{H}_\mu[X_n;t] \;=\; H_\mu[X_n;1/t]\, t^{n(\mu)} \;. \qquad 2.44$$

Now the first step changes 2.43 to the form

$$\Gamma_{1^k}\, H_\mu[X_n;t] \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le h(\mu)} t^{f(\mu,I)}\, H_{\mu^{(I)}}[X_n;t] \;, \qquad 2.45$$

and the second step reduces this to

$$\Gamma_{1^k}\, \tilde{H}_\mu[X_n;t] \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le h(\mu)} t^{-f(\mu,I)}\, \tilde{H}_{\mu^{(I)}}[X_n;t]\;\; t^{n(\mu)-n(\mu^{(I)})} \;. \qquad 2.46$$

We will evaluate the expression $-f(\mu,I)+n(\mu)-n(\mu^{(I)})$ by separately working out the contribution of each vertical segment of the diagram of μ. Let the indices $j_s<\cdots<j_2<j_1$ indicate here the successive heights of the squares that are removed from that portion to obtain the diagram of $p(\mu,I)$. Let the index h denote the height of the outer corner square of the diagram of μ which lies in that particular portion. This given, we see that to pass from $p(\mu,I)$ to $\mu^{(I)}$ we must move the first of these *holes* to height h the second to height $h-1$, etc. ..., the last to height $h-s+1$. The commutativity relation in 2.35 then yields that the contribution to $f(\mu,I)$ coming from this portion must be

$$h-j_1 \;+\; h-1-j_2 \;+\; \cdots \;+\; h-s+1-j_s \;.$$

On the other hand, in passing to $\mu^{(I)}$, the loss to $n(\mu)$ coming from this portion, is given by

$$h-1 \;+\; h-2 \;+\; \cdots \;+\; h_s-s \;.$$

In summary, we must conclude that the contribution of this portion to the exponent $-f(\mu,I)+n(\mu)-n(\mu^{(I)})$ reduces to

$$j_1-1 \;+\; j_2-1 \;+\; \cdots \;+\; j_s-1 \;.$$

Summing all these contributions gives

$$-f(\mu,I)+n(\mu)-n(\mu^{(I)}) \;=\; i_1-1 \;+\; i_2-1 \;+\; \cdots \;+\; i_k-1 \;.$$

Thus formula 2.40 must hold true precisely as asserted.

Theorem 2.7 yields recursions for the coefficients $\tilde{K}_{\lambda\mu}(t)$ that are worth a separate statement. To simplify our identities it will be good to let $I\subseteq\mu$ mean that

$$I \;=\; \big\{1\le i_1<i_2<\cdots<i_k\le h(\mu)\big\}$$

let us also set

$$s(I) \;=\; i_1-1+i_2-1+\cdots+i_k-1$$

and let $|I|$ denote the cardinality of I. This given we have.

Theorem 2.8

Given $k \leq n$ and two partitions $\mu \vdash n$ and $\rho \vdash n-k$ we have

$$\sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(t)\, \chi(\lambda/\rho \in V_k) \;=\; \sum_{I \subseteq \mu\ ,\ |I|=k} t^{s(I)}\, \tilde{K}_{\rho\mu^{(I)}}(t) \tag{2.47}$$

where, $\lambda/\rho \in V_k$ is to mean that λ/ρ is a vertical k-strip.

Proof

Note first that the defining equation 2.38 may be also written as

$$\Gamma_{1^k}\, S_\lambda[X_n] \;=\; \sum_{\rho \vdash n-k} S_\rho[X_n]\, \chi(\lambda/\rho \in V_k)\ . \tag{2.48}$$

Since 2.23 gives

$$\tilde{H}_\mu[X_n;t] \;=\; \sum_{\lambda \geq \mu} S_\lambda[X_n]\, \tilde{K}_{\lambda\mu}(t)\ , \tag{2.49}$$

The left hand side of 2.40 can be written as

$$\sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(t) \sum_{\rho \vdash n-k} S_\rho[X_n]\, \chi(\lambda/\rho \in V_k)$$

On the other hand making repeated uses of 2.49 we may rewrite the right hand side of 2.40 as

$$\sum_{I \subseteq \mu\ ,\ |I|=k} t^{s(I)} \sum_{\rho \vdash n-k} S_\rho[X_n]\, \tilde{K}_{\rho\mu^{(I)}}(t)\ .$$

Equating the coefficients of $S_\rho[X_n]$ in these two expressions gives 2.47 as desired.

Remark 2.3

The particular case $\mu = 1^n$ of 2.47 reveals an interesting combinatorial fact. In fact, the definition 2.44 applied to the special evaluation 2.30 b) gives

$$\tilde{H}_{1^n}[X_n;t] \;=\; (-1)^n\, (1-t)(1-t^2)\cdots(1-t^n)\, e_n[-\frac{X_n}{1-t}]\ .$$

Since 1.23 gives $(-1)^n e_n[-X_n/(1-t)] = h_n[X_n/(1-t)]$, formula 1.14 yields that

$$\tilde{H}_{1^n}[X_n,t] \;=\; (1-t)(1-t^2)\cdots(1-t^n)\, h_n[\frac{X_n}{1-t}] \;=\; \sum_{\lambda \vdash n} S_\lambda[X_n]\, f_\lambda(t)$$

where for convenience we have set

$$f_\lambda(t) \;=\; \sum_{T \in ST(\lambda)} t^{c(T)}\ . \tag{2.50}$$

Note further that when $\mu = 1^n$ then for $I \subseteq \mu$ and $|I| = k$, the partition $\mu^{(I)}$ is always equal to 1^{n-k}. This means that in this case 2.47 reduces to the suggestive identity

$$\sum_{\lambda \vdash n} f_\lambda(t)\, \chi(\lambda/\rho \in V_k) \;=\; t^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_t f_\rho(t) \;,$$

where the bracket $[\;]_t$ is to denote the t-analogue of the binomial coefficient. The search for a direct combinatorial argument yielding this property of the charge statistic appears to be a challenging project.

From the work of Lascoux and Schützenberger [] it follows that for each μ there is a class of standard tableaux $\mathcal{S}_\mu$ such that

$$\tilde{K}_{\lambda\mu}(t) \;=\; \sum_{\substack{T \in ST(\lambda) \\ T \in \mathcal{S}_\mu}} t^{c(T)} \;. \qquad 2.51$$

Thus an alternate approach to the proof of the Lascoux and Schützenberger result could be to establish that if we let $C_{\lambda\mu}(t)$ denote the right hand side of 2.51 then these polynomials satisfy the recursions in 2.47. Or, equivalently that the polynomials

$$C_\mu[X_n; t] \;=\; \sum_\lambda S_\lambda[X_n]\, C_{\lambda\mu}(t) \;,$$

satisfy the recursions

$$\Gamma_{1^k}\, C_\mu[X_n; t] \;=\; \sum_{1 \le i_1 < i_2 < \cdots < i_k \le h(\mu)} t^{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1}\, C_{\mu^{(I)}}[X_n; q] \;. \qquad 2.52$$

That this is sufficient to guarantee that $C_{\lambda\mu}(t) = \tilde{K}_{\lambda\mu}(t)$ is an immediate consequence of the following basic fact.

Proposition 2.3

Let $\{R_\mu[X_n]\}_\mu$ be a family of symmetric polynomials satisfying the recursions

$$\Gamma_{1^k}\, R_\mu[X_n] \;=\; \sum_\lambda R_\lambda[X_n]\, d^{(k)}_{\lambda\mu} \;. \qquad 2.53$$

then

$$R_\mu[X_n] \;=\; \sum_\rho \omega\, m_\rho[X_n]\, D_{\rho\mu} \;, \qquad 2.54$$

where m_ρ denotes the monomial symetric function and for a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$ we have

$$D_{\rho\mu} \;=\; \sum_{\lambda^{(1)} \vdash n - \rho_1} \sum_{\lambda^{(2)} \vdash n - \rho_1 - \rho_2} \cdots \sum_{\lambda^{(s-1)} \vdash n - \rho_1 - \rho_2 - \cdots - \rho_{s-1}} d^{(\rho_s)}_{\emptyset \lambda^{(s-1)}} \cdots d^{(\rho_2)}_{\lambda^{(2)} \lambda^{(1)}}\, d^{(\rho_1)}_{\lambda^{(1)} \mu}$$

Proof

Since the Hall-dual of the elementary symmetric function basis $\{e_\rho\}_\rho$ is the the image by ω of the monomial basis $\{m_\rho\}_\rho$ we have the expansion

$$R_\mu[X_n] \;=\; \sum_\rho \omega\, m_\rho[X_n] \,\langle\, R_\mu \,,\, e_\rho \,\rangle \;,$$

where $\langle\ ,\ \rangle$ denotes the ordinary Hall scalar product. On the other hand since the operator Γ_{1^k} is Hall-dual to multiplication by e_k, for a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$ we must have

$$\langle\, R_\mu \,,\, e_\rho \,\rangle \;=\; \langle\, R_\mu \,,\, e_{\rho_1} e_{\rho_2} \cdots e_{\rho_s} \,\rangle \;=\; \langle\, \Gamma_{1^{\rho_s}} \cdots \Gamma_{1^{\rho_2}} \Gamma_{1^{\rho_1}} R_\mu \,,\, 1 \,\rangle \;.$$

This given, repetitive uses of 2.53, immediately yield that the coefficient $\langle\, R_\mu \,,\, e_\rho \,\rangle$ evaluates to $D_{\rho\mu}$ precisely as asserted.

Remark 2.3

We should note that the raising operator formula 2.22 was also given by Milne in []. Moreover, our proof of the orthogonality result 2.10 and the introduction of the operator $\mathbf{H}(z)$ was inspired by a recent work of N. Jing [], where a formula for $Q_\mu[X;t]$, analogous to 2.33, is given in terms of the so called *vertex* operators. Finally, we are indebted to A. Lascoux for the suggestion that the theory of Hall-Littlewood polynomials gains elegance and clarity when presented in λ-ring notation.

3. Graded S_n modules giving the Kostka-Foulkes polynomials.

In view of 2.24 it is natural to consider the coefficients $K_{\lambda\mu}(t)$ as t-analogues of the classical Kostka numbers, they are sometimes referred to as Kostka-Foulkes polynomials []. We shall also refer to them here as the *t-Kostka* polynomials. The startling analogy between 2.23 and 2.25 suggests a representation theoretical approach to the proof that the $K_{\lambda\mu}(t)\,'s$ have non negative integer coefficients. Indeed, we can interpret 2.25 as expressing that the homogeneous symmetric function $h_\mu[X_n] = h_{\mu_1}[X_n] h_{\mu_2}[X_n] \cdots h_{\mu_k}[X_n]$ is the Frobenius image (see (1.15) of the class function

$$p^\mu \;=\; \sum_{\lambda \geq \mu} \chi^\lambda\, K_{\lambda\mu} \qquad 3.1$$

which, by Young's rule ([]), is the character of the action of S_n on the left cosets of the Young subgroup "$S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$". Thus we are naturally led to search for an S_n module yielding a graded version of this left coset action. Now the developments of chapter 2 yield a variety of ways of producing such a module. In fact, if we take any point $a = (a_1, a_2, \ldots, a_n)$ whose stabilizer is precisely such a subgroup, then theorem 1.1 yields that the two modules $gr\,\mathbf{R}_{[a]}$ and $\mathbf{H}_{[a]}$ have a graded character $p^\mu(t) = char_t\; gr\,\mathbf{R}_{[a]} = char_t\,\mathbf{H}_{[a]}$ whose expansion in terms of the irreducible characters must be of the form

$$p^\mu(t) \;=\; \sum_{\lambda \geq \mu} \chi^\lambda\, C_{\lambda\mu}(t) \;. \qquad 3.2$$

where the $C_{\lambda\mu}(t)\,'s$ are necessarily polynomials with non negative integer coefficients, satisfying the marginal conditions

$$C_{\lambda\mu}(t)\,|_{t=1} \;\;=\;\; K_{\lambda\mu} \;\;. \tag{3.3}$$

In this section we shall indicate how the methods we have developed in the previous chapters may be used to show that, when a is a generic point with such a stablilizer then we do, in fact, have

$$C_{\lambda\mu}(t) \;=\; \tilde{K}_{\lambda\mu}(t) \;\;, \tag{3.4}$$

with $\tilde{K}_{\lambda\mu}(t)$ as given in 2.36 b).

Before we can proceed, we need to have a visual way of representing the orbit of our point a and a systematic procedure for building a basis for the module $\mathbf{R}_{[a]}$ which reflects the combinatorics of $[a]$. Given a k-part partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$, a filling of its Ferrers' diagram with the integers $1, 2, \ldots, n$ which is increasing along the rows will be briefly referred to as a *row increasing (injective)* tableaux of shape μ. The collection of all such tableaux will be denoted by $\mathcal{RI}(\mu)$. Given $T \in \mathcal{RI}(\mu)$ we construct a point $b(T) = (b_1, b_2, \ldots, b_n)$ by letting $b_j = \alpha_i$ if the integer j is in the i^{th} row of T. Where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are given indeterminates which will remain fixed throughout this presentation. It is easily seen that, as long as $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all distinct, the point $a(T)$ has a stabilizer which is isomorphic to the group $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$. Thus the spaces $\mathbf{H}_{[a(T)]}$, $gr\ \mathbf{R}_{[a(T)]}$ are graded S_n-modules with a graded character of the form given by 3.2 and 3.3.

Clearly, the orbit $[a(T)]$ depends only on μ (and of the choice of the $\alpha_i\,'s$). To be more specific, it will be convenient to let, $a_\mu = a(T_o)$ where T_o is the row increasing tableau obtained by filling the (french) Ferrers' diagram of μ with $1, 2, \ldots, n$ starting from the top (shortest) row and ending with the bottom (longest) row. It is good to keep in mind that the orbit $[a_\mu]$ consists of all the points $a(T)$ obtained as T describes the collection $\mathcal{RI}_\mu$. We can thus use the elements of $\mathcal{RI}_\mu$ or the points of $[a_\mu]$ interchangeably in dealing with questions related to this orbit. It will also be convenient to set

$$\mathbf{H}_\mu \;\;=\;\; \mathbf{H}_{[a_\mu]} \qquad\qquad \text{and} \qquad\qquad \mathbf{R}_\mu \;\;=\;\; \mathbf{R}_{[a_\mu]} \;\;. \tag{3.5}$$

We shall start by determining the Hilbert series of $\mathbf{H}_\mu$ and $gr\,\mathbf{R}_\mu$ and identify $\mathbf{H}_\mu$ as a submodule of the space $\mathbf{H}_{S_n}$ of harmonic polynomials of the symmetric group. We should note that we have carried out this program for $\mu = 1^n$. Indeed, the developments of section 5. (see Remark 5.3) do show that for any choices of distinct $\alpha_1, \alpha_2, \ldots, \alpha_n$ the the space $\mathbf{H}_{1^n}$ coincides with $\mathbf{H}_{S_n}$. Now it develops that the method illustrated at the end of section 5. may be carried out in full generality, producing for any partition μ a remarkable monomial basis for the modules $\mathbf{R}_\mu$ and $gr\ \mathbf{R}_\mu$.

Imitating what we did in section 5, our program is to chose a suitable linear order $<_L$ of the orbit $[a_\mu]$ then for each point $b \in [a_\mu]$ construct a polynomial $\phi_b(x)$ which is of minimal degree among those which satisfy the conditions

$$\begin{array}{lrcl} a) & \phi_b(b) & = & 1 \\ b) & \phi_b(b') & = & 0 \quad \text{for all } \; b' <_L b \quad \text{in } [a_\mu] \end{array} \;\;. \tag{3.6}$$

This done, we plan to apply Proposition 5.2, and derive that the collection $\mathcal{B}_\mu = \{h(\phi_b)\}_{b\in[a_\mu]}$ is a basis for $gr\, \mathbf{R}_\mu$. Clearly, property 3.6 will assure that the collection $\{\phi_b\}_{b\in[a_\mu]}$ itself is a basis for $\mathbf{R}_\mu$. So once 3.6 is established we are left with the verification that the cardinalities $d_i = |\mathcal{B}_\mu^{=i}|$ yield the coefficients of the common Hilbert series of the modules $gr\, \mathbf{R}_\mu$ and $\mathbf{H}_\mu$. That is we must show that

$$\sum_{i\geq 0} q^i\, |\mathcal{B}_\mu^{=i}| \;\;=\;\; F_{\mathbf{H}_\mu}(q) \;\; . \tag{3.7}$$

Now it is precisely at this point that we need the solution of 3.6 to be as economical as possible with degree. In fact, the total order $<_L$ must also be appropriately chosen so that the minimal degree solutions of 3.6 will yield a solution of 3.7. It develops that the lexicographic order of coordinate vectors of the elements of $[a_\mu]$ is not the appropriate one in this case. The natural choice for $<_L$ here is a bit less obvious and will require a recursive construction. To begin with we shall represent our orbit points as the leaves of a tree $\mathcal{T}_\mu$. Each internal node of $\mathcal{T}_\mu$ at distance s from the root is indexed by a word $b = b_1 b_2 \cdots b_s$ whose letters are chosen from the alphabet $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ subject to the restriction that each α_i occurs with a multiplicity p_i, *which is at most* μ_i. This guarantees that we can chose $a = (a_1, a_2, \ldots, a_{n-s})$ so that

$$(a_1, a_2, \ldots, a_{n-s}, b_1, b_2, \ldots, b_s) \;\;\in\;\; [a_\mu] \tag{3.8}$$

Note that the latter is the case if and only if the number if occurrences of α_i in a is equal to $\mu_i - p_i$. Denoting by μ^b the weakly decreasing rearrangement of the positive components of the vector

$$(\mu_1 - p_1, \mu_2 - p_2, \ldots, \mu_k - p_k) \tag{3.9}$$

we see that the collection of vectors $a = (a_1, a_2, \ldots, a_{n-s})$ satisfying 3.8 is none other than the S_{n-s}-orbit $[a_{\mu^b}]$, corresponding to the subalphabet

$$\mathcal{A}(b) \;\;=\;\; \{\alpha_i \;:\; \mu_i > p_i\;\} \;\; ,$$

where the letters α_i must be taken in decreasing order of the corresponding differences $\mu_i - p_i$ (we may break ties according to the original order in $\mathcal{A}$). In particular there will be no α_i in a if $p_i = \mu_i$. This given, if $\mathcal{A}(b)$ has m elements, we give the node indexed by b precisely m children respectively indexed by the vectors

$$(\alpha_i, b_1, b_2, \ldots, b_s) \quad \alpha_i \in \mathcal{A}(b) \;\; .$$

Finally, to make $\mathcal{T}_\mu$ into a planar tree we total order these children by setting

$$(\alpha_i, b_1, b_2, \ldots, b_s) \;<_L\; (\alpha_j, b_1, b_2, \ldots, b_s) \Longleftrightarrow \begin{cases} \mu_i - p_i > \mu_j - p_j & \text{or} \\ \mu_i - p_i = \mu_j - p_j & \text{and} \quad i < j \end{cases} . \tag{3.10}$$

The construction of $\mathcal{T}_\mu$ will necessarily stop with nodes indexed by words with μ_i letters equal to α_i. Thus the leaves of $\mathcal{T}_\mu$ may be identified with the elements of $[a_\mu]$. We shall then adopt $<_L$ to be the order in which the leaves are encountered in the dept first traverse of the nodes of $\mathcal{T}_\mu$. It is easily seen that in the case $\mu = 1^n$, $<_L$ reduces to the lexicographic order. We should mention that the crucial property of $<_L$ needed here is that the children of any node of $\mathcal{T}_\mu$ are arranged in order of decreasing *total number of descendents*. In the figure below we have depicted the tree corresponding to the case $\mu = (2,2,1)$. For simplicity, we have not drawn the leaves, we have omitted some of the labels and have inserted at the node labelled by b the Ferrers' diagram of the corresponding partition μ^b.

We see that at distance one from the root the labels, in depth first order (left to right) are

$$\alpha_1 \ , \ \alpha_2 \ , \ \alpha_3$$

at distance two we have

$$\alpha_2\alpha_1 \ , \ \alpha_1\alpha_1 \ , \ \alpha_3\alpha_1 \ , \quad \text{then} \quad \alpha_1\alpha_2 \ , \ \alpha_2\alpha_2 \ , \ \alpha_3\alpha_2 \ , \quad \text{then} \quad \alpha_1\alpha_3 \ , \ \alpha_2\alpha_3 \ , \quad .$$

The reader should have no difficulty deducing the label of any of the nodes of this tree. For instance, at distance three, the labels of the children of the node labelled $\alpha_1\alpha_3$ are, from left to right, $\alpha_2\alpha_1\alpha_3$ and $\alpha_1\alpha_1\alpha_3$. We also see that the descendants of the node labelled $\alpha_2\alpha_2$ (vertically under the root) which are at distance 4 are labelled, from left to right,

$$\alpha_1\alpha_1\alpha_2\alpha_2 \ , \ \alpha_3\alpha_1\alpha_2\alpha_2 \ , \ \alpha_1\alpha_3\alpha_2\alpha_2 \ ,$$

and the orbit points hanging from them are respectively

$$(\alpha_3,\alpha_1,\alpha_1,\alpha_2,\alpha_2) \ , \ (\alpha_1,\alpha_3,\alpha_1,\alpha_2,\alpha_2) \ , \ (\alpha_1,\alpha_1,\alpha_3,\alpha_2,\alpha_2) \ .$$

It will be convenient to denote by $\mathcal{T}_\mu(b)$ the subtree of $\mathcal{T}$ whose root is labelled by b. For instance, with this notation the above three points may be identified with the leaves of the subtree $\mathcal{T}_\mu(\alpha_2\alpha_2)$. The construction of the collection of polynomials $\{\phi_b(x)\}_{b\in[a_\mu]}$ is best explained by working first on this specific example. Note that the linear factor $(x_i - \alpha_j)$ in a given $\phi_b(x)$ will make it vanish on all orbit points whose i^{th} coordinate is α_j. Thus in particular the factor $(x_5-\alpha_1)$ will make it vanish on all the leaves of the subtree $\mathcal{T}_\mu(\alpha_1)$. This eliminates, at the cost of one degree, 12 orbit points. The first orbit point in our total order $<_L$ at which $(x_5-\alpha_1)$ doesn't vanish is the leftmost leaf of $\mathcal{T}_\mu(\alpha_2)$ which is $(\alpha_3,\alpha_2,\alpha_1,\alpha_1,\alpha_2)$ thus the corresponding polynomial $\phi_b(x)$ may be chosen as $(x_5-\alpha_1)/(\alpha_2-\alpha_1)$. To obtain the polynomial corresponding to the leftmost leaf of $\mathcal{T}_\mu(\alpha_2\alpha_2)$ we need to introduce a factor that kills all the leaves of $\mathcal{T}_\mu(\alpha_1\alpha_2)$. This is achieved by the factor $(x_4-\alpha_1)$. So the polynomial $(x_5-\alpha_1)(x_4-\alpha_1)$ will kill all the leaves of $\mathcal{T}_\mu(\alpha_1)$ all the leaves of $\mathcal{T}_\mu(\alpha_1\alpha_2)$ and the first surviving orbit point will necessarily be $(\alpha_3,\alpha_1,\alpha_1,\alpha_2,\alpha_2)$. Thus its corresponding polynomial is $(x_5-\alpha_1)(x_4-\alpha_1)/(\alpha_2-\alpha_1)(\alpha_1-\alpha_1)$.

To proceed in full generality, we need to resort to a recursive procedure. We assume that we have constructed the basis $\{\phi_b(x)\}_{b\in[a_\nu]}$ for each partition ν whose Ferrers' diagram has $n-1$ squares then use the tree imagery to construct the basis $\{\phi_b(x)\}_{b\in[a_\mu]}$ for a partition with n squares. Note that although the resulting orbit does depend on the alphabet $\mathcal{A}=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$, reordering the alphabet, or even changing it, will not affect the structure of the corresponding tree. Only the labels will change, the tree itself is easily seen only to depend on μ. This means that the basis $\{\phi_b(x)\}_{b\in[a_\mu]}$ for the aplhabet $\mathcal{B}=\{\beta_1,\beta_2,\ldots,\beta_k\}$ can simply be obtained from that corresponding to the alphabet $\mathcal{A}=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ by simply replacing everywhere α_i by β_i. Let then $\mathcal{B}_\mu(\mathcal{A})$ denote the basis $\{\phi_b(x)\}_{b\in[a_\mu]}$ for the partition $\mu=(\mu_1\geq\mu_2\geq\cdots\geq\mu_k)$ and the alphabet $\mathcal{A}=\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$. This given, the general form of the construction illustrated in the above example may be simply expressed by the following symbolic recurrence

$$\mathcal{B}_\mu(\mathcal{A}) \;=\; \sum_{i=1}^{k} \mathcal{B}_{\mu^{\alpha_i}}(\mathcal{A}(\alpha_i)) \;\times\; \textstyle\prod_{j=1}^{i-1} (x_n-\alpha_j)/(\alpha_i-\alpha_j)\;. \tag{3.11}$$

Here the summation sign denotes disjoint union and the symbol "$\times$" represents multiplication of every element of $\mathcal{B}_{\mu^{\alpha_i}}(\mathcal{A}(\alpha_i))$ by the factor $\prod_{j=1}^{i-1}(x_n-\alpha_j)/(\alpha_i-\alpha_j)$.

Proposition 3.1

The collection $\mathcal{B}_\mu(\mathcal{A})$ defined by the recursion 3.11 is a basis for $\mathbf{R}_\mu$.

Proof

Clearly we need only show that $\mathcal{B}_\mu(\mathcal{A})$ consists of polynomials $\phi_b(x)$ with $b\in[a_\mu]$ which satisfy the conditions in 3.6. We proceed by induction on the number n of squares in the diagram of μ (the case $n=1$ being trivial). Note that our construction of the tree $\mathcal{T}_\mu$ yields that every leaf of $\mathcal{T}_\mu(\alpha_i)$ corresponds to an element of the S_{n-1} orbit $[a_{\mu^{\alpha_i}}]$ relative to the alphabet $\mathcal{A}(\alpha_i)$. This correspondence is obtained by sending the point

$$(b_1,b_2,\ldots,b_{n-1})$$

of $[a_{\mu^{\alpha_i}}]$ into the leaf of $\mathcal{T}_\mu$ indexed by the word

$$b_1b_2\ldots b_{n-1}\alpha_i\;.$$

Let then $\psi_{(b_1,b_2,\ldots,b_{n-1})}$ denote the element of $\mathcal{B}_{\mu^{\alpha_i}}(\mathcal{A}(\alpha_i))$ which corresponds to $(b_1,b_2,\ldots,b_{n-1})$. Since this correspondence preserves the total order $<_L$. We see that the polynomial

$$\psi_{(b_1,b_2,\ldots,b_{n-1})}(x)\prod_{j=1}^{i-1}(x_n-\alpha_j)/(\alpha_i-\alpha_j)$$

will necessarily vanish for all points of $[a_\mu]$ preceeding $(b_1,b_2,\ldots,b_{n-1},\alpha_i)$ in the total order $<_L$. In fact, the factor $\prod_{j=1}^{i-1}(x_n-\alpha_j)/(\alpha_i-\alpha_j)$ will make it vanish at all leaves preceeding the node indexed by α_i (in the depth first order) and the factor $\psi_{(b_1,b_2,\ldots,b_{n-1})}(x)$ will make it vanish at all leaves preceeding $b_1b_2\ldots b_{n-1}\alpha_i$ in the subtree $\mathcal{T}(\alpha_i)$. Since this polynomial evaluates to 1 at $x=(b_1,b_2,\ldots,b_{n-1},\alpha_i)$, we see that we may take it to be our choice of $\phi_b(x)$ when $b=(b_1,b_2,\ldots,b_{n-1},\alpha_i)$. This completes our proof.

Note that the recursive construction of the basis $\mathcal{B}_\mu(\mathcal{A})$ given by 3.11 makes it immediate that the collection of highest homogeneous components

$$\mathcal{B}_\mu \;=\; \{h(\phi_b)\;:\; b\in[a_\mu]\,\}$$

may be defined by the recursion

$$\mathcal{B}_\mu \;=\; \sum_{i=1}^{k}\mathcal{B}_{\mu^{(i)}} \;\times\; x_n^{i-1} \tag{3.12}$$

where $\mu^{(i)}$ is the partition obtained from μ by replacing μ_i by μ_i-1 and rearranging components in weakly decreasing order. We can clearly see then that $\mathcal{B}_\mu$ only depends on μ. An application of Proposition II.5.2 would then give us that $\mathcal{B}_\mu$ is a basis for $\mathbf{R}_\mu$ as well as *gr* $\mathbf{R}_\mu$. We can prove that it is a basis for $\mathbf{R}_\mu$ without resorting to this proposition, by means of an auxiliary result which will bring to evidence that $\mathcal{B}_\mu$ is indeed a remarkable collection of monomials.

Proposition 3.2

The collections $\mathcal{B}_\mu$ are nested into each other according to reverse dominance. More precisely we have

$$\nu\;\geq\;\mu\;\longrightarrow\;\mathcal{B}_\nu\;\subseteq\;\mathcal{B}_\mu \tag{3.13}$$

Proof

Clearly, this inequality need only be shown when ν is an immediate successor of μ in the dominance order. There are two cases to be considered. The first case occurs when ν is obtained from μ by lowering an outer corner square s to the inner corner in the column immediately to the right. The second case occurs when ν is obtained from μ by lowering an outer corner square s to the inner corner in the row immediately below. We shall treat only the first case since the argument is the same in both cases. For convenience let $h(\nu)$ and $h(\mu)$ denote the heights of the diagrams of nu and μ respectivelty. Let us say then that the lowered square is in position (j,b) in the diagram of μ and that it lands in position $(j+1,a)$ in the diagram of ν, with $a<b$ (note that we use cartesian coordinates here with $(0,0)$ the lowest lefthand position in the Ferrers' diagram). A look at the figure below should clarify the situation.

$$\mu \ \rightarrow \qquad\qquad\qquad\qquad \nu \ \rightarrow \tag{3.14}$$

It will simplify our language at this point if we identify a partition with its own diagram. In this vein, we can simply describe the operation $\mu \ \rightarrow \ \mu^{(i)}$ as the *removal from* μ *of the lowest corner square weakly above height* i. Note that, proceeding by induction, we need only show that, under these circumstances, we necessarily have $\mu^{(i)} \leq \nu^{(i)}$, for all $i \leq h(\nu)$. Clearly this inequality is trivial for $i > b$. Since then $\mu^{(i)}$ and $\nu^{(i)}$ are in the same relationship as μ and ν. Now for $i < a$ we shall have the same situation if the corner square of μ in row $a-1$ is not in column $j+1$. However, if it is, for some values of i, $\mu^{(i)}$ is obtained from μ by removing the square in position $(j+1, a-1)$ while $\nu^{(i)}$ is obtained from ν by removing the square in position $(j+1, a)$. Since the latter is higher than the former the inequality $\mu^{(i)} \leq \nu^{(i)}$ will necessarily follow. In all other cases, for $i < a$ we will be removing the same square from both μ and ν, and the inequality is trivial. For $i = a$ we remove the square in (j, b) from μ and that in $(j+1, a)$ from ν, that makes $\mu^{(i)} = \nu^{(i)}$. For $a < i < b$ we remove the square in (j, b) from μ and that in $(j, b-1)$ from ν. In this case, $\nu^{(i)}$ is obtained from $\mu^{(i)}$ by lowering the square in $(j, b-1)$ to position $(j+1, a)$ and so again we have $\mu^{(i)} \leq \nu^{(i)}$. Finally for $i = b$ we remove the square in (j, b) from μ and a higher square from ν, so the desired inequality holds also in this case. This completes our proof in the case that $h(\mu) = h(\nu)$. If $h(\mu) > h(\nu)$, that is when the lowered square, is the highest in the diagram of μ, then the recursion in 3.12 for μ has one more summand than that for ν and this only strengthens the inequality in 3.13.

This proposition yields that $\mathcal{B}_\mu$ is also a basis for $\mathbf{R}_\mu$. In fact we can prove something which is a bit stronger.

Proposition 3.3

Every polynomial in $\mathcal{B}_\mu(\mathcal{A})$ *may be written as a linear combination of the monomials in* $\mathcal{B}_\mu$

Proof

Note that for partitions of 2 we have two cases. Namely, $\mu = (1, 1)$ where $\mathcal{B}_\mu(\mathcal{A})$ consists of 1 and $x_2 - \alpha_1$ and $\mathcal{B}_\mu$ consists of 1 and x_2, and $\mu = (2)$ where both collections reduce to the constant polynomial 1. We see that in both cases the result is trivially true. We shall then assume the assertion true for partitions of $n-1$ and proceed to prove it for partitions of n. Note again that for $i = 1$ there is no extra factor in 3.11 and the induction hypothesis yields that all the polynomials produced by the first summand in 3.11 may be expressed as linear combinations of the monomials produced by the first summand in 3.12. The general summand in 3.11 requires only a little more. Indeed, let P be produced by the i^{th} summand of 3.11 and let

$$P(x) = Q(x) \prod_{j=1}^{i-1} {\scriptstyle (x_n - \alpha_j)/(\alpha_i - \alpha_j)}$$

with

$$Q \in \mathcal{B}_{\mu^{\alpha_i}}(\mathcal{A}(\alpha_i))$$

Since by induction Q may be expressed as a linear combination of monomials $m(x) \in \mathcal{B}_{\mu^{(i)}}$, we are reduced to shoving that the polynomials

$$m(x)\prod_{j=1}^{i-1}(x_n-\alpha_j)$$

may be expressed in terms of the monomials in $\mathcal{B}_\mu$. However, expanding the product yields that this polynomial is a linear combination of the monomials

$$m(x) \times x_n^{j-1}$$

with $j \leq i$. Now Proposition 3.2 gives that $\mathcal{B}_{\mu^{(i)}} \subseteq \mathcal{B}_{\mu^{(j)}}$ thus the recursion in 3.12 yields that all of these monomials belong to $\mathcal{B}_\mu$ as desired. This completes the induction and our proof.

For a given standard tableau S let $h(i,S)$ denote the height of the letter i in S. Let us then set

$$m(S) \;=\; \prod_{i\in S} x_i^{h(i,S)-1}$$

and refer to it as the *monomial of S*. It develops that Proposition 3.2 has also the following beautiful corollary.

Proposition 3.4

For each partition μ the collection $\mathcal{B}_\mu$ is a lower order ideal of monomials, whose maximal elements are the monomials of the standard tableaux of shape μ.

Proof

It is easily seen from the recursion 3.12 that a term of highest degree in $\mathcal{B}_\mu$ can be produced by the summand $\mathcal{B}_{\mu^{(i)}}$ only when the corner square that must be removed from the diagram of μ to obtain $\mu^{(i)}$ is precisely at height i. Thus we can recursively deduce from 3.12 that the elements of highest degree of $\mathcal{B}_\mu$ are all of degree

$$n(\mu) \;=\; \sum_{k=1}^{k}(i-1)\mu_i \;\; .$$

and indeed they are none other than the standard tableaux monomials defined above. Thus we are only left to show that $\mathcal{B}_\mu$ is a lower ideal of monomials. We shall show this again by induction on the number of squares of the diagram of μ. We recall that the partial order of monomials $<<$ we work with is defined by setting

$$x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n} \;\leq\; x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n} \quad \textit{if and only if} \quad p_i \leq q_i \quad \forall\, 1 \leq i \leq n \;\; .$$

This given, the induction hypothesis reduces us to show that if the monomial $x_n^{i-1}m(x_1,\ldots,x_{n-1})$ occurs in $\mathcal{B}_\mu$ then also the monomial $x_n^{j-1}m(x_1,\ldots,x_{n-1})$ can be found in $\mathcal{B}_\mu$ for any $1 \leq j < i$. By our construction, the latter monomial must come out of $x_n^{j-1}\mathcal{B}_\mu^{(j)}$. In other words the monomial $m(x_1,\ldots,x_{n-1})$ itself should be in $\mathcal{B}_\mu^{(j)}$. Since this monomial must have come out of $\mathcal{B}_\mu^{(i)}$, we are left to show that

$$\mathcal{B}_\mu^{(i)} \subseteq \mathcal{B}_\mu^{(j)} \;\; .$$

However this follows immediately from Proposition 3.2, since it is easy to see that for $j < i$ we have $\mu^{(j)} \leq \mu^{(i)}$ in the dominance order. This completes the proof.

We are now fully equipped to identify the harmonics of each orbit $[a_\mu]$ as a subspace of the S_n-harmonics. Before we can state the basic result here we need some notation. If A is an ordered subset of $X_n = \{x_1, x_2, \ldots, x_n\}$ we let $\Delta(A)$ denote the Vandermonde determinant in the elements of A. To be precise (since order counts), if $A = \{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\}$ we set

$$\Delta(A) \;=\; det\|x_{a_i}^{j-1}\|_{i,j=1}^{m} \;,$$

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ be a partition of n and $\mu' = (\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_h)$ denote its conjugate. Let T be a tableau T of shape μ with entries $1, 2, \ldots, n$ and let $C_1, C_2, \ldots, C_h$ denote the columns of T ordered from left to right. Note, that with this notation, column C_i is necessarily of lenght μ'_i. Now let A_i be the subalphabet of X_n obtained by selecting the variables x_j with indices in C_i in the order occurring in T (from bottom to top). This given, we set

$$\Delta_T(x) \;=\; \Delta(A_1)\Delta(A_2)\ldots\Delta(A_h) \;, \qquad 3.15$$

and refer to it as the *Garnir element* corresponding to T, we shall also say that $\Delta_T(x)$ is *standard* , *of shape* μ,...etc, if the same holds true for T itself.

For instance if

$$T_o \;=\; \begin{matrix} 1 & & \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix} \qquad 3.16$$

Then

$$\Delta_{T_o}(x) \;=\; det\begin{pmatrix} 1 & x_5 & x_5^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_1 & x_1^2 \end{pmatrix} \times det\begin{pmatrix} 1 & x_6 \\ 1 & x_3 \end{pmatrix} \times det\begin{pmatrix} 1 & x_7 \\ 1 & x_4 \end{pmatrix} \;. \qquad 3.17$$

These polynomials were used by Garnir in [], in his reconstruction of Young's natural representation. Recall that $\lambda(T)$ denotes the shape of a tableau T and $\mathcal{IT}(\mu)$ represents the collection of all injective tableaux of shape μ with entries $1, 2, \ldots, n$. It is not difficult to show (see []) that the space

$$\Gamma_\mu \;=\; \mathcal{L}\{\Delta_T \;:\; T \in \mathcal{IT}(\mu)\ \} \;, \qquad 3.18$$

linear span of the standard Garnir elements of shape μ, is an irreducible S_n-module with character given by χ^μ in the Young indexing.

It develops that the results of section II.3 immediately yield us the following remarkable fact.

Proposition 3.5

The Garnir elements of shape μ are contained in the space of harmonics of the orbit $[a_\mu]$ for any given alphabet $\mathcal{A}$. More precisely we have

$$\mathcal{L}\{\partial_x^p \Delta_T(x) \;:\; T \in \mathcal{IT}(\mu)\ \} \;\subseteq\; \mathbf{H}_{[a_\mu]} \;. \qquad 3.19$$

Proof

Since the space $\mathbf{H}_{[a_\mu]}$ is S_n-invariant and closed under differentiation, we need only show that it contains at least one Garnir element of that shape. To be specific let us show that

$$\Delta_{T_o} \in \mathbf{H}_{[a_\mu]} \quad , \tag{3.20}$$

where T_o is the tableaux we used to construct our basic orbit point a_μ. In order not to spoil the beauty of the argument with excessive notation we shall carry it out in the special case $\mu = (3,3,1)$. Note then that filling of the diagram of $(3,3,1)$ with $1, 2, \ldots, 7$ from top to bottom produces the tableau T_o given in 3.16. This gives that

$$a_\mu \;=\; a(T_o) \;=\; (\alpha_3, \alpha_2, \alpha_2, \alpha_2, \alpha_1, \alpha_1, \alpha_1) \ .$$

Now acting on this point with the Young subgroup

$$G \;=\; S_{[1,2,5]} \times S_{[3,6]} \times S_{[4,7]}$$

yields a total of $3 \times 2 \times 2 = 12$ distinct points. Thus a_μ is regular with respect to the action of G. From part (i) and (iii) of Theorem II.3.3 we then deduce that

$$\mathbf{H}_{[a_\mu]_G} \;=\; \mathcal{L}[\partial_x \Delta_G] \ , \tag{3.21}$$

where $[a_\mu]_G$ denotes the G-orbit of a_μ. Since $[a_\mu]_G$ is a subset of our S_n-orbit $[a_\mu] = [a_\mu]_{S_n}$ we necessarily have the reverse inclusion of the corresponding ideals. More precisely we must have

$$\mathcal{J}_{[a_\mu]_{S_n}} \;\subseteq\; \mathcal{J}_{[a_\mu]_G} \ ,$$

and a fortiori

$$gr\ \mathcal{J}_{[a_\mu]_{S_n}} \;\subseteq\; gr\ \mathcal{J}_{[a_\mu]_G} \ .$$

Taking orthogonal complemements yields that

$$\mathbf{H}_{[a_\mu]_G} \;\subseteq\; \mathbf{H}_{[a_\mu]_{S_n}} \ .$$

Combining with 3.21 we finally get that

$$\Delta_G \in \mathbf{H}_{[a_\mu]_{S_n}}$$

but this is our desired conclusion 3.20, since it is easily derived from II.5.2 and 3.17 that Δ_G and the Garnir element Δ_{T_o} are one and the same. We see no need to repeat the argument in the general case since this requires only routine notational changes.

We can actually do considerably better than we might suspect and quickly convert 3.19 into an equality. To this end note that Proposition 3.4 allows us to write each monomial $x^\epsilon \in \mathcal{B}_\mu$ in the form

$$x^\epsilon \;=\; m_{T_\epsilon}(x)/x^{\rho_\epsilon} \ , \tag{3.22}$$

where $m_{T_\epsilon}(x)$ is the monomial of a suitable a standard tableau of shape μ and ρ_ϵ is a non negative exponent vector. It will be convenient for our next argument that we assign to each $x^\epsilon \in \mathcal{B}_\mu$ unique standard tableau T_ϵ giving 4.22. For instance we can let T_ϵ be the lexicographically first such tableau. This given we have:

Theorem 3.1

The collection of polynomials

$$\{ \, \partial^{\rho_\epsilon} \, \Delta_{T_\epsilon}(x) \, \}_{x^\epsilon \in \mathcal{B}_\mu} \qquad 3.23$$

is a basis for $\mathbf{H}_\mu$. In particular $\mathbf{H}_\mu$ is the linear span of the derivatives of the Garnir elements of shape μ.

Proof.

Note that if A is an ordered subset of X_n then the corresponding Vandermonde determinant $\Delta(A)$ is an alternating sum of monomials all lexicographically larger that the monomial corresponding to the diagonal term in $\Delta(A)$. Thus if T is a standard tableau, the same will be true for each of the factors $\Delta(C_i)$ occuring in the definition of $\Delta_T(x)$. From this we easily deduce that $\Delta_T(x)$ has an expansion of the form

$$\Delta_T(x) \;=\; m_T(x) \;+ \sum_{x^p \;>_{lex}\; m_T(x)} c_p \;x^p \;\; ,$$

where "$>_{lex}$" denotes the lexicographic order of monomials defined in section I.4. Now, the definition of lexicographic order (see section I.4 page 13) immediately yields that the implication

$$x^p <_{lex} x^q \quad \rightarrow \quad \partial^\eta x^p <_{lex} \partial^\eta x^q$$

must hold for any differentiation ∂^η which doesn't kill x^q. However, this gives that the polynomials $\partial^{\rho_\epsilon} \, \Delta_{T_\epsilon}(x)$ may be be written in the form

$$\partial^{\rho_\epsilon} \, \Delta_{T_\epsilon}(x) \;=\; a x^\epsilon \;+ \sum_{x^p >_{lex} x^\epsilon} a_p x^p \;\; ,$$

with suitable coefficients a, a_p, and this is all that we need to establish their independence. Since by the previous proposition all these polynomials are in $\mathbf{H}_\mu$ and there are, by Proposition 3.1, a total of $dim\, \mathbf{R}_\mu = dim\; gr\; \mathbf{R}_m u = dim\, \mathbf{H}_\mu$ of them, they must necessarily from a basis. Q.E.D.

This remarkable result has a windfall of remarkable consequences.

Theorem 3.2

(i) The ideal $gr\; \mathcal{J}_{[a_\mu]}$ may be defined as the collection of polynomials which kill all the Garnir elements of shape μ. That is

$$gr\; \mathcal{J}_{[a_\mu]} \;=\; \mathcal{I}_\mu \;=\; \{ \; P \in \mathbf{R} \;:\; P(\partial_x)\Delta_T(x) = 0 \;\; \forall \;\; T \in \mathcal{IT}(\mu) \; \} \;\; . \qquad 3.24.$$

(ii) In particular this ideal and the two graded S_n-modules $\mathbf{H}_\mu$ and gr $\mathbf{R}_\mu$ are independent of the choice of the alphabet $\mathcal{A}$.

(iii) The collection of monomials $\mathcal{B}_\mu$ is a basis for gr $\mathbf{R}_\mu$.

(iv) The Hilbert series of $\mathbf{H}_\mu$ satisfies the recursion

$$F_{\mathbf{H}_\mu} \;=\; \sum_{i=1}^{h(\mu)} q^{i-1}\, F_{\mathbf{H}_{\mu^{(i)}}} \quad . \tag{3.25}$$

Proof

Since $gr\ \mathcal{J}_{[a_\mu]} = \mathbf{H}_\mu^\perp$, property (i) is an immediate consequence of Theorem 3.1 and Proposition II.4.1. Since there is no trace of any alphabet in the Garnir elements, we see that $(i) \rightarrow (ii)$. To show (iii) we note that our construction gives that the polynomial

$$\partial^{\rho_\epsilon}\, \Delta_{T_\epsilon}(x)$$

is homogeneous of the same degree as x^ϵ. Thus the collection in 3.23 is a homogeneous basis for $\mathbf{H}_\mu$. We can thus use formula I.2.5 and derive that the Hilbert series of BH_μ is given by the sum

$$F_{\mathbf{H}_\mu}(q) \;=\; \sum_{x^\epsilon \in \mathcal{B}_\mu} q^{degree\ x^\epsilon} \quad . \tag{3.26}$$

This allows us to apply Proposition II.5.2 and obtain that $\mathcal{B}_\mu$ is a basis for *gr* $\mathbf{R}_\mu$ as desired. Finally, property (iv) is an immediate consequence of (iii) and the recursion in 3.12.

Theorem 3.1 has also the following beautiful corollary.

Theorem 3.4

The S_n-modules $\mathbf{H}_\mu$ are nested into each other according to reverse dominance. More precisely we have

$$\nu \;\geq\; \mu \;\;\longrightarrow\;\; \mathbf{H}_\nu \;\subseteq\; \mathbf{H}_\mu \tag{3.27}$$

Proof

It is sufficient to show that at least one Garnir element of shape ν is in $\mathbf{H}_\mu$. Again we only need to prove this when ν is an immediate successor of μ in the dominance order. Suppose that ν is obtained from μ by lowering a square as depicted in 3.14. Given an injective tableau T_1 of shape μ let T_2 be the tableau of shape ν obtained by lowering the (j,b)-entry of T_1 into the square $(j+1,a)$ and leaving all the other entries unchanged. We see from the definition 3.15, that Δ_{T_2} differs from Δ_{T_1} only in the two factors corresponding to columns j and $j+1$. In the other case, when ν is obtained from μ by lowering a corner square from a row to the row immediately below, the situation is again the same. In summary we can pair off each Garnir element of shape μ with a Garnir element of shape ν by changing only two column factors. The change resulting from the transfer of just one of the variables from one the Vandermonde determinants to another one of smaller size. Now we can show that, as long as Δ_{T_2} is obtained from Δ_{T_1} in this manner, then Δ_{T_2} can be obtained from

Δ_{T_1} by a differentiation followed by a suitable S_n action. A moment of reflection should make it clear that this is an immediate consequence of the following identity.

Proposition 3.6

For any $0 \leq h < k$ we have

$$\begin{aligned}[k,k+1,\ldots,k+h]'\,\partial_{x_k}^{k-h-1}\,\Delta(x_1,x_2,\cdots,x_k)\,\Delta(x_{k+1},x_{k+2},\cdots,x_{k+h}) \;&=\\ (k-1)!\,\Delta(x_1,x_2,\ldots,x_{k-1})\,\Delta(x_{k+1},x_{k+2},\cdots,x_{k+h},x_k)\;&.\end{aligned} \tag{3.28}$$

Where for any set of integers A the symbol $[A]'$ is to denote (as in A. Young []) the alternating sum of all the elements of the symmetric group S_A.

Proof

The case $h=0$ is immediate. In fact, we should read 3.28 as

$$\partial_{x_k}^{k-1}\,\Delta(x_1,x_2,\cdots,x_k) \;=\; (k-1)!\,\Delta(x_1,x_2,\cdots,x_{k-1})\;,$$

and this clearly obtained by differentiating the last row of the Vandermonde determinant $\Delta(X_k) = det\|x_i^{j-1}\|_{i,j=1}^k$ $k-1$ times with respect to x_k. To deal with the case $h \geq 1$ we expand $\Delta(X_k)$ with respect to the last row and obtain

$$\Delta(X_k) \;=\; (-1)^k \sum_{i=0}^{k-1} (-1)^{i+1}\,\Delta_i(X_{k-1})\,x_k^i\;, \tag{3.29}$$

where $\Delta_i(X_{k-1})$ is a polynomial in $x_1,x_2,\ldots,x_{k-1}$ which reduces to $\Delta(X_{k-1})$ when $i=k-1$. Differentiating 3.29 $k-1-h$ times with respect to x_k, and noting that

$$\Delta(x_{k+1},x_{k+2},\ldots,x_{k+h}) \;=\; [k+1,k+2,\ldots,k+h]'\,x_{k+1}^0 x_{k+2}^1\cdots,x_{k+h}^{h-1}$$

we can write

$$\begin{aligned}&[k,k+1,\ldots,k+h]'\,\partial_{x_k}^{k-h-1}\,\Delta(x_1,x_2,\cdots,x_k)\,\Delta(x_{k+1},x_{k+2},\cdots,x_{k+h}) \;=\\ &\qquad (-1)^k\sum_{i=0}^{k-1}(-1)^{i+1}\,\Delta_i(X_{k-1})\;(i)_{k-1-h}\quad\times\\ &\qquad\qquad [k,k+1,\ldots,k+h]'\,[k+1,\ldots,k+h]'\;\;x_k^{i-k+h+1}\,x_{k+1}^0x_{k+2}^1\cdots,x_{k+h}^{h-1}\end{aligned} \tag{3.30}$$

However, using the group algebra identity

$$[k,k+1,\ldots,k+h]'\;\;[k+1,\ldots,k+h]' \;=\; h!\,[k,k+1,\ldots,k+h]'$$

the last factor in the i^{th} summand reduces to the determinant

$$[k,k+1,\ldots,k+h]\,x_k^{i-k+h+1}\,x_{k+1}^0x_{k+2}^1\cdots,x_{k+h}^{h-1}$$

which vanishes except for $i=k-1$ where it evaluates to

$$\Delta(x_{k+1},x_{k+2},\ldots,x_{k+h},x_k)\;.$$

Thus the right hand side of 3.30 collapses into the single term

$$(-1)^{2k}\, h!\, (k-1)_{k-1-h}\, \Delta(X_{k-1})\, \Delta(x_{k+1}, x_{k+2}, \ldots, x_{k+h}, x_k)\ ,$$

which is precisely the right hand side of 3.28. This establishes the proposition and completes the proof of Theorem 3.5

There are a number of relations which are easily derived from the combinatorics of the orbit $[a_\mu]$ which yield very useful elements of the ideal defined in 3.24. In particular we may derive in this manner a complete set of generators for $\mathcal{I}_\mu$. The mechanism for constructing these relations is best understood if we identify each row increasing injective tableau T of shape μ with the orbit point $b(T)$ defined at the beginning of this section. The idea is to construct polynomials whose failure to vanish at a given tableau forces conditions which cannot be satisfied by any orbit tableau. For instance if $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0)$ then the non vanishing of the polynomial

$$P(x) \;\; = \;\; (x_i - \alpha_1)(x_i - \alpha_2) \cdots (x_i - \alpha_k)\ ,$$

at the point $b(T)$ implies, that the entry i cannot be in any of the first k rows of T. But this is clearly impossible if T is to be of shape μ. In other words $P(x)$ must vanish at all the points of $[a_\mu]$. This gives that $P(x) \in \mathcal{J}_{[a_\mu]}$ and therefore its highest homogeneous component must belong to $\mathcal{I}_\mu = gr\ \mathcal{J}_{[a_\mu]}$. We thus obtain that $x_i^k \cong 0$ modulo $\mathcal{I}_\mu$ for any $i = 1, 2, \ldots, n$. In the same manner we can derive the following general result.

Proposition 3.7

For any any $s \in [1, k]$ and any subset $S \subseteq [1, n]$ of cardinality at least

$$1 + \mu_{s+1} + \cdots + \mu_k \tag{3.32}$$

we have

$$\textstyle\prod_{i \in S} x_i^s \;\; \cong \;\; 0 \qquad\qquad (\ mod\ \mathcal{I}_\mu\ )\ . \tag{3.33}$$

Proof

If the polynomial

$$P(x) \;\; = \;\; \prod_{i \in S} (x_i - \alpha_1)(x_i - \alpha_2) \cdots (x_i - \alpha_s)$$

fails to vanish at the point $b(T)$ then none of the elements of S can be in any of the first s rows of T. This forces them all in the remaining $k - s$ rows. However, since these rows have a total of $\mu_{s+1} + \cdots + \mu_k$ squares, we are left with an impossibility as soon as the cardinality of S exceeds this number. Thus P must vanish at all points of $[a_\mu]$. This places P in $\mathcal{J}_\mu$ and its highest homogeneous component in $\mathcal{I}_\mu$. QED

We should mention that the identity 3.3 was used in [] to study the action of S_n on the highest homogeneous component of the quotient $gr\ \mathbf{R}_\mu = \mathbf{Q}[X]/\mathcal{I}_\mu$. We shall show next that an ideal basis for $\mathcal{I}_\mu$, first introduced by Tanisaki in [], can also be obtained by a similar tableau argument. To this end we need to modify a bit our notation concerning partitions. Given $\mu \vdash n$, it will be convenient in this context to present the partition conjugate to μ as an n-component vector of the form

$$\mu' = (0 \leq \mu'_1 \leq \mu'_2 \leq \cdots \leq \mu'_n) \ .$$

We shall also set

$$d_s(\mu) \ = \ \mu'_1 + \mu'_2 + \cdots + \mu'_s \qquad\qquad (\ for \quad s = 1, \ldots, n \ ) \ . \tag{3.34}$$

For a given subalphabet $S = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$, let $e_r(S)$ denote the elementary symmetric function $e_r(x_{i_1}, x_{i_2}, \ldots, x_{i_s})$ if $r \leq s$ and 0 otherwise.

Proposition 3.8

The ideal $\mathcal{I}_\mu$ contains the collection of partial elementary symmetric functions

$$\mathcal{C}_\mu \ = \ \{ \ e_r(S) \ : \ S \subseteq X_n \ \& \ r > |S| - d_{|S|}(\mu) \ \} \tag{3.35}$$

Proof

Let S be of cardinality s. We are to show that

$$e_s(S) \ , \ e_{s-1}(S) \ , \ \ldots \ , \ e_{s-d_s(\mu)+1}(S) \tag{3.36}$$

all belong to $\mathcal{I}_\mu$. Since $\mathcal{I}_\mu$ is an S_n-invariant ideal, it is sufficient to show this for $S = \{x_1, x_2, \ldots, x_s\}$. This given, note that if $s > n - \mu_i$ then at least $s - (n - \mu_i)$ of the labels $1, 2, \ldots, s$ must necessarily spill into the i^{th} row of any orbit tableau of shape μ. This implies that the polynomial

$$P(t; x) \ = \ (t + x_1)(t + x_2) \cdots (t + x_s)$$

must contain the factor $(t + \alpha_i)^{s-n+\mu_i}$ at any orbit point. Letting u^+ denote u if $u > 0$ and zero otherwise we see that, at any orbit point, $P(t; x)$ must be divisible by the polynomial

$$M(t; \alpha) \ = \ \prod_{i=1}^{k} (t + \alpha_i)^{(\mu_i + s - n)^+} \ .$$

Set for a moment

$$m \ = \ deg_t \, M(t; \alpha) \ = \ \sum_{i=1}^{k} (\mu_i + s - n)^+ \ , \tag{3.37}$$

and let $Q(t; x, \alpha)$, respectively $R(t; x, \alpha)$, denote the quotient and the remainder of the long division of P by M. That is

$$P(t; x) \ = \ M(t; \alpha) \ Q(t; x, \alpha) \ + \ R(t; x, \alpha) \qquad\qquad (\text{ with } deg_t \, R \ < \ deg_t \, M \ ) \ . \tag{3.38}$$

Note that since $P(t;x)$ is homogeneous of total degree s in the variables $t, x_1, \ldots, x_s$ and $M(t;\alpha)$ is homogeneous of total degree m in $t, \alpha_1, \ldots, \alpha_k$ then $Q(t;x,\alpha)$ itself must be homogeneous of total degree $q = s - m$ in its arguments. Setting for convenience

$$M(t;\alpha) = \sum_{i=0}^{m} M_i(\alpha)\, t^i \qquad \text{and} \qquad Q(t;x,\alpha) = \sum_{i=0}^{q} Q_i(x,\alpha)\, t^i \ ,$$

we see that the coefficient $Q_i(x,\alpha)$ must be homogeneous of total degree $q - i = s - m - i$ as a polynomial in $\{x_1, x_2, \ldots, x_s, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. Clearly, the divisibility of P by M at any orbit point simply means that all the coefficients of $R(t;x,\alpha)$ belong to the ideal $\mathcal{J}_\mu$. Now from 3.38 we derive that the coefficient of t^u in $R(t;x,\alpha)$ has the expansion

$$R_u(x,\alpha) \;=\; e_{s-u}(S) \;-\; \sum_{i=0}^{u} Q_i(x,\alpha)\, M_{u-i}(\alpha) \ . \tag{3.39}$$

Since M contains no $x\,'s$, each of the terms coming out of the sum in 3.39 has x-degree at most $s - m$. Thus if $s - u > s - m$, the highest homogeneous component of $R_u(x,\alpha)$ must be $e_{s-u}(S)$. In summary we can conclude that the partial elementary symmetric functions

$$e_s(S), e_{s-1}(S), \ldots, e_{s-m+1}(S)$$

must all belong to the ideal $\mathcal{I}_\mu = gr\ \mathcal{J}_\mu$. Comparing with 3.36 we see that to complete our proof we need only show that $m = d_s(\mu)$. In other words (in view of 3.37) we must have

$$\sum_{i=1}^{k} (\mu_i - (n-s))^+ \;=\; \mu_1' + \mu_2' + \cdots + \mu_s' \ .$$

However this is geometrically obvious since both sides of this equation count the number of squares of the diagram of μ that are strictly to the right of column $n - s$. QED

The collection $\mathcal{C}_\mu$ was used by Tanisaki [] to simplify some of the arguments of DeConcini-Procesi in their study [] of the quotient ring $gr\,\mathbf{R}_\mu$. It is shown in [] that $\mathcal{C}_\mu$ generates the ideal $\mathcal{I}_\mu = gr\ \mathcal{J}_\mu$. Garsia and Procesi in [] made a crucial use of this fact in their calculation of the graded character of $gr\,\mathbf{R}_\mu$. Our final task in this section is to give at the very least all the basic steps in the identification of this character. We must therefore include the Tanisaki result that $\mathcal{I}_\mu = (\mathcal{C}_\mu)$. It develops that in proving it we shall have to establish a number of properties of the quotient ring $\mathbf{Q}[X_n]/(\mathcal{C}_\mu)$ which forcedly duplicate what we already know about $gr\,\mathbf{R}_\mu$. When we finally obtain that $\mathbf{Q}[X_n]/(\mathcal{C}_\mu)$ and $gr\,\mathbf{R}_\mu$ are one and the same we shall end up with alternate derivations of several propositions we have already established. We trust that the redundancy of this development is compensated by a deeper and multisided understanding of this remarkable S_n-module.

To carry out our program we need to establish a number of important properties of the collections $\mathcal{C}_\mu$. We shall express them as two separate propositions.

Proposition 3.9

(i) The collections $\mathcal{C}_\mu$ *are nested into each other according to the dominance order of partitions. That is we have*

$$\nu \ \leq \ \mu \ \ \longrightarrow \ \ \mathcal{C}_\nu \ \subseteq \ \mathcal{C}_\mu \ . \tag{3.40}$$

(ii) Each $\mathcal{C}_\mu$ *contains the elementary symmetric functions*

$$e_1(x_1,\ldots,x_n)\ ,\ e_2(x_1,\ldots,x_n)\ ,\ \ldots\ ,\ e_n(x_1,\ldots,x_n)\ . \tag{3.41}$$

(iii) For any partition μ *with* k *parts we have*

$$x_n^k \ \in \ \big(\mathcal{C}_\mu\big) \tag{3.42}$$

Proof

It is well known and easy to show that we have $\nu \leq \mu$ in dominance if and only if

$$\nu_1'+\nu_2'+\cdots+\nu_s' \ \ \leq \ \ \mu_1'+\mu_2'+\cdots+\mu_s' \qquad\qquad (\text{for} \quad s=1\ldots n)\ .$$

Thus, when $\nu \leq \mu$, the condition

$$r > s-\nu_1'-\nu_2'-\cdots-\nu_s'$$

is more stringent than

$$r > s-\mu_1'-\mu_2'-\cdots-\mu_s' \ . \tag{3.43}$$

In view of the definitions 3.34 and 3.35 we see that 3.40 must hold true as asserted. Note that for $s=n$ 3.43 reduces to $r>0$ thus all elementary symmetric functions in the full alphabet X_n must belong to each $\mathcal{C}_\mu$. This gives Property (ii). In a similar vein we note that for $s=n-1$ 3.43 reduces to

$$r \ > \ n-1-(n-\mu_n') \ \ = \ \ \mu_n'-1 \ .$$

Thus when μ has k parts this condition can be rewritten as

$$r \ \geq \ k \ .$$

Thus the definition 3.35 gives

$$e_k(x_1,\ldots,x_{n-1}) \ \ \in \ \ \mathcal{C}_\mu \ . \tag{3.44}$$

On the other hand from (ii) we can easily derive that

$$\frac{1}{(1-tx_n)} \ \cong \ (1-tx_1)(1-tx_2)\cdots(1-tx_{n-1}) \qquad\qquad \big(\text{mod} \quad (\mathcal{C}_\mu)\big)\ .$$

Equating coefficients of t^k we then get

$$x_n^k \ \cong \ e_k(x_1,x_2,\ldots,x_{n-1}) \ . \qquad\qquad \big(\text{mod} \quad (\mathcal{C}_\mu)\big)\ .$$

Combining with 3.44 yields 3.42 as desired.

The next result shows that each collection $\mathcal{C}_\mu$ is finely intermeshed with each of the collections $\mathcal{C}_{\mu^{(i)}}$. More precisely

Proposition 3.10

When μ is a partition with k parts then for any $i = 1, \ldots, k$ and for any $e_r(S) \in \mathcal{C}_{\mu^{(i)}}$ we have the following congruence modulo the ideal $(\mathcal{C}_\mu)$

$$x_n^{i-1}\, e_r(S) \;\cong\; \begin{cases} 0 & \text{if } |S| \leq n-1-\mu_i \text{ and} \\ -x_n^i\, e_{r-1}(S) & \text{if } |S| > n-1-\mu_i \end{cases} \tag{3.45}$$

Proof

It will be again helpful if we identify each partition with its own Ferrers' diagram. Recall then that $\mu^{(i)}$ is obtained by removing from μ the corner square in column μ_i. This means that strictly to the right of this column $\mu^{(i)}$ and μ have the same number of squares, while to the left $\mu^{(i)}$ has one less square than μ. This simple fact implies that we must have

$$d_s(\mu^{(i)}) \;=\; \begin{cases} d_{s+1}(\mu) & \text{if } s \leq n-1-\mu_i \\ d_{s+1}(\mu) - 1 & \text{if } s > n-1-\mu_i \end{cases} \quad . \tag{3.46}$$

To show the first equality in 3.45 we resort to the identity

$$e_r(S)\, z^m \;=\; (-1)^m\, e_{r+m}(S) \;+\; (-1)^{m+1} \sum_{j=0}^{m-1} (-z)^j\, e_{r+m-j}(S, z) \ , \tag{3.47}$$

which can be obtained by equating the coefficients of t^{r+m} in the trivial relation

$$\prod_{x\in S} (1-tx)\ (1-(tz)^m) \;=\; \prod_{x \in S} (1-tx)\ (1-tz) \sum_{i=0}^{m-1} (tz)^i \ .$$

This given, setting $z = x_n$ and $m = i-1$, 3.47 becomes

$$e_r(S)\, x_n^{i-1} \;=\; (-1)^{i-1}\, e_{r+i-1}(S) \;+\; (-1)^i \sum_{j=0}^{i-2} (-x_n)^j\, e_{r+i-1-j}(S, x_n) \ . \tag{3.48}$$

Now, if S is a subalphabet of cardinality $s \leq n-1-\mu_i$ then $r > s - d_s(\mu^{(i)})$ and 3.46 give

$$r > s - d_{s+1}(\mu) = s - d_s(\mu) - \mu'_{s+1} \ . \tag{3.49}$$

Note that each of the subscripts $r+i-1-j$ in the summands of 3.49 must then be greater than $s+1-d_{s+1}(\mu)$ and this forces all the summands in $(\mathcal{C}_\mu)$. Note next that since each column of μ strictly to the right of column μ_i has length less than i, then for $s+1 \leq n-\mu_i$, we must have $\mu'_{s+1} \leq i-1$. Thus 3.49 gives

$$r+i-1 > s - d_s(\mu) \ .$$

which forces the first term in 3.48 to lie in $\mathcal{C}_\mu$. This establishesthe first congruence in 3.45.

Note next that for $s>n-1-\mu_i$ the second case of 3.46 converts the inequality $r>s-d_s(\mu^{(i)})$ into

$$r>s+1-d_{s+1}(\mu)\ \ .$$

Thus when S is any s-subalphabet of X_{n-1} and $s>n-1-\mu_i$ then $e_r(S)\in\mathcal{C}_{\mu^{(i)}}$ implies that

$$e_r(S,x_n)\in\mathcal{C}_\mu\ \ .$$

But then the trivial identity $e_r(S,x_n)=e_r(S)+x_ne_{r-1}(S)$, multiplied by x_n^{i-1} yields

$$x_n^{i-1}e_r(S)\ \ =\ \ x_n^{i-1}e_r(S,x_n)\ \ -\ \ x_n^ie_{r-1}(S)\ \cong\ \ -\ x_n^ie_{r-1}(S)\ \ . \qquad \big(\mathrm{mod}\quad(\mathcal{C}_\mu)\big)\ \ .$$

This establishes the second congruence in 3.45 and completes our proof.

Remark 3.1

It will be convenient to denote by $\mathcal{C}_\mu(\alpha)$ the collection consisting of all the coefficients of all the remainders of the divisions

$$\prod_{i=1}^{s}(t+x_{j_i})\ /\ \prod_{i=1}^{k}(t+\alpha_i)^{(\mu_i+s-n)^+}\ \ ,$$

obtained as $\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}$ varies over all s-subalphabets of X_n and s runs from 1 to n. The above argument shows that the collection $\mathcal{C}_\mu$ is obtained by taking the highest homogeneous components of the elements of $\mathcal{C}_\mu(\alpha)$. It will be good to keep this in mind in the sequel. We shall refer to the elements of $\mathcal{C}_\mu$, respectively $\mathcal{C}_\mu(\alpha)$, as the *homogeneous* and *non-homogeneous Tanisaki polynomials*.

Theorem 3.5

The collections $\mathcal{C}_\mu(\alpha)$ and $\mathcal{C}_\mu$ generate the ideals $\mathcal{J}_\mu$ and $\mathcal{I}_\mu=gr\ \mathcal{J}_\mu$ respectively.

Proof

The argument is a beautiful example illustrating the power of Proposition II.4.3. We shall to use it here with the set S (occurring in the statement of the proposition) specialized to the orbit $[a_\mu]$, the $Q_i\,'s$, and the $P_i\,'s$, specialized to the elements of the collections $\mathcal{C}_\mu(\alpha)$ and $\mathcal{C}_\mu$. To apply the proposition we need to check that condition II.4.18 is satified. That is we are only left to verify that

$$dim\,\mathbf{R}\ /\ (\mathcal{C}_\mu)\ \ \le\ \ |[a_\mu]|\ \ .$$

We can actually prove a bit more and show that the collection $\mathcal{B}_\mu$ defined by the recursion 3.12 spans the quotient $\mathbf{R}\ /\ (\mathcal{C}_\mu)$. We shall give a recursive algorithm for expanding (modulo $(\mathcal{C}_\mu)$) every monomial x^p in terms of the monomials in the collection $\mathcal{B}_\mu$. We proceed by a double induction. We assume that, for each $i=1,\ldots,k$, we know how to expand (mod $(\mathcal{C}_{\mu^{(i)}})$) any monomial $x_1^{p_1}\cdots x_{n-1}^{p_{n-1}}$ in terms of the collection $\mathcal{B}_{\mu^{(i)}}$. We also assume that we know how to expand (mod $(\mathcal{C}_\mu)$) every monomial

$$x_1^{p_1}\cdots x_{n-1}^{p_{n-1}}x_n^{p_n}\qquad\quad (\text{ with}\quad p_n\ge i\ )\ \ .$$

in terms of monomials in the subcollection

$$\mathcal{B}_{\mu}^{\geq i+1} \;=\; \sum_{j=i+1}^{k} \mathcal{B}_{\mu^{(j)}} \; x_n^{j-1} \;,$$

then complete the induction by showing how to expand (modulo $(\mathcal{C}_\mu)$) any monomial

$$x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{i-1}$$

in terms of the collection $\mathcal{B}_{\mu}^{\geq i}$. We should note that for $n=1$ the result is trivial. Thus to verify that such an induction can be carried out, we need to check that for any given n and μ we have a starting point for the descent process indicated above. Now part (iii) of Proposition 3.9 enables us to start our descent at $i=k$. In fact, 3.42 yields that all the monomials $x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{k}$ expand to 0. It should be clear how the induction is completed. Indeed, given a monomial $x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{i-1}$, by induction we produce the expansion

$$x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} \;=\; \sum_{b \in \mathcal{B}_{\mu^{(i)}}} c_b \, b(X_{n-1}) \;+\; E$$

with an error term $E \in (\mathcal{C}_{\mu^{(i)}})$. This means that the difference

$$x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{i-1} \;-\; \sum_{b \in \mathcal{B}_{\mu^{(i)}}} c_b \, b(X_{n-1}) x_n^{i-1} \;=\; E \, x_n^{i-1} \qquad 3.50$$

has an expansion of the form

$$E \, x_n^{i-1} \;=\; \sum_{e_r(S) \in \mathcal{C}_{\mu^{(i)}}} A_{r,S} \, x_n^{i-1} e_r(S) \;,$$

where the coefficients $A_{r,S}$ are suitable polynomials in $x_1, x_2, \ldots, x_n$. But now Proposition 3.10, yields that this error term is a linear combination of monomials $x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n}$ with $p_n \geq i$, all of which, by the other induction hypothesis, can be expanded (mod $(\mathcal{C}_\mu)$) in terms of $\mathcal{B}_{\mu}^{\geq i+1}$. Combining these expansions with 3.50 gives the expansion of our monomial $x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{i-1}$ in terms of $\mathcal{B}_{\mu}^{\geq i}$ and completes the induction.

Remark 3.2

Some comments are in order concerning this theorem and its proof. First of all we see that a biproduct of the equality $(\mathcal{C}_\mu) = \mathcal{J}_\mu$ is that $gr\,\mathbf{R}_\mu = \mathbf{Q}[X_n]/(\mathcal{C}_\mu)$. In particular we must have

$$dim\, \mathbf{Q}[X_n]/(\mathcal{C}_\mu) \;=\; |\mathcal{B}_\mu| \;.$$

This forces $\mathcal{B}_\mu$ to be a basis for $\mathbf{Q}[X_n]/(\mathcal{C}_\mu)$ and for $gr\,\mathbf{R}_\mu$ as well. Thus we see that through the collections $\mathcal{C}_\mu$ we have been led to an alternate way of establishing part (iii) of Theorem 3.2. However,the argument here yields a bit more. Indeed, a close scrutiny of the expansion algorithm reveals (see []) that we have essentially proved that $\mathcal{B}_\mu$ is in fact the *standard* monomial basis $\mathcal{B}_{\mathcal{I}}$

discussed in the Remarks I.4.3 and I.4.4. This fact shows in particular where Propositions 3.2 and 3.4 are coming from. Indeed, we see that 3.13 can be derived (using 3.13) directly from Remark I.4.4. While Proposition 3.4 can be seen to be a natural consequence of the fact that a basis $\mathcal{B}_{\mathcal{J}}$ is by definition always a lower ideal of monomials. However, the most significant uses of the collections $\mathcal{C}_\mu$ are yet to come. Indeed, they play a crucial role in the identification of the graded characters of the modules $\mathbf{H}_\mu$. The next theorem may be seen as a first step in this direction.

It will be convenient here and in the rest of this section to set

$$p^\mu(t) \;=\; char_t\, gr\,\mathbf{R}_\mu \;=\; char_t\,\mathbf{H}_\mu \;.$$

Proposition 3.10 and Theorem 3.5 may now be used to derive the behaviour of this character upon restriction to S_{n-1}.

Theorem 3.6

If $\mu \vdash n$ has k parts then

$$p^\mu(t)\,|_{S_{n-1}} \;=\; \sum_{i=1}^{k} t^{i-1} p^{\mu^{(i)}}(t) \;. \tag{3.51}$$

Proof

Let the action of a permutation

$$\sigma = (\sigma_1, \ldots, \sigma_{n-1}) \in S_{n-1}$$

on a monomial $x^\epsilon = x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} \in \mathcal{B}_{\mu^{(i)}}$ have the expansion

$$\sigma\, x^\epsilon \;=\; \sum_{x^\eta \in \mathcal{B}_{\mu^{(i)}}} x^\eta\, a^{(i)}_{\eta,\epsilon}(\sigma) \;+\; E \;, \tag{3.52}$$

with the error term E in $(\mathcal{C}_{\mu^{(i)}})$. Then the graded character of the action of S_{n-1} on the module $gr\,\mathbf{R}_{\mu^{(i)}}$, evaluated at σ, is given by the expression

$$p^{\mu^{(i)}}(\sigma;t) \;=\; \sum_{x^\epsilon \in \mathcal{B}_{\mu^{(i)}}} \sigma\, x^\epsilon|_{x^\epsilon}\;\; t^{deg\;x^\epsilon} \;=\; \sum_{x^\epsilon \in \mathcal{B}_{\mu^{(i)}}} a^{(i)}_{\epsilon,\epsilon}(\sigma)\;\; t^{deg\;x^\epsilon} \;. \tag{3.53}$$

Now, 3.52 multiplied by x_n^{i-1} gives

$$\sigma\, x^\epsilon x_n^{i-1} \;=\; \sum_{x^\eta \in \mathcal{B}_{\mu^{(i)}}} x^\eta x_n^{i-1}\, a^{(i)}_{\eta,\epsilon}(\sigma) \;+\; E x_n^{i-1} \;. \tag{3.54}$$

From the identity 3.45 we can derive (as we did in the previous proof) that the error term Ex_n^{i-1} has an expansion (mod $(\mathcal{C}_\mu)$) which involves only monomials of the subcollection $\mathcal{B}_\mu^{\geq i+1}$. Since $\mathcal{B}_\mu$ is a basis for quotient ring $\mathbf{R}_\mu$ we must necessarily have

$$Ex_n^{i-1}\,|_{x^\epsilon x_n^{i-1}} \;=\; 0 \;.$$

Thus equating coefficients of $x^\epsilon x_n^{i-1}$ in both sides of 3.54 gives

$$\sigma\, x^\epsilon x_n^{i-1}\,\big|_{x^\epsilon x_n^{i-1}} \;\;=\;\; a_{\epsilon,\epsilon}^{(i)}(\sigma) \;\; .$$

Since this must hold true for every $i = 1, \ldots, k$ we derive that the character $p^\mu(t)$ evaluated at σ has the expansion

$$p^\mu(\sigma; t) \;\;=\;\; \sum_{i=1}^{k} t^{i-1} \sum_{x^\epsilon \in \mathcal{B}_{\mu^{(i)}}} a_{\epsilon,\epsilon}^{(i)}(\sigma)\, t^{deg\ x^\epsilon} \;\; .$$

Using 3.53 we can replace the internal sum here by $p^{\mu^{(i)}}(t)$ and obtain 3.51 as desired.

Now let

$$p^\mu(t) \;\;=\;\; \sum_{\lambda \geq \mu} \chi^\lambda\, C_{\lambda\mu}(t) \;\; .$$

be the expansion of our graded character in terms of the irreducible characters and let

$$C[X_n; t] \;\;=\;\; F\, p^\mu(t) \;\;=\;\; \sum_{\lambda} S_\lambda[X_n]\, C_{\lambda\mu}(t) \qquad 3.55$$

denote its Frobenius image. As we have indicated in 3.3 it follows immediately from the construction of the orbit $[a_\mu]$ that

$$C_{\lambda\mu}(t)\,\big|_{t=1} \;\;=\;\; K_{\lambda\mu} \;\; . \qquad 3.56$$

Our final goal in this section is to present the basic steps of the Garsia-Procesi [] proof that

$$C_{\lambda\mu}(t) \;=\; \tilde{K}_{\lambda\mu}(t) \;\; . \qquad 3.57$$

We shall use the same notation as in [] so that we may freely refer to that paper for some of the more tedious technical details.

We start by observing that the Frobenius image of operation of *restriction* to S_n is precisely the operator Γ_1 defined in 2.38. More precisely we have

Proposition 3.11

If $\mathbf{M}$ *is an* S_n*-module with character* $\chi^{\mathbf{M}}$ *then the Frobenius image of the character of the action of* S_{n-1} *on* $\mathbf{M}$ *is given by the identity*

$$F_{n-1}\,(\chi^{\mathbf{M}}\,|_{S_{n-1}}) \;\;=\;\; \Gamma_1\, F_n\, \chi^{\mathbf{M}} \;\; . \qquad 3.58$$

Proof

To avoid ambiguity, since two symmetric groups are involved here, we have denoted by F_n and F_{n-1} the Frobenius maps corresponding to S_n and S_{n-1}. This given, since both Γ_1 and the Frobenius maps are linear operators, it is sufficient to verify 3.58 when $\mathbf{M}$ is an irreducible module. So let $\chi^{\mathbf{M}} = \chi^\lambda$. Now it is well known since the work of Young [] that

$$\chi^\lambda\,|_{S_{n-1}} \;\;=\;\; \sum_{\lambda^-} \chi^{\lambda^-} \;\; ,$$

where the sum is over all partitions λ^- which immediately precede λ in Young's lattice (i.e. obtained by removing one of the corner squares from the diagram of λ). So the left hand side of 3.58 in this case reduces to

$$\sum_{\lambda^-} S_{\lambda^-} \ .$$

On the other hand the classical Pieri rule [] giving the multiplication of a Schur function by the elementary symmetric function $e_1[X_n]$ has a dual version which may be written as

$$S_{\lambda/1} \ = \ \sum_{\lambda^-} S_{\lambda^-} \ .$$

In view of our definition 2.38 we see that in this case both sides of 3.58 evaluate to the same thing. This establishes the Proposition.

Using this proposition and the definition 3.55, the recursion in 3.51 may be rewritten as

$$\Gamma_1 \, C_\lambda[X_n;t] \ = \ \sum_{i=1}^{h(\mu)} t^{i-1} \ p^{\mu^{(i)}}(t) \ .$$

Comparing with Theorem 2.7, we see that we have just established the case $k = 1$ of 2.40 when $\tilde{H}_\mu[X_n;t]$ is replaced by $C_\mu[X_n;t]$. Although this equation by itself is not sufficient to imply that

$$C_\mu[X_n;t] \ = \ \tilde{H}_\mu[X_n;t] \ , \tag{3.59}$$

we have shown (Proposition 2.3) that all of the equations in 2.52 do indeed suffice. Thus our approach to the proof of 3.57 will be to show that the graded characters $p^\mu(t)$ satisfy all the recursions

$$\Gamma_{1^k} \, p^\mu(t) \ = \ \sum_{1\leq i_1<i_2<\cdots<i_k\leq h(\mu)} t^{i_1-1+i_2-1+\cdots+i_k-1} \ p^{\mu^{(I)}}(t) \ . \tag{3.60}$$

Where with some abuse of notation we are using the symbol Γ_{1^k} also for the linear operator on class functions defined by setting

$$\Gamma_{1^k} \, \chi^\lambda \ = \ \chi^{\lambda/1^k} \ . \tag{3.61}$$

Clearly, the equations in 2.52 and 3.60 are equivalent since the former are the Frobenius images of the latter. However, before we can proceed to verify them we need to find out what kind of operation on the module $gr\,\mathbf{R}_\mu$ produces a module with character $\Gamma_{1^k}\, p^\mu(t)$. To answer this question we need some notation.

For a given alphabet A let us denote by $S_{[A]}$ the group of permutations of the letters of A. If A and B are disjoint alphabets we shall also denote by $S_{[A]} \otimes S_{[B]}$ the subgroup of $S_{[A+B]}$ which leaves A and B invariant (as sets). We can assume here, without loss, that $A = 1, 2, \ldots, h$ and $B = h+1, h+2, \ldots, h+k$ with $h + k = n$. If $\sigma_A \in S_{[A]}$ and $\sigma_B \in S_{[B]}$ with

$$\sigma_A \ = \ \begin{pmatrix} 1 & 2 & \cdots & h \\ \sigma_1 & \sigma_2 & \cdots & \sigma_h \end{pmatrix} \quad \text{and} \quad \sigma_B \ = \ \begin{pmatrix} h+1 & h+2 & \cdots & h+k \\ \sigma_{h+1} & \sigma_{h+2} & \cdots & \sigma_{h+k} \end{pmatrix} \ ,$$

then we shall set

$$\sigma_A \cdot \sigma_B \;=\; \begin{pmatrix} 1 & 2 & \cdots & h & h+1 & h+2 & \cdots & h+k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_h & \sigma_{h+1} & \sigma_{h+2} & \cdots & \sigma_{h+k} \end{pmatrix} .$$

Finally, for an alphabet B we shall let $N[B]$ denote the sum of the *signed* elements of $S_{[B]}$. More precisely,

$$N[B] \;=\; \sum_{\sigma_B \in S_{[B]}} sign(\sigma_B)\, \sigma_B \;\; .$$

The idempotent $N[B]$ is what Young refers to as the *negative symmetric group of* B. We shall also view $N[B]$ as an idempotent in $S_{[A]} \otimes S_{[B]}$. In fact, we shall systematically identify each element of $\sigma_A \in S_{[A]}$ with the element $\sigma_A \cdot \epsilon_B$ and each element $\sigma_B \in S_{[B]}$ with the element $\epsilon_A \cdot \sigma_B$. Here of course ϵ_A and ϵ_B denote the identity elements of $S_{[A]}$ and $S_{[B]}$ respectively.

This given, it can be shown that if $\mathbf{M}$ is an $S_{[A+B]}$-module with character χ then the restriction of $N[B]\,\mathbf{M}$ to the subgroup $S_{[A]} \otimes \epsilon_A$ is an $S_{[A]}$-module with character $\Gamma_{1^k}\,\chi$. In fact, we can even be more constructive and state that

Proposition 3.12

Let $\mathcal{M} = \{m_1, m_2, \ldots, m_M\}$ *be a basis for an* $S_{[A+B]}$*-module* $\mathbf{M}$ *with character* χ *Then the action of* $S_{[A]}$ *on the linear span of* $N(B)\mathcal{M} = \{N(B)m_1, N(B)m_2, \ldots, N(B)m_M\}$ *induces a representation with character* $\Gamma_{1^k}\,\chi$.

This is not difficult to show and it is but a very special case of a more general result whose statement may be found in []. Thus we refer the reader to [] for the proof.

To obtain the general recursion in 3.60 we aim to use this proposition with

$$\begin{aligned} \mathbf{M} \;&=\; gr\,\mathbf{R}_\mu \quad , \quad \mathcal{M} \;=\; \mathcal{B}_\mu, \\ A = \{1, 2, \ldots, h\} \quad &, \quad B = \{h+1, h+2, \ldots, h+k\} \;\; , \end{aligned}$$

where for convenience we have set $h = n - k$. Let us also set

$$X_A \;=\; \{x_1, x_2, \ldots, x_h\} \quad \text{and} \quad X_B \;=\; \{x_{h+1}, x_{h+2}, \ldots, x_{h+k}\} \;\; .$$

We shall use the same notation as in section 2 and follow closely the notation in []. Our point of departure is the equation 3.12 which in this context should be written in the form

$$\mathcal{B}_\mu \;=\; \sum_{i=1}^{h(\mu)} \mathcal{B}_{\mu^{(i)}} \;\times\; x_n^{i-1} \;\; . \tag{3.62}$$

Carrying out k recursive uses of this equation yields the expansion

$$\mathcal{B}_\mu \;=\; \sum_{p_1 p_2 \cdots p_k} \mathcal{B}_{\mu^{(p)}} \;\times\; x_n^{p_1-1} x_{n-1}^{p_2-1} \cdots x_{n-k+1}^{p_k-1} \;\; . \tag{3.63}$$

Since the monomials in $\mathcal{B}_{\mu^{(p)}}$ only depend on the variables in X_A the application of the antisymmetrizer $N(B)$ to each monomial in $\mathcal{B}_\mu$ results in the expansion

$$N(B)\,\mathcal{B}_\mu \;=\; \sum_{p_1p_2\cdots p_k} \mathcal{B}_{\mu^{(p)}} \;\times\; N(B)\, x_n^{p_1-1}x_{n-1}^{p_2-1}\cdots x_{n-k+1}^{p_k-1}\ . \qquad 3.64$$

Now, the expression $N(B)\, x_n^{p_1-1}x_{n-1}^{p_2-1}\cdots x_{n-k+1}^{p_k-1}$ vanishes identically whenever two of the $p_i\,'s$ are equal and when they are all distinct it evaluates to the determinant

$$\Delta_p(X_B) \;=\; det\,\|x_i^{p_j-1}\| \qquad (\text{with} \quad i\in B\ ,\ j\in[1..k])\ . \qquad 3.65$$

Thus 3.64 may be rewritten as

$$N(B)\,\mathcal{B}_\mu \;=\; \sum_{\substack{p_1p_2\cdots p_k\\ distinct}} \mathcal{B}_{\mu^{(p)}} \;\times\; \Delta_p(X_B)\ . \qquad 3.66$$

It is important to have a close look at the exponent vectors $(p_1,p_2,\cdots,p_k)$ that occur in this summation and the nature of the corresponding partitions $\mu^{(p)}$. It will be helpful if we visualize the mechanism that produces the terms in 3.63 as *paths* in the tree $\tilde{\mathcal{T}}_\mu$ of the recursion 3.62. Basically, $\tilde{\mathcal{T}}_\mu$ is the same as the tree $\mathcal{T}_\mu$ we used at the beginning of the section. Only here we label the nodes of $\tilde{\mathcal{T}}_\mu$ by partitions and the edges by the exponents p_i. More precisely, we label the root by μ. Then recursively, we label the out edges of a node labelled by ν, from left to right, by

$$1\ ,\ 2\ ,\ \ldots\ ,\ h(\nu)$$

and its children, in the same order, by

$$\nu^{(1)}\ ,\ \nu^{(2)}\ ,\ \ldots\ ,\ \nu^{(h(\nu))}\ .$$

We can see then that each term in 3.63 corresponds to a path of length k out the root of $\tilde{\mathcal{T}}_\mu$. Then by reading the labels of the edges along the path we get the exponent vector $p_1,p_2,\ldots,p_k$ and the label of the end node gives $\mu^{(p)}$. We also see that, when $p_1p_2\cdots p_k$ are all distinct, the vector

$$I_p = (i_1 < i_2 < \cdots < i_k) \qquad 3.67$$

obtained by rearranging $p_1,p_2,\cdots,p_k$ in increasing order represents a subset of $[1,h(\mu)]$. This given we can construct the partition $\mu^{(I_p)}$ by the operation described in section 2. Now it develops that the partitions $\mu^{(p)}$ and $\mu^{(I_p)}$ are related in a manner which is crucial in our proof of 3.60.

Proposition 3.13

If $p_1 > p_2 > \cdots > p_k$ then

$$\mu^{(p)} \;=\; \mu^{(I_p)}\ . \qquad 3.68$$

More generally if $p_1,p_2,\ldots,p_k$ are all distinct then in the dominance order we have

$$\mu^{(p)} \;\geq\; \mu^{(I_p)}\ . \qquad 3.69$$

Proof

If we follow the path of $\tilde{\mathcal{T}}_\mu$ which yields the exponent sequence $p = (p_1, p_2, \ldots, p_k)$ we pass through the sequence of partitions

$$\mu^{(p_1)} \ , \ \mu^{(p_1 p_2)} \ , \ \ldots \ , \ \mu^{(p_1 p_2 \cdots p_k)} \ .$$

To get $\mu^{(p_1)}$ we must *hit* (from the right) the diagram of μ at height p_1, climb up its vertical face to the nearest corner square and then remove it. Next to get $\mu^{(p_1 p_2)}$ we *hit* the diagram of $\mu^{(p_1)}$ at height p_2 climb up its vertical face to the nearest corner square and then remove it, $\ldots$ *etc.* Let us imagine that while we do this we label by p_i the i^{th} square that is being removed. Let us refer to this as the *standard labeling process* corresponding to the construction of $\mu^{(p)}$.

In the case that $p_1 > p_2 > \cdots > p_k$ let us carry out an alternate process. Let us first label the squares at the end of rows $p_1, p_2, \ldots, p_k$ by $p_1, p_2, \ldots, p_k$ respectively. Then let the labelled squares *bubble up* to the top of their column (without changing the order). We claim that in this case, we obtain the same final labelling as from the standard labeling process. The reason for this is quite simple. When $p_1 > p_2 > \cdots > p_k$, the standard labeling process places these labels in the diagram of μ exactly in this order when read from top to bottom. Moreover, these labels will occur tightly packed at the top of their column. But this is precisely, the labeling that results from the alternate process. Now recall that given a subset

$$I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)\}$$

the partition $\mu^{(I)}$ was obtained from μ by replacing its parts $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k}$ by $\mu_{i_1} - 1, \mu_{i_2} - 1, \ldots, \mu_{i_k} - 1$ respectively, then rearrange the parts back in weakly decreasing order. Now it is not difficult to recognize that the partition thus obtained is the same as that obtained by applying to μ the alternate labeling process described above for $p = (i_k, \ldots, i_2, i_1)$ and then remove the labelled squares. A moment's reflection should reveal that this yields the equality in 3.68.

In the general case when $p_1, p_2, \ldots, p_k$ are only known to be distinct the reason $\mu^{(p)}$ must dominate $\mu^{(I_p)}$ should now appear quite clear. In this construction of $\mu^{(I_p)}$ the labelled squares only bubble up their columns to reach their final destination. On the other hand, at the i^{th} step in the construction of $\mu^{(p)}$, the square which bears the label p_i is either on the same column as can be found in the construction of $\mu^{(I_p)}$ or, in the the worst case, in a column **west** of it. The reason for this is simple. The removal of squares due to the previous $i-1$ steps could have so depleted the column which originally contained the label p_i that there no longer is a square at level p_i in that column. Thus to reach the boundary of this new diagram, at that level, we are forced to go further west. This ultimately results in the label p_i landing westward of its final position in the construction of $\mu^{(I_p)}$. It follows then that, on and to the right of any column, the diagram of $\mu^{(p)}$ contains as many squares as the diagram of $\mu^{(I_p)}$. And this implies that $\mu^{(p)}$ dominates $\mu^{(I_p)}$. QED

This yields us the following basic result.

Theorem 3.7

The collection of polynomials

$$\sum_{h(\mu)\geq i_k>\cdots>i_2>i_1\geq 1} \mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}} \;\; \Delta_{i_k\cdots i_2 i_1}(X_B) \tag{3.70}$$

is a basis for the S_n-module $N(B)\ gr\ \mathbf{R}_\mu$

Proof

Proposition 3.12 yields that the expression in 3.66 is a spanning set for $N(B)\ gr\ \mathbf{R}_\mu$. However, note that 3.69 combined with Proposition 3.2 gives the containment

$$\mathcal{B}_{\mu^{(p)}} \;\subseteq\; \mathcal{B}_{\mu^{(I_p)}} \;\; .$$

Moreover, rearranging the components of $p=(p_1,p_2,\ldots,p_k)$ only changes the sign of $\Delta_p(X_B)$. This means that when $p_1,p_2,\ldots,p_k$ is not a decreasing sequence and $I_p=\{i_1,i_2,\ldots,i_k\}$ then the collection $\mathcal{B}_{\mu^{(p)}}\ \Delta_p(X_p)$ only adds terms that can already be obtained from $\mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}}\ \Delta_{(i_k\cdots i_2 i_1)}(X_B)$. This shows that there is no loss in dropping the former terms. Thus our spanning set is also given by the summation in 3.70. To complete the argument we need only show that the number of terms produced by 3.70 is equal to the dimension of the module $N(B)\ gr\ \mathbf{R}_\mu$. To this end note that since for $t=1$ $\tilde{H}_\mu[X_n;t]$ reduces to the homogeneous symmetric function

$$\tilde{H}_\mu[X_n;1] \;\; = \;\; h_\mu[X_n]$$

by setting $t=1$ in 2.40 we derive that

$$\Gamma_{1^k}\ h_\mu[X_n] \;\; = \sum_{h(\mu)\geq i_k>\cdots i_2>i_1\geq 1} h_{\mu^{(i_k\cdots i_2 i_1)}} \;\; .$$

Moreover, 3.56 gives that h_μ is the Frobenius image of the *ungraded* character of $gr\ \mathbf{R}_\mu$. Thus this ungraded character must satisfy the recursion

$$\Gamma_{1^k}\ p^{\mu}(1) \;\; = \sum_{h(\mu)\geq i_k>\cdots i_2>i_1\geq 1} p^{\mu^{(i_k\cdots i_2 i_1)}}(1) \;\; . \tag{3.71}$$

Now Proposition 3.12 gives us that this is precisely the ungraded character of $N(B)gr\ \mathbf{R}_\mu$. Thus its dimension must be given by the formula

$$dim\ N(B)gr\ \mathbf{R}_\mu \;\; = \sum_{h(\mu)\geq i_k>\cdots i_2>i_1\geq 1} dim\ gr\ \mathbf{R}_{\mu^{(i_k\cdots i_2 i_1)}} \;\; . \tag{3.72}$$

Since $\mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}}$ is a basis for $gr\ \mathbf{R}_{\mu^{(i_k\cdots i_2 i_1)}}$ we see that the sum in 3.72 yields precisely the cardinality of our spanning set 3.70 which is the only fact we needed to complete the proof.

Before we can proceed with our proof of 3.60 we need to relate the ideals $(\mathcal{C}_{\mu^{(I)}})$ to the ideal $(\mathcal{C}_\mu)$. Recall that by our conventions, the partition μ', conjugate to μ, has parts

$$\mu'_1, \leq \mu'_2 \leq \cdots \leq \mu'_n$$

with μ'_i giving the length of column $n+1-i$ in the diagram of μ. To simplify our language, let us refer to column $n+1-i$ simply as $bar[i]$ *of the diagram of* μ. Now given a subset $I=\{i_1<i_2<\cdots<I_k\}\subseteq[1,h(\mu)]$, we see from our construction of $\mu^{(I)}$, that we may write,

$$d_j(\mu^{(I)}) \;=\; d_{k+j}(\mu) \;-\; m_j \qquad (\; for \quad j=1,2,\ldots,n-k\;) \tag{3.73}$$

where m_j denotes the number of squares, we have removed from μ to obtain $\mu^{(I)}$, that *are weakly to the right of* $bar[k+j]$. The reason for this is that, since $\mu^{(I)}$ is a partition of $n-k$ what we should call $bar[j]$ of $\mu^{(I)}$ coincides with $bar[k+j]$ of μ. Thus the number of squares of the diagram of $\mu^{(I)}$ that are weakly to the right of its $bar[j]$ is precisely given by the right hand side of 3.73. This given we have the following extension of Proposition 3.10

Proposition 3.14

Let $I=\{i_1<i_2<\cdots<I_k\}\subseteq[1,h(\mu)]$, and S be any s-subset of $X_A=\{x_1,x_2,\ldots,x_h\}$, (with $h=n-k$). Then

$$e_r(S)\in\mathcal{C}_{\mu^{(I)}} \qquad or \quad equivalently \quad r \;>\; s-d_s(\mu^{(I)}) \tag{3.74}$$

forces

$$x_{h+1}^{i_1-1}x_{h+2}^{i_2-1}\cdots x_{h+k}^{i_k-1}\; e_r(S,x_{h+1},x_{h+2},\ldots,x_{h+m_s}) \;\in\; (\mathcal{C}_\mu)\;. \tag{3.75}$$

Note that using 3.73, condition 3.74 becomes

$$r \;>\; s+m_s-d_{s+k}(\mu)$$

so when all the removed squares are weakly to the right of $bar[s+k]$, that is when $m_s=k$, then

$$r \;>\; s+k-d_{s+k}(\mu)$$

and we necessarily have the stronger result

$$e_r(S,x_{h+1},x_{h+2},\ldots,x_{h+k}) \;\in\; (\mathcal{C}_\mu)\;.$$

We see that this extends the second case of 3.45. The proof in the general case requires manipulations with symmetric functions similar in nature to those that yielded the identity in 3.48. Since it is somewhat tedious we shall omit it here and refer the reader to [] where it is already given in the fullest detail.

To proceed we need to break the collection in 3.70 into the summands

$${}^k\mathcal{NB}_\mu{}^{=d} \;=\; \sum_{\substack{n(\mu)\geq i_k>\cdots>i_1\geq 1\\ i_1-1+\cdots+i_k-1=d}} \mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}}\;\Delta_{i_k\cdots i_2 i_1}(X_B)\;,$$

and set

$${}^k\mathcal{NB}_\mu{}^{\geq d} \;=\; \sum_{d'\geq d} {}^k\mathcal{NB}_\mu{}^{=d'} \quad,\quad {}^k\mathcal{NB}_\mu{}^{>d} \;=\; \sum_{d'>d} {}^k\mathcal{NB}_\mu{}^{=d'}\;.$$

Note that since all the elements of $\mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}}$ are monomials in the alphabet $X_A=\{x_1,x_2,\ldots,x_h\}$ with $h=n-k$, we shall represent the generic element of $\mathcal{B}_{\mu^{(i_k\cdots i_2 i_1)}}$ by $x_A^\epsilon = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_h^{\epsilon_h}$. This given, we can convert Proposition 3.13 into an algorithm for expanding the elements of $N(B)\, gr\, \mathbf{R}_\mu$ in terms of the basis in 3.70.

Proposition 3.14

For any subset $I=\{i_1<i_2<\cdots<i_k\}\subseteq[1,h(\mu)]$ and any polynomial $P(X_A)=P(x_1,x_2,\ldots,x_h)$ we have the expansion

$$P(X_A)\,\Delta_I(X_B) \;=\; \sum_{x_A^\epsilon\in\mathcal{B}_{\mu^{(I)}}} c_\epsilon\, x_A^\epsilon\, \Delta_I(X_B) \;+\; E \qquad (mod\ (\mathcal{C}_\mu)\,) \tag{3.76}$$

with

$$\begin{cases} a) \quad c_\epsilon \;=\; P(X_A)\,|_{x_A^\epsilon} & \text{in } \mathbf{R}_{\mu^{(I)}} \\ b) \quad E \;\in\; \mathcal{L}[\ {}^k\mathcal{N}\mathcal{B}_\mu{}^{>i_1-1+\cdots+i_k-1}] & \end{cases}\ . \tag{3.77}$$

Proof

Note that if P is a constant there is nothing to prove. Indeed, in this case 3.77 a) and b) become trivialities since the constant 1 is in all bases $\mathcal{B}_{\mu^{(I)}}$ and 3.76 reduces to a constant multiple of $\Delta_I(X_B)$ and $E=0$. We shall thus proceed by induction on the degree of P. To this end let

$$P(X_A) \;=\; \sum_{x_A^\epsilon\in\mathcal{B}_{\mu^{(I)}}} c_\epsilon\, x_A^\epsilon \qquad (mod\ (\mathcal{C}_{\mu^{(I)}})\,) \tag{3.78}$$

be the expansion of P in $gr\ \mathbf{R}_{\mu^{(I)}}$. We are to show that 3.76 holds in $\mathbf{R}_\mu$ with an error term which satisfies 3.77 b).

Let us then write 3.78 as

$$P(X_A) \;-\; \sum_{x_A^\epsilon\in\mathcal{B}_{\mu^{(I)}}} c_\epsilon\, x_A^\epsilon \;=\; \sum_{e_r(S)\in\mathcal{C}_{\mu^{(I)}}} Q_{r,S}(X_A)\, e_r(S)\ . \tag{3.79}$$

Multiplying by $x_{h+1}^{i_1-1}x_{h+2}^{i_1-2}\cdots x_{h+k}^{i_k-1}$ and applying the antisymmetrizer $N(B)$ to both sides of this equation we derive that the error term in 3.76 has the expansion

$$E \;=\; \sum_{e_r(S)\in\mathcal{C}_{\mu^{(I)}}} Q_{r,S}(X_A)\, N(B)\, x_{h+1}^{i_1-1}x_{h+2}^{i_2-1}\cdots x_{h+k}^{i_k-1}\, e_r(S)\ . \tag{3.80}$$

Now from 3.75 we deduce that, when $e_r(S)\in\mathcal{B}_{\mu^{(I)}}$ and $|S|=s$

$$x_{h+1}^{i_1-1}\cdots x_{h+k}^{i_k-1}\, e_r(S) \;=\; -\sum_{j=1}^{s} e_{r-j}(S)\, e_j(x_{h+1},\ldots,x_{h+m_s})\, x_{h+1}^{i_1-1}\cdots x_{h+k}^{i_k-1} \qquad (mod\ (\mathcal{C}_\mu)\,)\ .$$

Note that we may write

$$e_j(x_{h+1},\ldots,x_{h+m_s}) \;=\; \sum_\eta x_{h+1}^{\eta_1}\cdots x_{h+k}^{\eta_k}$$

with $\eta_i = {0 \atop 1}$ and $\eta_1 + \cdots + \eta_k = j$. Thus the error term in 3.80 is (mod $(\mathcal{C}_\mu)$) equal to a sum of terms of the form

$$E_{r,S,j,\eta} \;=\; Q_{r,S}(X_A)\, e_{r-j}(S)\, N(B)\, x_{h+1}^{i_1-1+\eta_1} \cdots x_{h+k}^{i_k-1+\eta_k} \ . \qquad 3.81$$

Since $(\mathcal{C}_{\mu^{(I)}})$ is a homogeneous ideal, we see that if P is homogeneous of degree m then it can be assumed that the factor $Q_{r,S}(X_A)$ is homogeneous of degree $m-r$. This implies that the factor $Q_{r,S}(X_A)e_{r-j}(S)$ in 3.81 is of degree $m-j<m$. On the other hand the term

$$N(B)\, x_{h+1}^{i_1-1+\eta_1} \cdots x_{h+k}^{i_k-1+\eta_k} \qquad 3.82$$

is different from zero only if the exponents $i_1-1+\eta_1\ ,\ \ldots\ ,\ i_k-1+\eta_k$ are all distinct. Moreover, by Proposition 3.7, the expression in 3.82 will necessarily vanish if any of these exponents is greater or equal to $h(\mu)$. This means that if a term $E_{r,S,j,\eta}$ survives, then denoting by $J=\{j_1<j_2<\cdots j_k\}$ the increasing rearrangement of $\{i_1+\eta_1,\ldots,i_k+\eta_k\}$ we shall have $J\subseteq[1,h(\mu)]$ and

$$E_{r,S,j,\eta} \;=\; Q(X_A)\,\Delta_{j_1-1,\cdots,j_k-1}(X_B)$$

with a polynomial $Q(X_A)$ of degree *less* than m. We can thus use the induction hypothesis and conclude that

$$E_{r,S,j,\eta} \;\in\; \mathcal{L}[\ {}^k\mathcal{N}\mathcal{B}_\mu{}^{\geq j_1-1+\cdots+j_k-1}]\ .$$

But since

$$j_1-1+\cdots+j_k-1 \;=\; i_1-1+\cdots+i_k-1+j \;>\; i_1-1+\cdots+i_k-1\ ,$$

we can be sure that

$$E_{r,S,j,\eta} \;\in\; \mathcal{L}[\ {}^k\mathcal{N}\mathcal{B}_\mu{}^{>i_1-1+\cdots+i_k-1}]\ .$$

Thus the same must hold true for E as well. This completes the induction and our proof.

We are finally in a position to establish the main and final result of this section.

Theorem 3.8

The t-Kostka polynomials $K_{\lambda\mu}(t)$ give the multiplicities of the irreducibles in the graded module $\mathbf{R}_\mu$. More precisely we have

$$p^\mu(t) \;=\; \sum_{\lambda\geq\mu} \chi^\lambda\, \tilde{K}_{\lambda\mu}(t) \qquad 3.83$$

Proof

As we have observed we need only establish the recurrence in 3.60. To this end, given a subset $I=\{i_1<i_2<\cdots<i_k\}\subseteq[1,h(\mu)]$, let us express the action of $S_{[A]}$ on $gr\ R_{\mu^{(I)}}$ in the form

$$\sigma_A\, x_A^\epsilon \;=\; \sum_{x_A^{\epsilon'}\in\mathcal{B}_{\mu^{(I)}}} x_A^{\epsilon'}\ a^{(I)}_{\epsilon',\epsilon}(\sigma_A)\ . \qquad (mod\ (\mathcal{C}_{\mu^{(I)}})\,) \qquad 3.84$$

This gives that

$$p^{\mu^{(I)}} \;=\; \sum_{x_A^{\epsilon}\in \mathcal{B}_{\mu^{(I)}}} t^{degree\ x_A^{\epsilon}}\ a_{\epsilon,\epsilon}^{(I)}\ . \qquad 3.85$$

From Proposition 3.14 we then get that

$$\sigma_A x_A^{\epsilon}\Delta_I(X_B) \;=\; \sum_{x_A^{\epsilon'}\in \mathcal{B}_{\mu^{(I)}}} x_A^{\epsilon'}\,\Delta_I(X_B)\ a_{\epsilon',\epsilon}^{(I)}(\sigma_A) \;+\; E\ . \qquad (mod\,(\mathcal{C}_{\mu})\,)$$

with

$$E \;\in\; \mathcal{L}[\ {}^{k}\mathcal{N}\mathcal{B}_{\mu}{}^{>i_1-1+\cdots+i_k-1}]\ .$$

Thus in $N(B)gr\ R_{\mu}$ we have

$$\sigma_A\ x_A^{\epsilon}\Delta_I(X_B)\ |_{x_A^{\epsilon}\Delta_I(X_B)} \;=\; a_{\epsilon,\epsilon}^{(I)}(\sigma_A)\ . \qquad 3.86$$

Now since $\Delta_I(X_B)$ is a polynomial of degree $i_1-1+\cdots+i_k-1$, Proposition 3.12 gives that

$$\Gamma_{1^k}\ p^{\mu}(t) \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le h(\mu)} t^{i_1-1+i_2-1+\cdots+i_k-1} \sum_{x_A^{\epsilon}\in \mathcal{B}_{\mu^{(I)}}} t^{degree\ x_A^{\epsilon}}\ a_{\epsilon,\epsilon}^{(I)}\ .$$

Combining this with 3.85 the recursion in 3.60 follows as desired. This completes our identification of the graded character $p^{\mu}(t)$.