UCSD LECTURE NOTES 1991-92

ORBIT HARMONICS
AND
GRADED REPRESENTATIONS

A. M. Garsia and M. Haiman (*)

CHAPTER 1.

Polynomials and differential operators.

1.0 Preliminaries

Let $X_n = \{x_1, x_2, \dots, x_n\}$ be an alphabet of commuting variables. Throughout this writing we shall denote by

$$\mathbf{R} \;=\; \mathbf{Q}[x_1, x_2, \dots, x_n]$$

the ring of polynomials

$$P(X_n) \;=\; P(x_1, x_2, \dots, x_n) \;=\; \sum_p c_p\, x^p \;\;,$$

with rational coefficients c_p. Here and in the following we let

$$x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \quad \text{and} \quad \partial^p = \partial_1^{p_1} \partial_2^{p_2} \cdots \partial_n^{p_n}$$

where

$$\partial_1 = \frac{\partial}{\partial x_1} \;\;, \partial_2 = \frac{\partial}{\partial x_2} \;\;, \dots \;, \quad \partial_n = \frac{\partial}{\partial x_n}$$

are the partial derivatives with respect to the given variables.

Given a formal power series $f(X) = \sum_q f_q\, x^q$ we let

$$f(\partial) \;=\; f(\partial_1, \partial_2, \dots, \partial_n) \;=\; \sum_q f_q\, \partial^q \qquad 0.1$$

(*) Work carried out under NSF grant support.

denote the corresponding differential operator. Of course the action of $f(\partial)$ on a polynomial is simply a linear extension of the action of the monomial ∂^q on the monomial x^p. That is we set

$$\partial_1^{q_1}\partial_2^{q_2}\cdots\partial_1^{q_n}\ x_1^{p_1}x_2^{p_2}\cdots x_1^{p_n} \;=\; \prod_{i=1}^{n} p_i(p_i-1)\cdots(p_i-q_i+1)\,x_i^{p_i-q_i}\ .$$

Note that if P is a polynomial of degree M and f is given by 0.1 then

$$f(\partial)P \;=\; \sum_{|q|\le M} f_q\,\partial^q\ P\ ,$$

where we have set $|q| = q_1+q_2+\cdots+q_n$. Thus no questions of convergence arise when dealing with formal power series as differential operators acting on polynomials or with polynomial differential operators acting on formal power series.

In particular if we set

$$e^{t_1\partial_1+t_2\partial_2+\cdots+t_n\partial_n} \;=\; \sum_q t^q\frac{\partial^q}{q!} \qquad\qquad (\ where\ \ q!=q_1!q_2!\cdots q_n!\ )$$

then it is an immediate consequence of Taylor's theorem that for any polynomial P we shall have

$$e^{t_1\partial_1+t_2\partial_2+\cdots+t_n\partial_n}P(x_1,x_2,\ldots,x_n) \;=\; P(x_1+t_1,x_2+t_2,\ldots,x_n+t_n)\ . \qquad 0.2$$

1.1 The invariant scalar product.

There is a natural scalar product on $\mathbf{R}$ which will play a basic role in our later developments. To define it, let us denote by L_o the linear operation of evaluating a polynomial at 0. That is

$$L_o\ P(x_1,x_2,\ldots,x_n) \;=\; P(x_1,x_2,\ldots,x_n)\,|_{x_1=x_2=\cdots x_n=0} \qquad 1.1$$

This given we shall set

$$\langle P\ ,\ Q\rangle \;=\; L_oP(\partial)Q(x)\ . \qquad 1.2$$

It is easy to see that this scalar product makes the monomials $\{x^p\}_p$ into an orthogonal basis. In fact, we have

$$\langle x^p\ ,\ x^q\rangle \;=\; \begin{cases} 0 & \text{if } p\neq q \\ p! & \text{if } p=q \end{cases}\ .$$

This scalar product has a number of useful properties. Let us recall that the action of a permutation

$$\sigma \;=\; \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$$

on a polynomial is simply defined by setting

$$\sigma\ P(x_1,x_2,\ldots,x_n) \;=\; P(x_{\sigma_1},x_{\sigma_2},\ldots,x_{\sigma_n})\ . \qquad 1.3$$

Since for any differential operator $f(\partial)$ and any polynomial Q we have

$$\sigma\, f(\partial)Q(X_n) \;=\; f(\partial_{\sigma_1}\partial_{\sigma_2},\dots,\partial_{\sigma_n})Q(x_{\sigma_1},x_{\sigma_2},\dots,x_{\sigma_n}) \tag{1.4}$$

it follows that $\langle P,Q\rangle$ is invariant under permutation action. That is, for all σ we have

$$\langle \sigma P\ ,\ \sigma Q\rangle \;=\; \langle P\ ,\ Q\rangle\ . \tag{1.5}$$

We can proceed a bit more generally and define an action of $GL(n)$ on $\mathbf{R}$ by setting, for any $n\times n$ matrix $A=\|a_{ij}\|_{i,j=1}^n$ and any monomial x^p

$$T_A\ x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n} \;=\; \prod_{j=1}^n (T_A x_j)^{p_j}\ , \tag{1.6}$$

where, for each $j=1\dots n$ we let

$$T_A\ x_j \;=\; \sum_{i=1}^n x_i a_{ij}\ .$$

This action extends, by linearity, to all polynomials $P(x_1,x_2,\dots,x_n)\in\mathbf{R}$. Perhaps it is best to think that the argument x in a polynomial $P(x)$ is a row vector $x=(x_1,x_2,\dots,x_n)$ and then write

$$T_A P(x) \;=\; P(xA) \tag{1.7}$$

where xA denotes matrix multiplication of the $1\times n$ vector x by A. We can easily see that this is in complete agreement with 1.6.

An important special case of 1.6 is obtained when A is taken to be the matrix which corresponds to the action of a permutation. To be precise, note that if for a permutation $\sigma\in S_n$ we set

$$P(\sigma) \;=\; \|\chi(i=\sigma_j)\|_{i,j=1}^n \tag{1.8}$$

then

$$T_{P(\sigma)}x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n} \;=\; x_{\sigma_1}^{p_1}x_{\sigma_2}^{p_2}\cdots x_{\sigma_n}^{p_n}\ .$$

Thus in this case 1.6 reduces to 1.3.

This given we see that 1.4 and 1.5 are but special cases of the following general identities:

Proposition 1.1

For any orthogonal matrix A and any pair $P,Q\in\mathbf{R}$ we have

$$T_A\ P(\partial_x)Q(x) \;=\; (T_A P)(\partial_x)\,(T_A Q)(x) \tag{1.9}$$

in particular

$$\langle T_A\ P\ ,\ T_A\ Q\rangle \;=\; \langle P\ ,\ Q\rangle\ . \tag{1.10}$$

Proof

Clearly it is sufficient to show 1.9 when P and Q are monomials. That is we need only show that for all pairs of exponent vectors p, q we have

$$T_A\ (\partial_x)^p\ x^q \ = \ (\partial_x A)^p\ (xA)^q \ . \tag{1.11}$$

Letting $a = \{a_1, a_2, \dots, a_n\}$ and $b = \{b_1, b_2, \dots, b_n\}$ be two new commuting alphabets, we can represent all of the identities in 1.11 by the single identity

$$T_A e^{(a,\partial_x)} e^{(b,x)} \ = \ e^{(a,\partial_x A)}\ e^{(b,xA)} \ , \tag{1.12}$$

where (x, y), for any two vectors x and y, denotes the customary inner product

$$(x, y) \ = \ x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \ .$$

Indeed, 1.11 may be derived from 1.12 by equating the coefficient of the monomial $a^p\ b^q$ on both sides of 1.12. This given, note that the orthogonality of A gives

$$(a, \partial_x A) = (aA^{-1}, \partial_x) \quad \text{and} \quad (b, xA) = (bA^{-1}, x) \ , \tag{1.13}$$

thus 1.12 may be rewritten as

$$T_A e^{(a,\partial_x)} e^{(b,x)} \ = \ e^{(aA^{-1},\partial_x)}\ e^{(bA^{-1},x)} \ . \tag{1.14}$$

But now, 0.2 gives

$$e^{(a,\partial_x)} e^{(b,x)} \ = \ e^{(b,x+a)} \ = \ e^{(b,a)} e^{(b,x)} \ ,$$

similarly

$$e^{(aA^{-1},\partial_x)} e^{(bA^{-1},x)} \ = \ e^{(bA^{-1},x+aA^{-1})} \ = \ e^{(bA^{-1},aA^{-1})} e^{(bA^{-1},x)} \ ,$$

and 1.14, using the definition 1.7 and 1.13, becomes

$$e^{(b,a)} e^{(b,xA)} \ = \ e^{(bA^{-1},aA^{-1})} e^{(b,xA)} \ .$$

However, this holds true because the orthogonality of A also yields that

$$(bA^{-1}, aA^{-1}) \ = \ (b, a) \ .$$

Finally, note that since the definitions 1.1 and 1.7 give that for any formal power series f

$$L_o T_A\ f \ = \ L_o\ f$$

we see that 1.10 is simply the special case $x = 0$ of 1.9. This completes our proof.

1.2 Homogeneous subspaces.

It will be convenient here and in the following to denote by π_m the linear operator on polynomials defined by setting

$$\pi_m\, x^p \;=\; \begin{cases} x^p & \text{if } |p| = m, \\ 0 & \text{if } |p| \neq m\ . \end{cases} \tag{2.1}$$

Clearly, π_m extracts the homogeneous component of degree m out of a polynomial. In particular, given a collection $\mathbf{V}$ of polynomials we shall let $\mathcal{H}_m(\mathbf{V})$ denote the image of $\mathbf{V}$ by π_m. In other words $\mathcal{H}_m(\mathbf{V})$ simply consists of the collection of homogeneous components of degree m from elements of $\mathbf{V}$.

This given, we have the following

Definition 2.1

A subspace $\mathbf{V} \subseteq \mathbf{R}$ is said to be homogeneous if and only if

$$\mathcal{H}_m(\mathbf{V}) \;\subseteq\; \mathbf{V} \qquad\qquad (\ \forall\ m \geq 0\ ) \tag{2.2}$$

We can easily see that every homogeneous subspace $\mathbf{V}$ admits the direct sum decomposition

$$\mathbf{V} \;=\; \mathcal{H}_0(\mathbf{V}) \;\oplus\; \mathcal{H}_1(\mathbf{V}) \;\oplus\; \mathcal{H}_2(\mathbf{V}) \;\oplus\; \cdots \tag{2.3}$$

A homogeneous subspace $\mathbf{V} \subseteq \mathbf{R}$, although possibly infinite dimensional, behaves in many ways like a finite dimensional subspace. Indeed, each of its homogeneous components $\mathcal{H}_m(\mathbf{V})$ is finite dimensional. In fact,

$$dim(\mathcal{H}_m(\mathbf{V})) \;\leq\; dim(\mathcal{H}_m(\mathbf{R})) = \binom{n+m-1}{n-1}$$

This circumstance yields, as we shall see, a number of properties that are not available for more general subspaces. First of all, we can write down the generating function of dimensions of the components $\mathcal{H}_m(\mathbf{V})$. This is the formal series

$$F_{\mathbf{V}}(q) \;=\; \sum_{m\geq 0} q^m\ dim(\mathcal{H}_m(\mathbf{V})) \tag{2.4}$$

which is usually referred to as the *Hilbert series* of $\mathbf{V}$. Secondly, because of 2.3, we can always produce a basis $\mathcal{B}(\mathbf{V})$ for $\mathbf{V}$ consisting of homogeneous polynomials. We shall also have the decomposition

$$\mathcal{B}(\mathbf{V}) \;=\; \sum_{m\geq 0} \mathcal{B}_m(\mathbf{V})$$

where $\mathcal{B}_m(\mathbf{V})$ denotes the collection of elements of degree m in $\mathcal{B}(\mathbf{V})$, and " $\sum$ " here simply means disjoint union. This gives us the useful formula

$$F_{\mathbf{V}}(q) \;=\; \sum_{b\in\mathcal{B}(\mathbf{V})} q^{degree(b)}\ . \tag{2.5}$$

In particular we see that

$$F_{\mathbf{R}}(q) \;=\; \sum_p q^{degree(x^p)} \;=\; \sum_p q^{p_1+p_2+\cdots+p_n} \;=\; \frac{1}{(1-q)^n}\ . \tag{2.6}$$

Another facet of the finite dimensional qualities of homogeneous subspaces can be stated as follows

Proposition 2.2

The orthogonal complement $\mathbf{U} = \mathbf{V}^\perp$ *of a homogeneous subspace* $\mathbf{V}$*, with respect to the scalar product in 1.2 , is also homogeneous. Moreover, we have*

$$\begin{aligned} &a)\quad \mathbf{V}^{\perp\perp} \;=\; \mathbf{V} \\ &b)\quad \mathbf{R} \;=\; \mathbf{U} \;\oplus\; \mathbf{V} \end{aligned} \tag{2.7}$$

Proof

Let $\mathcal{B}_m$ be an orthonormal basis for $\mathcal{H}_m(\mathbf{V})$. Let us then apply the Gramm-Schmidt orthonormalization procedure to the ordered collection consisting of the elements of $\mathcal{B}_m$ arranged in any order followed by the monomials $\{x^p\}_{|p|=m}$ in lex order. This produces an orthonormal basis $\mathcal{C}_m$ for $\mathcal{H}_m(\mathbf{R})$ which consists of $\mathcal{B}_m$ followed by an orthonormal set $\mathcal{A}_m$. This given, any polynomial $P \in \mathbf{R}$ may be written in the form

$$P \;=\; \sum_{m\ge 0} \Big(\sum_{a\in\mathcal{A}_m} \langle P\ ,\ a\rangle\, a \quad + \quad \sum_{b\in\mathcal{B}_m} \langle P\ ,\ b\rangle\, b \Big)\ . \tag{2.8}$$

Thus the polynomials P that are orthogonal to $\mathbf{V}$ are those of the form

$$P \;=\; \sum_{m\ge 0} \sum_{a\in\mathcal{A}_m} \langle P\ ,\ a\rangle\, a\ .$$

In other words $\{\mathcal{A}_m\}_{m\ge 0}$ is an orthonormal basis for $\mathbf{U} \;=\; \mathbf{V}^\perp$. This implies that $\mathbf{U}$ is a homogeneous subspace of $\mathbf{R}$ and that 2.7 a) & b) are immediate consequences of 2.8.

A basic example of homogeneous subspace, is the collection of solutions of a fixed system of homogeneous differential equations. To be precise, if $f = \{f_1, f_2, \ldots, f_k\}$ is any collection of homogeneous polynomials, let us set

$$\mathbf{S}(f_1, f_2, \ldots, f_k) \;=\; \{\ P \in \mathbf{R}:\ \ f_1(\partial)P = 0\ ,\ f_2(\partial)P = 0\ ,\ldots,\ f_k(\partial)P = 0\ \}\ .$$

It is easy show that if $f_1, f_2, \ldots, f_k$ are all homogeneous then $\mathbf{S}(f_1, f_2, \ldots, f_k)$ itself is a homogeneous subspace. Indeed, if

$$P \;=\; \sum_{m=0}^{M} P_m \qquad\qquad (\ with\ \ P_m = \pi_m P\ )$$

and $degree\ (f_i) = m_i$ then $degree(f_i(\partial)P_m) = m - m_i$, and we cannot have

$$0 \;=\; f_i(\partial)P \;=\; \sum_{m=0}^{M} f_i(\partial)P_m$$

without each of the individual summands $f_i(\partial)P_m$ being equal to zero as well.

Note further that every $P \in \mathbf{S}(f)$ is also annihilated by every differential operator in the ideal generated by the polynomials $f_1, f_2, \ldots, f_k$. That is we have $Q(\partial)P = 0$ for every polynomial Q of the form

$$Q \;=\; A_1 f_1 \;+\; A_2 f_2 \;+\; \cdots \;+\; A_k f_k \tag{2.9}$$

with $A_1, A_2, \ldots, A_k$ arbitrary polynomials. As customary we shall denote the ideal generated by $f_1, f_2, \ldots, f_k$ by the symbol

$$(\ f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k\ )\ .$$

It is easy to see that if $f_1, f_2, \ldots, f_k$ are homogeneous then $(f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)$ is also a homogeneous subspace. Moreover we have

Proposition 2.3

The orthogonal complement of the ideal $(f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)$ *is simply given by the solution space* $\mathbf{S}(f_1, f_2, \ldots, f_k)$.

Proof

It is clear that $\mathbf{S}(f_1, f_2, \ldots, f_k)$ is contained in $(f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)^{\perp}$. Thus we need only show the converse. To this end note that if $f \in (f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)$ then also the product $x^q f$ is in $(f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)$ for any monomial x^q. This implies that if $P \in (f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)^{\perp}$ then P must be orthogonal not only to f but to $x^q f$ as well . In other words we must have

$$L_o \partial^q f(\partial)P(X_n) \;=\; \langle f, P \rangle \;=\; 0$$

for arbitrary q. That is, the polynomial $f(\partial)P(X_n)$ and all its derivatives must be equal to zero at the origin. From Taylor's theorem we deduce then that $f(\partial)P(X_n)$ must be identically zero itself. This proves the reverse inclusion

$$(f_1\ ,\ f_2\ ,\ \ldots\ ,\ f_k)^{\perp} \;\subseteq\; \mathbf{S}(f_1, f_2, \ldots, f_k)\ .$$

Q.E.D.

1.3 Graded representations.

Let G be a finite group of $n \times n$ matrices and let $\mathbf{V}$ be a homogeneous subspace of $\mathbf{R}$ which is invariant under the action of G. More precisely, we assume that for each $A \in G$ and $P \in \mathbf{V}$ we have $T_A P \in \mathbf{V}$. Since the operator T_A preserves the degree of a polynomial each subspace $\mathcal{H}_m(\mathbf{V})$ is also invariant under G. Thus the action of G on $\mathbf{V}$ decomposes into a direct sum of finite dimensional representations. Let then $\chi^{\mathbf{V}}_m$ denote the character of the representation corresponding to $\mathcal{H}_m(\mathbf{V})$ and set

$$\chi^{\mathbf{V}}(q) \;=\; \sum_{m \geq 0} q^m \chi^{\mathbf{V}}_m \;. \tag{3.1}$$

This is in complete analogy with what we did in the definition 2.4. Indeed, $F_{\mathbf{V}}(q)$ is none other than the evaluation of $\chi^{\mathbf{V}}(q)$ at the identity element of G. We shall refer to $\mathbf{V}$ as a *graded* G-module, and to $\chi^{\mathbf{V}}(q)$ as the corresponding *graded* character. When convenient we shall also use the symbols *char* and $char_q$ to denote characters and graded characters respectively. In particular we may write $char_q \mathbf{V}$ instead of $\chi^{\mathbf{V}}(q)$.

Note that if $\mathcal{B}$ is any homogeneous basis of $\mathbf{V} \subseteq \mathbf{R}$ we may obtain the corresponding graded character at an element A of G, by the formula

$$\chi^{\mathbf{V}}(A;q) \;=\; \sum_{b \in \mathcal{B}} q^{degree(b)}\, T_A b \,|_b \;. \tag{3.2}$$

We shall also refer to the representation of G corresponding to the basis $\mathcal{B}$ as a graded representation of G.

Remark 3.1

We have another way of obtaining graded representations, that is in a sense *dual* to what we did above. We start by giving a G-invariant homogeneous ideal $\mathbf{I}$ and then let $\mathbf{R}_I = \mathbf{R}/\mathbf{I}$ denote the corresponding quotient polynomial ring. The homogeneity of $\mathbf{I}$ makes $\mathbf{R}_I$ into a graded ring and the G-invariance of $\mathbf{I}$ allows us to define an action of G on $\mathbf{R}_I$ as well as on each of its homogeneous subspaces $H_m(\mathbf{R}_I)$. In complete analogy to what we did before we can then define the graded character of $\mathbf{R}_I$ by setting

$$char_q\, \mathbf{R}_I \;=\; \sum_m q^m char\; H_m(\mathbf{R}_I) \;.$$

For any two polynomials P, Q set

$$\big\langle P\,,\, Q \big\rangle_G \;=\; \frac{1}{|G|} \sum_{A \in G} \big\langle\, T_A P\,,\, T_A Q \,\big\rangle \;,$$

and let $\mathbf{H}_I$ be the orthogonal complement of $\mathbf{I}$ with respect to this scalar product. This is the standard construction of a G-invariant complement of a G-invariant subspace. In our particular situation $\mathbf{H}_I$ turns out to be a graded G-module with the same graded character as $\mathbf{R}_I$. More precisely, we have

Proposition 3.1

Every homogeneous basis for $\mathbf{H}_I = \mathbf{I}^\perp$ *is also a basis for* $\mathbf{R}_I = \mathbf{R}/\mathbf{I}$ *and*

$$char_q\, \mathbf{R}_I \;=\; char_q\, \mathbf{H}_I \;=\; char_q\, \mathbf{R} \;-\; char_q\, \mathbf{I}\ . \tag{3.3}$$

Proof

Let $H_m(\mathbf{H}_I)$ be the homogeneous component of degree m of $\mathbf{H}_I$ and let $\mathcal{B}_m = \{P_1, P_2, \dots, P_M\}$ be any basis of $H_m(\mathbf{H}_I)$. From the direct sum decomposition

$$H_m(\mathbf{R}) \;=\; H_m(\mathbf{H}_I) \,\oplus\, H_m(\mathbf{I})$$

we derive that every homogeneous polynomial P of degree m can be uniquely written in the form

$$P \;=\; \sum_{i=1}^{M} c_i\, P_i \;+\; E$$

with E an element of $\mathbf{I}$. This shows that $\mathcal{B}_m$ spans $\mathcal{H}_m(\mathbf{R}_I)$. On the other hand the relation

$$\sum_{i=1}^{M} c_i\, P_i \;=\; E \,\in\, \mathbf{I}$$

can only hold for $E = 0$. Thus $P_1, P_2, \dots, P_M$ are also independent as elements of $\mathcal{H}_m(\mathbf{R}_I)$. This implies that $\mathcal{B}_m$ is a basis for $\mathcal{H}_m(\mathbf{R}_I)$. This establishes our first assertion. Finally, the identity in 3.3 must hold true because the action of G on $\mathcal{B}_m$ is expressed by one and the same matrix whether we consider $\mathcal{B}_m$ in $\mathcal{H}_m(\mathbf{H}_I)$ or in $\mathcal{H}_m(\mathbf{R}_I)$.

From 1.6 we derive that for a monomial x^q of degree m we have

$$T_A\, x^q \;=\; \sum_{|p|=m} x^p\, (T_A x^q)\,|_{x^p}\ .$$

Thus the action of T_A on the subspace $\mathcal{H}_m(\mathbf{R})$, when expressed in terms of the basis

$$\mathcal{M}_m = \{x^p : |p| = m\}\ ,$$

is given by the matrix

$$A^{(m)} \;=\; \|(T_A x^q)\,|_{x^p}\| \qquad\qquad (\ |p|, |q| = m\ )\ . \tag{3.4}$$

This is what is usually referred to as the m^{th} *symmetric power* of A.

The celebrated *Master Theorem* of Macmahon [25] may be viewed as a formula for the generating function of the traces of the successive symmetric powers of a given matrix. In other words, the Master Theorem yields a closed form for the graded character of the action of the operator T_A. Using this notation we can state

Theorem 3.1*(Macmahon)*

$$\sum_{m\geq 0} q^m trace\ A^{(m)} \;=\; \sum_p q^{|p|}\,(T_A x^p)\mid_{x^p} \;=\; \frac{1}{det(I-qA)}\ . \tag{3.5}$$

Proof

We note first that the definitions 1.6 and 1.7 immediately imply that for any two $n\times n$ matrices A, B we have

$$A^{(m)}B^{(m)} \;=\; (AB)^{(m)}\ . \tag{3.6}$$

This given, since the trace function is invariant under similarity, we only have to check the validity of 3.5, for triangular matrices. However, if A is triangular and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are its diagonal elements, then 3.5 reduces to the trivial identity

$$\sum_p q^{|p|}\ \lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_n^{p_n} \;=\; \prod_{i=1}^n \frac{1}{1-q\lambda_i}\ . \tag{3.7}$$

This proves the theorem.

By letting $A=P(\sigma)$ and setting $\chi^{\mathbf{R}}(P(\sigma);q)=\chi^{\mathbf{R}}(\sigma;q)$ we derive the important corollary:

Theorem 3.2

For any permutation $\sigma\in S_n$

$$\chi^{\mathbf{R}}(\sigma,q) \;=\; \sum_p q^{|p|}\ \sigma x^p\mid_{x^p} \;=\; \prod_{i=1}^n \frac{1}{(1-q^i)^{m_i(\sigma)}} \;=\; p_{\lambda(\sigma)}(1,q,q^2,\ldots) \tag{3.8}$$

where $m_i(\sigma)$ denotes the number of cycles of length i in σ, and $\lambda(\sigma)$ denotes the shape of σ that is

$$\lambda(\sigma) \;=\; \prod_{i=1}^n i^{\,m_i(\sigma)}\ .$$

Proof.

For the second equality it is sufficient to note that the characteristic polynomial of the matrix corresponding to a cycle of length i is precisely given by $1-q^i$. We recall here that p_λ denotes the power symmetric function indexed by partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$. That is

$$p_\lambda(x_1,x_2,x_3,\ldots) \;=\; \prod_{j=1}^n p_{\lambda_j}(x_1,x_2,x_3,\ldots)\ . \tag{3.9}$$

with

$$p_k(x_1,x_2,x_3,\ldots) \;=\; \sum_{i\geq 1} x_i^k\ .$$

Thus the last equality in 3.8 is obtained by making the replacements $\lambda\rightarrow\lambda(\sigma)$ and $x_i\rightarrow q^{i-1}$ in 3.9.

Making use of the Frobenius expansion of the power symmetric function we can transform 3.8 into a combinatorial identity which will turn out to be very useful for us here:

Theorem 3.3

$$\chi^{\mathbf{R}}(\sigma,q) \;=\; \sum_{\lambda\vdash n} \chi^{\lambda}(\sigma)\,\frac{\sum_{T\in ST(\lambda)}\; q^{c(T)}}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{3.10}$$

where $c(T)$ denotes the "cocharge" of T.

Proof.

The Frobenius expansion [24] gives that

$$p_{\lambda(\sigma)}(1,q,q^2,\ldots) \;=\; \sum_{\lambda\vdash n} \chi^{\lambda}(\sigma)\, S_\lambda(1,q,q^2,\cdots) \tag{3.11}$$

and 3.10 follows from the well known ([24] ex. 14 p. 49) identity

$$S_\lambda(1,q,q^2,\cdots) \;=\; \frac{\sum_{T\in ST(\lambda)}\; q^{c(T)}}{(1-q)(1-q^2)\cdots(1-q^n)}\;. \tag{3.12}$$

Remark 3.1

Given a standard tableau T the *cocharge tableau* $C(T)$ corresponding to T is the column-strict tableau of same shape obtained by the following procedure. We first replace the entry 1 of T by 0. Then, inductively, having replaced the entry i of T by c, the entry $i+1$ is replaced by c again if $i+1$ is east of i in T and by $c+1$ otherwise. The cocharge $c(T)$ may then be defined as the sum of the entries in $C(T)$. This given, it is easy to see that the smallest possible value of $c(T)$ as T varies among all standard tableaux of shape λ is given by the integer

$$n(\lambda) \;=\; \sum_{i=0}^{n-1} i\,\lambda_{n-i}\;, \tag{3.13}$$

which is the cocharge of the so called *superstandard* tableau of shape λ. This is the tableau obtained by filling the successive rows of the shape λ with the integers $1,2,\ldots,n$ from left to right starting with the longest row. We see then from 3.10 that the smallest m for which $H_m(\mathbf{R})$, as S_n-module, contains the irreducible representation with character χ^λ is precisely given by $n(\lambda)$.

1.4 Ideals and their associated graded versions.

Our ultimate goal here is the calculation of the graded character of certain remarkable S_n-invariant homogeneous subspaces of polynomials. Thus we shall be concerned with the case $G = S_n$. Nevertheless, before getting into our material, it will be good to review a few basic facts and constructions concerning general graded representations. Our calculations of graded characters hinges on a few elementary facts concerning ungraded quotients and their associated graded versions. We shall include them here since we need them in a more constructive form than they usually appear in the literature.

We need to adopt a partial order on monomials $x^p = x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}$ which is defined by setting $x^p << x^q$ and saying that x^p is *contained* in x^q if and only if

$$p_1 \le q_1\ ,\ p_2 \le q_2\ ,\ \cdots\ ,\ p_n \le q_n\ \ .$$

We also recall that an element x^p of a collection $\mathcal{C}$ of monomials is said to be minimal if it does not properly contain any other element of $\mathcal{C}$.

This given, we shall say that $\mathcal{U}$ is an *upper ideal* of monomials if and only if

$$x^p \in \mathcal{U}\ \ \&\ \ x^q\ >>\ x^p\ \ \rightarrow\ \ \ x^q \in \mathcal{U}\ \ .$$

In words, every monomial which contains an element of $\mathcal{U}$ is in $\mathcal{U}$. Clearly, if we take any finite collection of monomials $\mathcal{C} = \{m_1, m_2, \ldots, m_k\}$ then the collection

$$\mathcal{U}(\mathcal{C})\ \ =\ \ \{x^q\ \ :\ \ x^q >> m_i\ \ for\ \ some\ \ \ m_i \in \mathcal{C}\ \ \}$$

is an upper ideal. Remarkably the converse is also true. More precisely, we have

Theorem 4.1 *(Gordan) J. de Math. Pures et Apll. 6 (1900) 141-156, see also Grace-Young # 151 p. 178)*

Every upper ideal of monomials has a finite number of minimal elements

Proof.

We shall proceed by induction on the number of variables. Clearly, the assertion is true for $n = 1$. Assume then we have shown it to be true whenever the alphabet has cardinality less than n, and let $\mathcal{V}$ be a given upper ideal of monomials in the alphabet $X_n = \{x_1, x_2, \ldots, x_n\}$. Choose once and for all an element $x^q = x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n} \in \mathcal{V}$. For a given $i = 1..n$ and for $s < q_i$ let $\mathcal{V}_{is}$ be a collection of monomials in the alphabet $X_n - x_i$

$$\mathcal{V}_{is}\ \ =\ \ \{x^p/x_i^s\ \ :\ \ p_i = s\ \ \&\ \ x^p \in \mathcal{V}\ \ \}\ \ .$$

In words, $\mathcal{V}_{is}$ is obtained by removing x_i^s from every monomial in $\mathcal{V}$ which has x_i raised precisely to the power s. It easy to see that $\mathcal{V}_{is}$ is an upper ideal of monomials, and since the alphabet involved has less than n letters, by induction, we can find in $\mathcal{V}_{is}$ a finite set of monomials

$$\mathcal{C}_{is}\ \ =\ \ \{m_{is}^{(1)}, m_{is}^{(2)}, \ldots, m_{is}^{(k_{is})}\}$$

with the property that every element of $\mathcal{V}_{is}$ contains an element of $\mathcal{C}_{is}$. That is we shall have

$$\mathcal{V}_{is}\ \ =\ \ \mathcal{U}(\mathcal{C}_{is})\ \ .$$

We claim that each element x^p of $\mathcal{V}$ contains one of the monomials in the collection

$$\{\ x^q\}\cup\{\ x_i^s m_{is}^{(j)}\ \ for\ \ i = 1..n\ \ \ ,\ \ \ s < q_i\ \ \&\ \ j = 1..k_{is}\ \ \} \tag{4.1}$$

Indeed, if x^p does not contain x^q then for some $i \in [1..n]$ we must have $p_i < q_i$ and this places $x^p/x_i^{p_i}$ in the ideal $\mathcal{V}_{ip_i}$. Thus each minimal element of $\mathcal{V}$ must belong to the collection in 4.1. This completes the induction and the proof of the theorem.

Remark 4.1

We should note that Theorem 4.1 could have been stated for any collection whatsoever of monomials. Indeed, if $\mathcal{C}$ is such a collection then since $\mathcal{C}$ and $\mathcal{U}(\mathcal{C})$ have the same minimal elements, the result for $\mathcal{C}$ follows by applying Theorem 4.1 to $\mathcal{U}(\mathcal{C})$.

We shall derive next a somewhat sharper version of Hilbert's basis theorem than is usually given. To this end we need a total order on monomials that refines both containment and degree. More precisely, we shall order our monomials by increasing degrees, breaking equal degree ties lexicographically. Perhaps we should recall that the *lexicographic* order of monomials is defined by setting $x^p \leq_L y^q$ if and only if we have

$$p_1 = q_1 \ , \ p_2 = q_2 \ , \ \cdots \ , \ p_{i-1} = q_{i-1} \ \ \& \ \ p_i < q_i \quad \textit{for some} \ \ i \in [1, n] \ .$$

Thus according to the order we adopt here, x^p comes before x^q if and only if, either $|p| < |q|$ or $|p| = |q|$ and $x^p \leq_L y^q$. To avoid confusions we shall denote our order by the symbol "$<_{DL}$" and refer to it briefly as the *dlex order*.

Remark 4.2
Up to this moment we have tacitly assumed that the degree of a monomial $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ is given by the sum $p_1 + p_2 + \cdots + p_n$. It should be noted that in most of our considerations the degree can just as well be given by the sum $d_1 p_1 + d_2 p_2 + \cdots + d_n p_n$, where the coefficients d_i are some given fixed positive numbers. In other words under this notion of degree the variable x_i is assigned degree d_i. Of course, when a group action is involved then the coefficients d_i must be restricted so that the operators T_A (for A in the given group) do retain their degree-preserving property. We shall have occasion in the sequel to use to advantage this additional freedom in the choice of degree. A typical situation in which such freedom is possible is when the matrices $A \in G$ all break up in the same block diagonal fashion. In this case, we can assure that the action of G will preserve degree, by simply requiring that the degree function assigns the same degree to the variables indexing any given block.

For a given polynomial P let us then denote by $\lambda(P)$ the dlex largest monomial appearing in the expansion of P. We shall refer to $\lambda(P)$ as the *leading* monomial of P. Note then that if $\mathcal{J}$ is an ideal of polynomials then the collection of leading monomials

$$\mathcal{M}(\mathcal{J}) \ = \ \{ \, \lambda(P) \ : \ P \in \mathcal{J} \, \}$$

is an upper ideal of monomials. By Gordan's theorem the minimal elements of $\mathcal{M}(\mathcal{J})$ will form a finite collection. Let us denote it by $\mathcal{V}(\mathcal{J})$. Clearly, we can find elements $\{f_1, f_2, \ldots, f_k\}$ in $\mathcal{J}$ which give

$$\mathcal{V}(\mathcal{J}) \ = \ \{\lambda(f_1), \lambda(f_2), \ldots, \lambda(f_k)\} \ . \qquad 4.2$$

It is good to think that the $f_1, f_2, \ldots, f_k$ are listed in increasing dlex order of their leading monomials. Keeping this in mind we have

Theorem 4.2

If $\mathcal{J}$ is an ideal of polynomials and the elements $f_1, f_2, \ldots, f_k$ are constructed so as to satisfy 4.2 then every element $P \in \mathcal{J}$ may be expressed in the form

$$P \;=\; \sum_{i=1}^{k} A_i \; f_i \tag{4.3}$$

with the coefficients A_i polynomials satisfying the conditions

$$degree(A_i f_i) \le degree(P) \qquad (for \;\; i = 1..k \;) \;. \tag{4.4}$$

Proof

Given $P \in \mathcal{J}$ let us denote by $f(P)$ the first f_i whose leading monomial is contained in $\lambda(P)$, and set $\mu(P) = \lambda(P)/\lambda(f(P))$. Note that if c is the ratio of the coefficient of $\lambda(P)$ in P by the coefficient of $\lambda(f(P))$ in $f(P)$ then the polynomial $P - c\mu(P)\, f(P)$ as an expansion of the form

$$P \;-\; c\mu(P)\, f(P) \;=\; \sum_{x^q <_{DL} \lambda(P)} c_q x^q \;. \tag{4.5}$$

This is because $\lambda(P)$ cancels out and all the other terms in $\mu(P)\, f(P)$ involve monomials which are dlexicographically less than $\lambda(P)$. We can now prove 4.3 and 4.4 by induction on the lexicographic order of $\lambda(P)$. Note first that since every $\lambda(P)$ contains one of the $\lambda(f_i)$, the lexicographically smallest $\lambda(P)$ is $\lambda(f_1)$. But then we can have $\lambda(P) = \lambda(f_1)$ only if $P = f_1$ for otherwise, because of 4.5, $P - c\, f(P)$ would have a leading monomial dlex-smaller than $\lambda(f_1)$ and this would contradict the choice of f_1. Now let the assertion of the theorem hold true for all $P \in \mathcal{J}$ whose leading monomials are dlexicographically less than the monomial x^q and let $P \in \mathcal{J}$ have leading monomial x^q. From this assumption and 4.5 we can then write

$$P \;=\; c\mu(P)\, f(P) \;+\; \sum_{i=1}^{k} A_i \; f_i$$

with the A_i satisfying 4.4. But since $degree(\mu(P)\, f(P)) = degree(\lambda(P)) = degree(P)$ we see that the expansion in 4.3 will hold for P with coefficients satisfying 4.5 as well. This completes our developments.

The method of proof we adopted here has some very desirable consequences. Indeed, not only it provides an algorithm for the actual construction of generators of given ideals, but it also yields a number of additional results. A useful example is given by the following corollary of our proof of Theorem 4.2.

Theorem 4.3

For a given ideal $\mathcal{J}$ let $\mathbf{B}_{\mathcal{J}}$ denote the collection of monomials that is complementary to the monomial ideal $\mathcal{M}(\mathcal{J})$. Then $\mathbf{B}_{\mathcal{J}}$ is a basis for the quotient ring

$$\mathbf{R}_{\mathcal{J}} \;=\; \mathbf{Q}[x_1, x_2, \ldots, x_n]/\mathcal{J} \tag{4.6}$$

Proof.

Let $\mathbf{M} = \sum_k \mathbf{M}_k$ denote the basis of $\mathbf{R}_{\mathcal{J}}$ obtained by applying Gauss elimination (mod $\mathcal{J}$) to the monomials arranged in dlex order. That is, proceeding in this order, a given monomial x^p is taken if and only if it is not a linear combination (mod $\mathcal{J}$) of the dlexicographically preceeding ones. Here $\mathbf{M}_k$ denotes the collection of monomials of degree k that survive the process. We claim that $\mathbf{M}$ and $\mathbf{B}_{\mathcal{J}}$ are one and the same. Indeed, the inclusion

$$\mathbf{M} \;\subseteq\; \mathbf{B}_{\mathcal{J}} \tag{4.7}$$

is immediate. For if an element $x^p \in \mathbf{M}$ were to be in $\mathcal{M}(\mathcal{J})$ then (using notation in the proof of theorem 4.2), for suitable $c's$, we would have

$$x^p \;=\; c\mu(x^p)f(x^p) \;+\; \sum_{x^q <_{DL} x^p} c_q x^q \;\cong\; \sum_{x^q <_{DL} x^p} c_q x^q \qquad (mod\ \mathcal{J})\ ,$$

and this plainly contradicts the very choice of x^p. In the presence of 4.7, to complete our proof we need only show that any finite collection of elements of $\mathbf{B}_{\mathcal{J}}$ is independent. We proceed again by contradiction. Let there be a finite linear combination

$$\sum_{x^p \in \mathbf{B}_{\mathcal{J}}} c_p x^p \;\in\; \mathcal{J}$$

but then the very definition $\mathcal{M}(\mathcal{J})$ yields that the dlex largest monomial in this sum must lie in $\mathcal{M}(\mathcal{J})$. This contradicts the definition of $\mathbf{B}_{\mathcal{J}}$.

Remark 4.3

The basis $\mathbf{B}_{\mathcal{J}}$ takes a particularly nice form when for some integer k_o all monomials of degree larger than k_o belong to $\mathcal{M}(\mathcal{J})$. In this case we can construct a finite antichain of monomials (with respect to containment)

$$\mathbf{M} = \{m_1\ ,\ m_2\ ,\ \ldots\ ,\ m_h\}$$

with the property that a monomial x^p belongs to $\mathbf{B}_{\mathcal{J}}$ if and only if it is contained in one of the monomials of $\mathbf{M}$. In other words $\mathbf{B}_{\mathcal{J}}$ reduces in this case to the (containment) lower order ideal generated by the elements of $\mathbf{M}$. The reason for this is quite simple. If a monomial x^p belongs to $\mathbf{B}_{\mathcal{J}}$, then every monomial below x^p must also belong to $\mathbf{B}_{\mathcal{J}}$. In other words $\mathbf{B}_{\mathcal{J}}$ is, in any case, a lower order ideal. When all monomials of degree larger than k_o are in $\mathcal{M}(\mathcal{J})$ then each monomial in $\mathbf{B}_{\mathcal{J}}$ must be contained in a maximal element of $\mathbf{B}_{\mathcal{J}}$. Thus in this case we may take $\mathbf{M}$ to be the set of maximal elements of $\mathbf{B}_{\mathcal{J}}$.

Remark 4.4

It will be good to note here that the simple definition of $\mathbf{B}_{\mathcal{J}}$ yields the remarkable fact that for any two ideals $\mathcal{I}, \mathcal{J}$ we have

$$\mathcal{I} \subseteq \mathcal{J} \implies \mathbf{B}_{\mathcal{J}} \subseteq \mathbf{B}_{\mathcal{I}} .$$

This is immediate since $\mathcal{I} \subseteq \mathcal{J}$ implies that $\mathcal{M}(\mathcal{I}) \subseteq \mathcal{M}(\mathcal{J})$, and the desired inclusion follows by taking complements.

We are now in a position to derive the identities we need in the calculation of the graded characters we plan to study. Basically, these identities are obtained by relating the characters of a quotient module $\mathbf{A}$ and its associated graded version $gr\,\mathbf{A}$.

Let then

$$\mathbf{A} = \mathbf{Q}[x_1, x_2, \ldots, x_n] \,/\, \mathcal{J} .$$

for some ideal $\mathcal{J} \subseteq \mathbf{Q}[x_1, x_2, \ldots, x_n]$. Our point of departure is to define $gr\,\mathbf{A}$ simply as the quotient ring

$$gr\,A = \mathbf{Q}[x_1, x_2, \ldots, x_n] \,/\, gr\,\mathcal{J} \tag{4.8}$$

where $gr\,\mathcal{J}$ is the ideal generated by the leading homogeneous components of the elements of $\mathcal{J}$. More precisely, for a polynomial P, let $h(P)$ denote its homogeneous component of highest degree. Then

$$gr\,\mathcal{J} = \big(h(P) \,:\, P \in \mathcal{J} \big) . \tag{4.9}$$

Since the dlex leading monomial of a polynomial f must lie in $h(f)$, and the equality $h(h(f)A) = h(f)h(A)$ holds true for any pair of polynomials A, f, we can easily deduce that

$$\mathcal{M}(\mathcal{J}) = \mathcal{M}(gr\mathcal{J})$$

Thus Theorem 4.3 immediately yields the following beautiful corollary.

Theorem 4.4

The two rings $\mathbf{A} = \mathbf{R}/\mathcal{J}$ *and* $gr\,\mathbf{A} = \mathbf{R}/gr\mathcal{J}$ *have* $\mathbf{B}_{\mathcal{J}}$ *as common basis. In particular, in the bounded degree case, they have the same dimension.*

It will be good to be able to refer by name to the collection $\{f_1, f_2, \ldots, f_k\}$ satisfying 4.2 and the corresponding basis of monomials $\mathbf{B}_{\mathcal{J}}$. To this end we shall respectively call them a *standard basis* for $\mathcal{J}$ and the *standard monomial basis* for $\mathbf{R}_{\mathcal{J}}$ and $gr\,\mathbf{R}_{\mathcal{J}}$

Our next application involves an additional ingredient. We shall assume that $\mathcal{J}$ is invariant under a finite group G of matrices A acting on polynomials according to the definition 1.6. For simplicity, we shall restrict ourselves to the bounded degree case. That is we shall assume that every monomial of degree larger than k_o lies in $\mathcal{J}$.

Clearly, by the degree preserving properties of the operators T_A we necessarily have that $gr\,\mathcal{J}$ itself is also G-invariant. This allows us to define a G-action on the two quotient rings $\mathbf{A}$ and $gr\mathbf{A}$. Since $gr\mathbf{A}$ is a graded ring it is a direct sum of its homogeneous components. Denoting by $H_k(gr\mathbf{A})$ its homogeneous component of degree k, we can define as before the graded character of $gr\mathbf{A}$ by setting

$$char_q\, gr\mathbf{A} \;=\; \sum_{k\geq 0} q^k\, char\, H_k(gr\mathbf{A}) \;. \tag{4.10}$$

Going back to the proof of theorem 4.3, we can easily see that the component $\mathbf{M}_k$ of the common basis $\mathbf{M} = \mathbf{B}_{\mathcal{J}}$ is a basis for $H_k(gr\mathbf{A})$. Let us then denote by $\mathbf{A}^{\leq k}$ the linear span (modulo $\mathcal{J}$) of the elements of the union $\sum_{h\leq k}\mathbf{M}_h$. It is clear, again by the degree preserving property of the G-action that $\mathbf{A}^{\leq k}$ is also a G-invariant subspace of $\mathbf{A}$. This given, we have the following important fact.

Theorem 4.5

The characters of the G-modules $\mathbf{A}^{\leq k}$ are related to the graded character of $gr\mathbf{A}$ by the following identity

$$char_q\, gr\mathbf{A} \;=\; (1-q)\sum_{k\geq 0} q^k\, char\, \mathbf{A}^{\leq k} \;. \tag{4.11}$$

In particular, in the finite degree case, when $\mathbf{A} = \mathbf{A}^{\leq k_\circ}$, we shall have

$$char_q\, gr\mathbf{A} \;=\; q^{k_\circ} char\, \mathbf{A} \;+\; (1-q)\sum_{k=0}^{k_\circ-1} q^k\, char\, \mathbf{A}^{\leq k} \;. \tag{4.12}$$

Proof.

Let $\{f_1, f_2, \ldots, f_s\}$ be chosen as to satisfy 4.2 (with k replaced by s), and let $\{e_1, e_2, \ldots, e_s\}$ be their respective highest degree homogeneous components. Since, $\lambda(f_i) = \lambda(e_i)$, from 4.9 and theorem 4.2 we derive that $\{e_1, e_2, \ldots, e_s\}$ generate $gr\mathcal{J}$. Let us then express the action of G on $\mathbf{A}^{\leq k}$ in terms of the basis $\mathbf{B}_{\mathcal{J}}$. We shall have for any monomial $b \in \sum_{h\leq k}\mathbf{M}_h$ and a generic element $\gamma = T_A \in G$:

$$\gamma\, b \;=\; \sum_{h\leq k}\;\sum_{b'\in\mathbf{M}_h} b'\, a_{b',b}(\gamma) \;. \qquad (\ mod\ \mathcal{J}) \tag{4.13}$$

This gives that

$$char\, \mathbf{A}^{\leq k} \;=\; \sum_{h\leq k}\;\sum_{b\in\mathbf{M}_h} a_{b,b} \tag{4.14}$$

On the other hand from 4.13 and theorem 4.2 we derive that if $degree(b) = k$ then

$$\gamma\, b \;=\; \sum_{h\leq k}\;\sum_{b'\in\mathbf{M}_h} b'\, a_{b',b}(\gamma) \;+\; \sum_{i=1}^{s} A_i f_i \;, \tag{4.15}$$

with $degree(A_i f_i) \leq k$. Letting $g_i = f_i - e_i$ and substituting in 4.15 we finally get

$$\gamma\, b \;=\; \sum_{h\leq k}\;\sum_{b'\in\mathbf{M}_h} b'\, a_{b',b}(\gamma) \;+\; \sum_{i=1}^{s} A_i g_i \qquad (mod\ gr\mathcal{J}), \tag{4.16}$$

However, since $degree(A_i g_i) < degree(A f_i) \leq k$, the last summation in 4.16 yields a polynomial of degree stricly less than k. But since $gr\mathbf{A}$ is graded, we can equate homogeneous components of degree k on both sides of 4.16 and get identity

$$\gamma\, b \;=\; \sum_{b' \in \mathbf{M}_k} b'\, a_{b',b}(\gamma) \qquad\qquad (mod\ gr\mathcal{J})\ .$$

This gives that the coefficient of b in $\gamma\, b$ in $gr\mathbf{A}$ is also given by $a_{b,b}(\gamma)$. Thus we must have

$$char_q\, gr\mathbf{A} \;=\; \sum_{b \in \mathbf{B}_{\mathcal{J}}} q^{degree(b)}\, a_{b,b}\ , \tag{4.17}$$

and the desired formulas 4.11 and 4.12 are immediately obtained by combining 4.14 and 4.17.

Remark 4.5

We should note that this theorem gives useful information even in the case when G reduces to the $n \times n$ identity matrix, for then formulas 4.11 and 4.12 reduce to the following expressions for the Hilbert series of $gr\ \mathbf{A}$:

$$F_{gr\mathbf{A}}(q) \;=\; (1-q) \sum_{k \geq 0} q^k\ dim\ \mathbf{A}^{\leq k}\ , \tag{4.18}$$

and in the finite case

$$F_{gr\mathbf{A}}(q) \;=\; q^{k_\circ} dim\ \mathbf{A} \;+\; (1-q) \sum_{k=0}^{k_\circ - 1} q^k\ dim\ \mathbf{A}^{\leq k}\ , \tag{4.19}$$

We terminate this section with two immediate but useful consequences of the definition of $gr\ \mathcal{J}$:

Proposition 4.1

A collection of homogeneous polynomials which is independent in gr $\mathbf{A}$ *is necessarily also independent in* $\mathbf{A}$*. In particular we get that any homogeneous basis for gr* $\mathbf{A}$ *is also a basis for* $\mathbf{A}$*.*

Proof

Let $b_1, b_2, \ldots, b_m$ be given homogeneous polynomials, and for some constants $c_1, c_2, \ldots, c_m$, set

$$P \;=\; c_1 b_1 + c_2 b_2 + \cdots + c_m b_m$$

and let

$$h(P) \;=\; c_{i_1} b_{i_1} + c_{i_2} b_{i_2} + \cdots + c_{i_k} b_{i_k}$$

be the highest homogeneous component of P. Clearly, $P \in \mathcal{J}$ implies that $h(P) \in gr\ \mathcal{J}$. Thus if $b_1, b_2, \ldots, b_m$ are independent in $gr\ \mathbf{A}$, then P cannot be equal to zero in $\mathbf{A}$ without all the c_i vanishing. But this is precisely the first of our assertions. Clearly, the second assertion is trivial in

the finite dimensional case. In the infinite dimensional case, we resort to formula 4.18 and derive that for any k we must have

$$\sum_{h\le k} dim\ H_h(gr\mathbf{A}) \ = \ dim\ \mathbf{A}^{\le k} \ . \tag{4.20}$$

This given we see that if $b_1, b_2, \ldots, b_m$ are the elements of degree $\le k$ in a homogeneous basis for $gr\ \mathbf{A}$ then m must be equal to $dim\ \mathbf{A}^{\le k}$. By what we just proved, $b_1, b_2, \ldots, b_m$ must then be independent in $\mathbf{A}$. They also all lie in $\mathbf{A}^{\le k}$ and their number is equal to the dimension of $\mathbf{A}^{\le k}$. Thus they must be a basis for $\mathbf{A}^{\le k}$. Since this must hold true for arbitrary k our proof is complete.

Combining this result with Proposition 3.1 we deduce the following useful fact.

Theorem 4.6

Let $\mathcal{J} \subseteq \mathbf{R}$ be an ideal and $\mathbf{A} = \mathbf{R}/\mathcal{J}$. Let $\mathbf{H} = (gr\ \mathcal{J})^{\perp}$. Then any homogeneous basis for $\mathbf{H}$ is also a basis for $\mathbf{A}$ and $gr\ \mathbf{A}$. Moreover, if $\mathcal{J}$ is G-invariant then $gr\ \mathbf{A}$ and $\mathbf{H}$ are equivalent graded G-modules and

$$char_q\ \mathbf{H} \ = \ char_q\ gr\mathbf{A} \ = \ (1-q)\sum_{k\ge 0} q^k\ char\ \mathbf{A}^{\le k} \ . \tag{4.21}$$

Proof

The proof of Proposition 3.1 yields that if $\mathcal{B}$ is a homogeneous basis for $\mathbf{H}$ then its elements of degree m yield a basis for $H_m(gr\ \mathbf{A})$. Thus $\mathcal{B}$ itself must be a basis for $gr\ \mathbf{A}$. Proposition 4.1 then gives that $\mathcal{B}$ is also a basis for $\mathbf{A}$. The last assertion follows from Proposition 3.1 and the identity in 3.3.

Remark 4.6

It might be good to point out here that formula 4.11 is equivalent to the sequence of identities

$$char\ \mathbf{A}^{\le k} \ = \ \sum_{i=0}^{k} char\ H_i(gr\ \mathbf{A}) \tag{4.22}$$

UCSD LECTURE NOTES 1991-92

CHAPTER 2.

Orbits and their associated graded modules

1. The G-module $\mathbf{R}_{[a]}$ and its graded versions.

Here and in the following we shall work with a finite group G of $n \times n$ real orthogonal matrices. We shall also assume that we are given a distribution of degrees $d_1, d_2, \ldots, d_n$ for our variables $x_1, x_2, \ldots, x_n$ so that the degree of a monomial is given by the expression

$$degree\ x^p \ = \ \sum_{i=1}^{n} p_i d_i \ , \tag{1.1}$$

and that degrees computed in this manner are invariant under G.

Given a point

$$a \ = \ (a_1, a_2, \ldots, a_n) \ ,$$

we shall then denote by $[a]_G$ or simply by $[a]$ the finite set

$$[a] \ = \ \{\, aA \ : \ A \in G \,\} \tag{1.2}$$

where aA denotes the matrix multiplication of the row vector a on the right by A. In other words $[a]$ is simply the orbit of a under the action of G on ambient space. For a given point a the subgroup

$$G_a \ = \ \{\, A \in G \ : \ aA = a \,\} \tag{1.3}$$

is usually referred to as the *stabilizer* of a. We shall say that a is *regular* if G_a reduces to the identity. We see that this happens if and only if and only if the cardinality of $[a]$ is equal to the order of G. We may write this as

$$|[a]| \ = \ |G| \ . \tag{1.4}$$

In any case we set

$$\mathcal{J}_{[a]} \ = \ \{\, P(x) \ : \ P(aA) = 0 \quad \forall A \in G \,\} \ . \tag{1.5}$$

In other words $\mathcal{J}_{[a]}$ denotes the ideal of polynomials that vanish in the G-orbit of a. We also let

$$\mathbf{R}_{[a]} \ = \ \mathbf{Q}[x]/\mathcal{J}_{[a]} \tag{1.6}$$

This quotient may be viewed as the *coordinate ring* of the orbit $[a]$ considered as an algebraic variety.

There are a number of algebraic constructs associated with the orbit $[a]$ that will be our object of study here. Our first goal is to produce a working basis for the associated quotient ring $\mathbf{R}_{[a]}$. To this end we assume that we have been able to construct a polynomial $\phi_e(x)$ (*) which is of minimal degree among those which have the value 1 on a and vanish on all the other points of the orbit $[a]$. It will be convenient to denote by G/G_a a set of representatives for the right G_a-cosets of G so that every element of $[a]$ may be uniquely represented in the form $a\tau^{-1}$ with $\tau \in G/G_a$. Of course, when a is regular then G/G_a reduces to G itself. With this notation, the conditions on $\phi_e(x)$ may be expressed in the form

$$\phi_e(x) \;=\; \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x = a\tau^{-1} \text{ with } \tau \in G/G_a \text{ and } a\tau^{-1} \neq a\ . \end{cases} \qquad 1.7$$

This given we set for any $\gamma \in G/G_a$

$$\phi_\gamma(x) \;=\; T_\gamma\, \phi_e(x) \;=\; \phi_e(x\gamma)\ . \qquad 1.8$$

Clearly

$$\phi_\gamma(x) \;=\; \begin{cases} 1 & \text{if } x = a\gamma^{-1}, \\ 0 & \text{if } x = a\tau^{-1} \text{ with } \tau \in G/G_a \text{ and } a\tau^{-1} \neq a\gamma^{-1}\ . \end{cases} \qquad 1.9$$

This implies that for any polynomial $P(x)$ we have

$$P(x) \;-\; \sum_{\gamma \in G/G_a} P(a\gamma^{-1})\, \phi_\gamma(x) \;=\; 0 \qquad\qquad \forall\, x \in [a]\ .$$

In other words

$$P(x) \;=\; \sum_{\gamma \in G/G_a} P(a\gamma^{-1})\, \phi_\gamma(x) \qquad\qquad (\ mod\ \mathcal{J}_{[a]}\ )\ . \qquad 1.10$$

This means that the polynomials $\{\phi_\gamma\}_{\gamma \in G/G_a}$ are a basis for $\mathbf{R}_{[a]}$ and that the dimension of $\mathbf{R}_{[a]}$, as a vector space, is equal to the cardinality of $[a]$. The definition 1.8 gives also that

$$T_B\, \phi_A(x) \;=\; \phi_e(xBA) \;=\; \phi_{BA}(x)\ .$$

Thus we see that when a is regular, the action of G on $\mathbf{R}_{[a]}$, when expressed in terms of this basis is none other than the *left regular representation*. When a has a non trivial stabilizer, we see that we shall have

$$T_{B_1}\phi_A(x) \;=\; T_{B_2}\phi_A(x) \qquad\qquad (\ \forall\, x \in [a]\ )\ ,$$

if and only if

$$aA^{-1}B_1^{-1} \;=\; aA^{-1}B_2^{-1}\ .$$

In other words these two polynomials are equal $mod\ \mathcal{J}_a$, if and only if B_1A and B_2A represent the same right G_a-coset. Clearly, the polynomial ϕ_A itself only depends ($mod\ \mathcal{J}_a$) upon the right coset of

(*) Here and after e will refer to the identity element

A. Putting all this together we see that, in terms of our basis $\{\phi_\gamma\}_{\gamma\in G/G_a}$, G acts on $\mathbf{R}_{[a]}$, precisely as it acts on the right G_a-cosets by left multiplication.

Note further that if $degree(\phi_e) = n_{[a]}$ then 1.10 shows that each $P \in \mathbf{R}$ can be expressed $(mod\ \mathcal{J}_{[a]})$ as linear combinations of polynomials of degree at most $n_{[a]}$. In fact, the minimality of ϕ_e yields that we must have

$$\mathbf{R}_{[a]}{}^{\leq k} \;=\; \mathbf{R}_{[a]} \qquad\Longleftrightarrow\qquad k \geq n_{[a]}\ . \tag{1.11}$$

This is easily seen, for if equality in 1.11 did take place earlier than $n_{[a]}$, then ϕ_e itself would be equal $(mod\ \mathcal{J}_{[a]}\ )$ to a polynomial $\psi_e(x)$ of degree smaller than $n_{[a]}$. But this means that $\psi_e(x)$ must be equal to 1 on a and zero on all other points of $[a]$. However, this would contradict the minimality of ϕ_e.

We can now bring into this setting all the ingredients we introduced in the previous sections and set

$$gr\ \mathbf{R}_{[a]} \;=\; \mathbf{R}/gr\ \mathcal{J}_{[a]} \qquad\text{and}\qquad \mathbf{H}_{[a]} \;=\; (gr\ \mathcal{J}_{[a]})^{\perp}\ . \tag{1.12}$$

These observations combined with the facts we gathered in sections 3. and 4. immediately yield us the following basic result.

Theorem 1.1

For any G-orbit $[a]$ the coordinate ring $\mathbf{R}_{[a]}$ has a maximum degree given by $n_{[a]}$. Moreover, each of the G-modules $\mathbf{R}_{[a]}$, $gr\ \mathbf{R}_{[a]}$ and $\mathbf{H}_{[a]}$ is a version of the action of G on the right cosets of G_a, with

$$char_q\ \mathbf{H}_{[a]} \;=\; char_q\ gr\ \mathbf{R}_{[a]} \;=\; q^{n_{[a]}}\ char\mathbf{R}_{[a]} \;+\; (1-q)\sum_{k=0}^{n_{[a]}-1} q^k\ char\ \mathbf{R}_{[a]}^{\leq k}\ . \tag{1.13}$$

The important point here is that we now have a mechanism for constructing a variety of *graded* versions of right coset actions for any finite matrix group. As we shall see, this mechanism has some rather remarkable applications. We shall start by deriving some applications to reflection groups. To this end we need to review some basic facts from invariant theory.

2. Invariants and harmonics

We recall that a polynomial P is said to be G-invariant if and only if

$$T_A\ P \;=\; P \qquad\qquad \forall\ A \in G\ .$$

It is customary to denote the ring of invariants by the symbol $\mathbf{R}^G$. Clearly, $\mathbf{R}^G$ is a homogeneous subspace of $\mathbf{R}$. In fact, if we set

$$\rho_G \;=\; \frac{1}{|G|}\sum_{A\in G} T_A\ ,$$

then the image $\rho_G Q$ of any polynomial Q is necessarily G-invariant. Thus $\mathbf{R}^G$ may be viewed as the projection of $\mathbf{R}$ by the degree preserving linear operator ρ_G. We recall that ρ_G is usually referred to as the *Reynolds* operator corresponding to G. It should be noted that invariants behave like constants with respect to ρ_G. More precisely, if P is any polynomial and Q is G-invariant then

$$\rho_G\, QP \;=\; Q\, \rho_G\, P \tag{2.1}$$

It will be convenient, here and after to let

$$\mathbf{R}^G_+ \;=\; \oplus_{m\geq 1}\, H_m(\mathbf{R}^G) \qquad \text{and} \qquad \mathcal{J}_G \;=\; (\mathbf{R}^G_+)\ .$$

In other words $\mathcal{J}_G$ is the ideal generated by the homogeneous invariants of positive degree. This given, we set

$$\mathbf{H}_G \;=\; (\mathbf{R}^G_+)^{\perp} \;=\; \mathcal{J}_G^{\perp}\ . \tag{2.2}$$

Note that, the polynomial

$$\|x\| \;=\; x_1^2 + x_2^2 + \cdots + x_n^2$$

belongs to $\mathbf{R}^G$ for any of the groups under consideration here. This is simply due to the fact that an orthogonal matrix does not change the length of a vector. Thus it follows from the definition 2.2 that every polynomial $P \in \mathbf{H}_G$ satisfies the Laplace equation

$$\sum_{i=1}^{n} \partial^2_{x_i}\, P \;=\; 0\ .$$

For this reason the elements of $\mathbf{H}_G$ are usually referred to as the *harmonics* corresponding to G. We shall later see that in the cases we consider these polynomials are harmonic in an even stronger sense.

The following basic fact is one of the crucial reasons why the spaces $\mathbf{H}_{[a]}$ are worth a special study

Theorem 2.1

For any finite group G of orthogonal matrices and any orbit $[a]$ we have

$$\mathbf{H}_{[a]} \;\subseteq\; \mathbf{H}_G \qquad (\textit{or equivalently}) \qquad \mathcal{J}_G \;\subseteq\; gr\, \mathcal{J}_{[a]}\ . \tag{2.3}$$

In particular we deduce that

$$dim\, \mathbf{H}_G \;\geq |G| \tag{2.4}$$

Proof

We simply note that if $P(x)$ is invariant under G then it is constant in $[a]$ and the polynomial

$$P(x) \;-\; P(a)$$

must therefore belong to $\mathcal{J}_{[a]}$. On the other hand if P is also homogeneous and of positive degree then it is the highest homogeneous component of $P(x) - P(a)$ and as such it must also belong to $gr\ \mathcal{J}_{[a]}$. In other words $\mathbf{R}_+^G$ is contained in $gr\ \mathcal{J}_{[a]}$ and 2.3 immediately follows from the definitions of $\mathbf{H}_{[a]}$, $\mathbf{H}_G$, $gr\ \mathcal{J}_G$ and $\mathcal{J}_G$. To establish 2.4 we simply use 2.3 with a a regular point and note that we always have

$$dim\ \mathbf{H}_{[a]} \ = \ |[a]| \tag{2.5}$$

Remark 2.1

We should perhaps point out here that for any of our groups G the regular points form a non empty open set. Indeed, if a is not requlr then for some element $A \in G$ we have $aA = a$. This implies that the non-regular elements of G form a closed set consisting of a finite number of hyperplane intersections. The assertion follows then immediately from the fact that ambient space is not a union of a finite number of hyperplanes.

Proposition 2.1

For any finite matrix group G we have

$$x_i^{|G|} \in \mathcal{J}_G \ . \tag{2.6}$$

Moreover, we can always construct a finite collection of monomials $\mathcal{M}$ and a set of invariants $I_1, I_2, \ldots, I_m$ such that every polynomial P may be written in the form

$$P \ = \ \sum_{x^p \in \mathcal{M}} Q_p(I_1, I_2, \ldots, I_m)\ x^p \tag{2.7}$$

with $Q_p(I_1, I_2, \ldots, I_m)$ a polynomial in its arguments.

Proof

For a given $a = (a_1, a_2, \ldots, a_n)$ let

$$E_{[a]}(t) \ = \ \prod_{A \in G} (t - (xA, a))$$

It is clear that

$$E_{[a]}(t) \ = \ t^{|G|} \ + \ \sum_{k=1}^{|G|} t^{|G|-k}\ (-1)^k\ e_k(x; a) \ .$$

with $e_k(x; a) \in \mathbf{R}_+^G$. Since $E_{[a]}(t)\ |_{t=(x,a)} = 0$, we get that

$$(x, a)^{|G|} \ = \ \sum_{k=1}^{|G|} (x, a)^{|G|-k}\ (-1)^{k-1}\ e_k(x; a) \ .$$

Taking a to be the i^{th} unit coordinate vector $u^{(i)} = (0, 0, \ldots, \underset{i \uparrow}{1}, \ldots, 0)$ gives

$$x_i^{|G|} \ = \ \sum_{k=1}^{|G|} x_i^{|G|-k}\ (-1)^{k-1}\ e_k(x; u^{(i)}) \ . \tag{2.8}$$

In particular, 2.6 must hold true as asserted. Moreover, this relation yields that every $P \in \mathbf{R}$ can be written in the form

$$P \;=\; \sum_{x^p \in \mathcal{M}} c_p\, x^p \;+\; \sum_{i=1}^{m} A_i\, I_i \;\;, \tag{2.9}$$

with $\mathcal{M}$ the set of monomials x^q with $q_i < |G|$, $A_i \in \mathbf{R}$ and $\{I_1, I_2, \ldots, I_m\}$ given by the collection

$$\{e_k(x; u^{(i)}) \;:\; i = 1..n \;\&\; k = 1..|G|\} \;\;.$$

Since the coefficients A_i themselves may be expanded in this manner as well, we can easily see that 2.7 can be derived by successive recursive uses of 2.9.

There are three immediate consequences of this Proposition:

Theorem 2.2 *For any finite matrix group G*

(i) The ring $\mathbf{R}^G$ is finitely generated.

(ii) The space $\mathbf{H}_G$ is finite dimensional.

(iii) If $\mathcal{B}_H$ is any homogeneous basis of $\mathbf{H}_G$ then every polynomial P has an expansion of the form

$$P \;=\; \sum_{b \in \mathcal{B}_H} A_b\, b \qquad\qquad (\, A_b \in \mathbf{R}^G \,) \tag{2.10}$$

Proof

To derive (i) we simply apply the operator ρ_G to 2.7 for an invariant P and obtain (using 2.1)

$$P \;=\; \sum_{x^p \in \mathcal{M}} Q_p(I_1, I_2, \ldots, I_m)\;\; \rho_G\, x^p \;\;. \tag{2.11}$$

However, this gives that $\mathbf{R}^G$ is generated by the collection $\{J_1, J_2, \ldots, J_M\}$ consisting of $I_1, I_2, \cdots, I_m$ followed by the invariants $\rho_G\, x^q$ with $x^q \in \mathcal{M}$.

To get (ii) we simply note that 2.9 also says that the ring $\mathbf{R}/\mathcal{J}_G$ is finite dimensional. Thus the finite dimensionality of $\mathbf{H}_G$ is simply due to the fact that $\mathbf{H}_G$ and $\mathbf{R}/\mathcal{J}_G$ have the same dimension.

Finally, to obtain (iii) we use Proposition 1.3.1 and deduce that $\mathcal{B}_H$ must also be a basis for $\mathbf{R}/\mathcal{J}_G$. Since $J_1, J_2, \ldots, J_M$ also generate $\mathcal{J}_G$ we must be able to express every $P \in \mathbf{R}$ in the form

$$P \;=\; \sum_{b \in \mathcal{B}_H} c_b\; b \;+\; \sum_{i=1}^{M} A_i\, J_i \qquad\qquad (\, A_i \in \mathbf{R} \,) \;\;.$$

Using this recursively, as we did in the proof of 2.7, yields the expansion in 2.10 with the coefficients A_b polynomials in the invariants $J_1, J_2, \ldots, J_M$.

Proposition 2.1 has a variant that is worth recording at this point.

Proposition 2.2

If G is a finite matrix group and $I_1, I_2, \ldots, I_m$ are homogeneous G-invariants which generate $\mathcal{J}_G$ then every $P \in \mathbf{R}^G$ may be written in the form

$$P \;=\; Q(I_1, I_2, \ldots, I_m)$$

with Q a polynomial in its arguments.

Proof

For convenience let $d_s = degree(I_s)$. Note that we need only show that if P is a homogeneous G-invariant of degree $k > 0$, then P may be written in the form

$$P \;=\; \sum_{d_s \le k} B_s I_s \;\;,$$

with B_s a homogeneous G-invariant of degree $k - d_s$. For then an obvious induction argument yields the desired result.

By hypothesis every homogeneous G-invariant P of positive degree has the expansion

$$P \;=\; \sum_{s=1}^{m} A_s I_s \;\;.$$

with A_s suitable polynomials. However, we can assume that the coefficients A_s are also invariant, for if not an application of ρ_G to both sides of this equation, replaces A_s by $\rho_G A_s$, and the latter is G-invariant. Now if P is also homogeneous and of degree k then applying the operator π_k (defined in 1.2.1) to both sides of the equation gives,

$$P \;=\; \sum_{s=1}^{m} B_s I_s \;\;,$$

where,

$$B_s \;=\; \begin{cases} 0 & \text{if } d_s > k \\ \pi_{k-d_s} A_s & \text{if } d_s \le k \end{cases} \;\;.$$

This completes our proof since the G-invariance of A_s implies that each of its homogeneous components, in particular each coefficient B_s must be invariant as desired.

Note that if f is an idempotent element of the group algebra of G then the linear operator

$$\rho_f \;=\; \sum_{A \in G} f(A)\, T_A \;\;.$$

acts as an idempotent on each of the spaces $H_m(\mathbf{R})$, in particular, its range $\rho_f \mathbf{R}$ is a homogeneous subspace. This simple observation combined with Macmahon's Master theorem yields a beautiful formula for the Hilbert series of the spaces $\rho_f \mathbf{R}$.

Theorem 2.3

For any finite matrix group G and any idempotent $f \in \mathcal{A}(G)$ we have

$$F_{\rho_f \mathbf{R}}(q) \;=\; \sum_{A \in G} \frac{f(A)}{det\,(I - qA)} \;. \tag{2.12}$$

In particular, the Hilbert series of the ring of invariants $\mathbf{R}^G$ is given by the formula

$$F_{\mathbf{R}^G}(q) \;=\; \frac{1}{|G|} \sum_{A \in G} \frac{1}{det\,(I - qA)} \;. \qquad (Moliens) \tag{2.13}$$

Proof

Since, $H_m(\rho_f \mathbf{R}) = \rho_f H_m(\mathbf{R})$ we see that the dimension of the space $H_m(\rho_f \mathbf{R})$ is given by the rank of the matrix

$$\rho_f^{(m)} \;=\; \sum_{A \in G} f(A)\, A^{(m)} \;.$$

However, the idempotency of f together with the fact that the map $A \rightarrow A^{(m)}$ is a representation (see I.3.6), yields that $\rho_f^{(m)}$ itself is an idempotent. Since the the rank of idempotent matrix is given by the sum of its eigenvalues, we get that

$$rank\; \rho_f^{(m)} \;=\; trace\; \rho_f^{(m)} \;.$$

In summary we must have

$$dim\; H_m(\rho_f \mathbf{R}) \;=\; trace\; \rho_f^{(m)} \;=\; \sum_{A \in G} f(A)\, trace\; A^{(m)} \;.$$

Multiplying, by q^m and summing gives

$$F_{\rho_f \mathbf{R}}(q) \;=\; \sum_{A \in G} f(A) \sum_{m \geq 0} q^m\; trace\; A^{(m)} \;,$$

and I.3.5 then yields 2.12 as desired. Note that if we let f be the trivial idempotent $1/|G|$, then ρ_f reduces to the operator ρ_G defined at the beginning of this section. Thus 2.13 is a special case of 2.12. This completes our proof.

Our proof of part (ii) of Theorem 2.2 yields the extravagant bound

$$dim\; \mathbf{H}_G \;\leq\; |G|^n \;. \tag{2.14}$$

On the other hand, Theorem 2.1 yields that the smallest possible value for this dimension is the order of G itself and in that case, for each regular point a we must have

$$\mathbf{H}_{[a]} \;=\; \mathbf{H}_G \;. \tag{2.15}$$

It develops that when this takes place the three spaces $\mathbf{R}$, $\mathbf{R}^G$ and $\mathbf{H}_G$ have a remarkably tight relationship. This can be stated as follows.

Theorem 2.4

Let

$$dim\, \mathbf{H}_G \;=\; |G| \tag{2.16}$$

and let $\mathcal{B}_G = \{b_\alpha(x)\}_{\alpha\in G}$ be a homogeneous basis of $\mathbf{H}_G$ indexed by elements of G. Then every polynomial P may be uniquely expressed in the form

$$P \;=\; \sum_{\alpha\in G} b_\alpha\, A_\alpha \qquad\qquad (\, A_\alpha \in \mathbf{R}^G\,)\ . \tag{2.17}$$

In particular, the Hilbert series of $\mathbf{H}_G$ and $\mathbf{R}^G$ are related by the identity

$$\frac{1}{(1-q)^n} \;=\; F_{\mathbf{H}_G}(q)\, F_{\mathbf{R}^G}(q)\ . \tag{2.18}$$

Proof

Part (*iii*) of Theorem 2.2 asserts that the expansion in 2.17 is always possible. We are left to show that 2.16 implies uniqueness of the coefficients. To this end we start by expanding in this manner each interpolating polynomial ϕ_β and get

$$\phi_\beta \;=\; \sum_{\alpha\in G} b_\alpha\, A_{\alpha,\beta} \qquad\qquad (\, A_{\alpha,\beta} \in \mathbf{R}^G\,)\ . \tag{2.19}$$

Evaluating this identity at the orbit point $a\gamma^{-1}$, 1.9 and the G-invariance of $A_{\alpha,\beta}$ yields

$$\chi(\beta=\gamma) \;=\; \sum_{\alpha\in G} b_\alpha(a\gamma^{-1})\, A_{\alpha,\beta}(a\gamma^{-1}) \;=\; \sum_{\alpha\in G} b_\alpha(a\gamma^{-1})\, A_{\alpha,\beta}(a)\ . \tag{2.20}$$

This identity may be interpreted as saying that the matrices

$$||b_\alpha(a\gamma^{-1})||_{\gamma,\alpha} \quad \text{and} \quad ||A_{\alpha,\beta}(a)||_{\alpha\beta}$$

multiply to the identity matrix. Thus they are both invertible. Suppose then that for some invariants A_α we have

$$0 \;=\; \sum_{\alpha\in G} b_\alpha\, A_\alpha\ .$$

Evaluating again at the orbit point $a\gamma^{-1}$ yields

$$0 \;=\; \sum_{\alpha\in G} b_\alpha(a\gamma^{-1})\, A_\alpha(a)\ .$$

But now, the invertibility of the matrix $||b_\alpha(a\gamma^{-1})||_{\gamma,\alpha}$ implies that

$$A_\alpha(a) \;=\; 0 \qquad\qquad \forall \alpha \in G\ .$$

This means that the polynomials A_α must vanish at all regular orbit points. Since the latter, as we already observed constitue a non empty open set, this can only hold true if all the coefficients A_α vanish identically. Thus the expansion in 2.17 must be unique as asserted. Let now, $\mathcal{A}_m$ and $\mathcal{B}_m$ denote bases for $H_m(\mathbf{H}_G)$ and $H_m(\mathbf{R}^G)$. Uniqueness in 2.17 implies that the dimension of $H_m(\mathbf{R})$ is given by the expression

$$\sum_{k\geq 0} |\mathcal{A}_k|\,|\mathcal{B}_{m-k}| \ .$$

Multiplying by q^m and summing gives 2.18 as desired.

3. On groups generated by reflections

In view of Theorem 2.4 it is worthwhile investigating under what conditions equality holds in 2.16. To this end we need to review some additional standard facts. Given a vector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

we set for any $x = (x_1, x_2, \ldots, x_n)$

$$\sigma_\alpha\, x \ = \ x \ - \ 2(x,\alpha)\,\alpha/(\alpha,\alpha) \ . \tag{3.1}$$

We recall that an orthogonal matrix A is said to be a *reflection* if for some α we have

$$x\,A \ = \ \sigma_\alpha\, x \qquad\qquad \forall\, x \ . \tag{3.2}$$

Clearly, this condition does not uniquely determine α. To get uniqueness we need to properly restrict the vectors α that are used to represent reflections. To this end we shall denote by $\mathcal{PU}$ the collection of unit vectors α (that is $(\alpha,\alpha) = 1$) whose first non vanishing component is positive. We may refer to the elements of $\mathcal{PU}$ as *positive unit normals*. For any of our groups G we shall then set

$$\mathcal{N}(G) \ = \ \{\, \alpha \in \mathcal{PU} \ : \ \sigma_\alpha \in G\} \tag{3.3}$$

where (with a slight abuse of notation) we write $\sigma_\alpha \in G$ to mean that the matrix defined by 3.2 belongs to G. In words, we may say that $\mathcal{N}(G)$ is the set of positive unit normals to the *reflecting hyperplanes of* G. We shall also let

$$\mathcal{R}(G) \ = \ \{\, \sigma_\alpha \ : \ \alpha \in \mathcal{N}(G) \ \} \ . \tag{3.4}$$

denote the set of reflections of G. It develops that equality in 2.4 has a very beautiful characterization given by Steinberg in [].

Theorem 3.1

The dimension of the space $\mathbf{H}_G$ *is equal to the order of* G *if and only if* G *is generated by reflections.*

The proof will be obtained by combining a number of propositions we shall present in this and the coming sections. We shall start by deriving a number of results that are interesting in their own right. If G is a group generated by reflections (GGR in brief) we set

$$\Delta_G(x) \;=\; \prod_{\alpha\in\mathcal{N}(G)} (x,\alpha)\ . \tag{3.5}$$

The polynomial Δ_G is usually referred to as the *discriminant* of G. To study Δ_G and for later purposes we need the following useful fact.

Proposition 3.1

For any α and $P\in\mathbf{R}$ the polynomial

$$P \;-\; \sigma_\alpha P \tag{3.6}$$

is divisible by (x,α).

Proof

Let us chose an orthonormal set of vectors $\beta_2,\dots,\beta_n$ orthogonal to α and set

$$y_1=(x,\alpha)\ ;\quad y_i=(x,\beta_i)\quad\text{for}\quad i=2,\dots,n$$

In terms of the coordinate system $y_1,y_2,\dots,y_n$ the action of σ_α reduces to

$$\sigma_\alpha y_1 \;=\; -y_1\quad;\quad \sigma_\alpha y_i=y_i\quad\forall\ i=2,\dots,n$$

Thus for any monomial $y^p=y_1^{p_1}y_2^{p_2}\cdots y_n^{p_n}$ we get that

$$y^p \;-\; \sigma_\alpha y^p \;=\; (1-(-1)^{p_1})y^p \;\neq\; 0$$

if and only if p_1 is odd. This implies that y_1 always divides $y^p-\sigma_\alpha y^p$ and thus the same will hold true for any polynomial. QED

This result enables us to establish three crucial properties of the discriminant.

Theorem 3.2

If G is a GGR then

(a) $\sigma_\alpha\Delta_G \;=\; -\Delta_G \qquad \forall\quad \alpha\in\mathcal{N}(G)$

(b) If $P\in\mathbf{R}$ is such that $\sigma_\alpha P=-P\quad\forall\alpha\in\mathcal{N}(G)$ then $P=A\Delta_G$ with $A\in\mathbf{R}^G$.

(c) Every element $P\in\mathcal{J}_G$ kills Δ_G (that is $P(\partial_x)\Delta_G=0$).

(d) If $P\in\mathbf{R}$ kills Δ_G then $P\in\mathcal{J}_G$.

Proof

Note that for any $\alpha\in\mathcal{N}(G)$ we can write

$$\Delta_G(x) \;=\; c(x,\alpha)\prod_{\substack{\beta\in\mathcal{N}(G)\\ \beta\perp\alpha}}(x,\beta)\prod_{\beta\in\mathcal{N}^*(G)}(x,\beta)(x,\sigma_\alpha\beta)\ . \tag{3.7}$$

where $\mathcal{N}^*(G)$ is to represent a subset of $\mathcal{N}(G)$ with the property that every element of $\mathcal{R}(G)$ that does not commute with σ_α is given by σ_β or by $\sigma_{\sigma_\alpha\beta}$ for some $\beta \in \mathcal{N}^*(G)$. The reason we can find such a subset is that σ_α acts on $\mathcal{R}(G)$ (by conjugation) as an involution with fixed point set $\{\sigma_\beta \ : \ \beta \in \mathcal{N}(G) \& \beta \perp \alpha \}$. The constant factor c in 3.7 (which must be ± 1) must be included to compensate for the fact that for some elements $\beta \in \mathcal{N}^*(G)$ the vector $\sigma_\alpha\beta$ may not be in $\mathcal{PU}$. This given, property (a) follows from the fact that the products $(x,\beta)(x,\sigma_\alpha\beta)$ are left invariant by σ_α and $\sigma_\alpha(x,\alpha) = -(x,\alpha)$.

Property (b) is an immediate consequence of Proposition 3.1. In fact, $\sigma_\alpha P = -P$ gives that $P = \frac{1}{2}(P - \sigma_\alpha P)$ and thus P itself must be divisible by each factor (x,α) appearing in the definition of Δ_G. Denoting by A the quotient of P by Δ_G we can easily see that A must be left invariant by every element of $\mathcal{R}(G)$ and thus also by every other element of G.

To prove property (c) we note that if P is a homogeneous G-invariant polynomial of positive degree then the polynomial $Q = P(\partial_x)\Delta_G$ must also be *skew* (that is $\sigma_\alpha Q = -Q \quad \forall \ \sigma_\alpha \in \mathcal{R}(G)$). But then property (b) yields that Q must be a multiple of Δ_G. However, since the operator $P(\partial_x)$ lowers degree, to avoid a contradiction we must conclude that $Q = 0$. In other words every generator of $\mathcal{J}_G$ kills Δ_G and thus the same must hold true for every element of $\mathcal{J}_G$.

We are left to verify that (d) holds true as well. To this end we start by observing that we need only prove (d) for P homogeneous. Furthermore, in view of proposition 2.1 (see 2.6) there is nothing to prove if the customary degree of P is greater than $n|G|$. So we may proceed by reverse induction on degree. Assume that any polynomial of degree larger than d which kills Δ_G is in $\mathcal{J}_G$. Suppose that P kills Δ_G and is homogeneous of degree d. Note then that, for any given α the polynomial

$$Q(x) \ = \ (x,\alpha)P(x)$$

must kill Δ_G as well. Since Q is of degree $d+1$ the induction hypothesis gives that we must have

$$(x,\alpha)P(x) \ = \ \sum_{i=1}^{M} A_i(x)\, J_i$$

with $A_i \in \mathbf{R}$ and $\{J_i\}_{i=1..M}$ the set of invariants constructed in the proof of Theorem 2.2. This implies that

$$(x,\alpha)(P + \sigma_\alpha P) \ = \ \sum_{i=1}^{M} (A_i - \sigma_\alpha A_i)\, J_i \ ,$$

but proposition 3.1 gives that (x,α) divides each factor $A_i - \sigma_\alpha A_i$. Letting B_i denote the result of this division, we deduce that

$$P + \sigma_\alpha P \ = \ \sum_{i=1}^{M} B_i J_i \ ,$$

In other words

$$P \ = \ -\sigma_\alpha P \qquad (mod \ \mathcal{J}_G \ ) \ .$$

Since this must hold true for every $\sigma_\alpha \in \mathcal{R}(G)$ and G is supposedly generated by $\mathcal{R}(G)$ we must have

$$P \ = \ det(A)\, T_A P \qquad (\, mod \ \mathcal{J}_G \,)$$

for all $A \in G$. Averaging this relation over all elements of G we get

$$P \;=\; \frac{1}{|G|}\sum_{A\in G} det(A)\, T_A P \qquad (\ mod\ \mathcal{J}_G\ )\ .$$

Note that the polynomial

$$P^* \;=\; \frac{1}{|G|}\sum_{A\in G} det(A)\, T_A P$$

is skew. Thus, property (b) gives that one of the following alternatives must hold true

$$P^* \;=\; \begin{cases} 0 & (\text{at least when } d < degree(\Delta_G)) \\ c\Delta_G & \text{with } d = degree(\Delta_G) \quad and\ \ c \neq 0) \\ A\Delta_G & \text{with } d > degree(\Delta_G) \quad and\ \ A \in \mathbf{R}^G_+ \end{cases}\ . \tag{3.8}$$

Now, since P kills Δ_G and $P - P^*$ is in $\mathcal{J}_G$ property (c) gives that P^* itself must kill Δ_G. This eliminates the second alternative in 3.8 since then Δ_G would have to kill itself, which is absurd. If the last alternative holds then P^* lies in $\mathcal{J}_G$, and since $P - P^*$ is already in $\mathcal{J}_G$, P itself must lie in $\mathcal{J}_G$. Thus we see that in any case we get that $P = 0\ \ (mod\ \ \mathcal{J}_G)$ as desired.

This theorem has several remarkable consequences connecting the invariants and the harmonics of groups generated by reflections.

Theorem 3.3
If G is GGR then

(i) For every regular point a we have

$$\mathbf{H}_{[a]} \;=\; \mathbf{H}_G\ .$$

(ii) The G-module $\mathbf{H}_G$ is a graded version of the left regular representation of G. In particular

$$dim\, \mathbf{H}_G \;=\; |G|$$

(iii) $\mathbf{H}_G$ is the linear span of the partial derivatives of the discriminant, and we shall express this by writing

$$\mathbf{H}_G \;=\; \mathcal{L}[\partial_x^p\ \Delta_G]$$

(iv) If $\mathcal{B}_H$ is a homogeneous basis of $\mathbf{H}_G$ then every polynomial P may be uniquely expressed in the form

$$P \;=\; \sum_{b\in\mathcal{B}_H} b\, A_b \qquad (\ A_b \in \mathbf{R}^G\ ) \tag{3.9}$$

(v) For any regular point $[a]$ the minimum degree $n_{[a]}$ is equal to the degree of the discriminant.

(vi) The graded character of $\mathbf{H}_G$ is related to the Hilbert series of $\mathbf{R}^G$ by the identity

$$\frac{1}{det(I - qA)} \;=\; F_{\mathbf{R}^G}(q)\, \chi^{\mathbf{H}_G}(A;q) \qquad (\ \forall\ A \in G\ ) \tag{3.10}$$

Proof

Part a) of Theorem 3.2 shows that the map $A \to det\ A$ yields a non trivial one dimensional irreducible representation of G. Thus, for any regular a, $\mathbf{H}_{[a]}$ must contain a non-trivial skew polynomial Δ_1. That is one which doesn't vanish and satisfies

$$T_A \Delta_1 \ = \ det\ A\ \Delta_1 \qquad\qquad (\forall\ A \in G)\ .$$

This polynomial must be homogeneous for otherwise the multiplicity of $det\ A$ in $\mathbf{H}_{[a]}$ would be greater than one. Part b) of Theorem 3.2 then yields that we must have

$$\Delta_1 \ = \ Q\ \Delta_G$$

with $Q = const \neq 0$ or with $Q \in \mathbf{R}^G_+$. However, the second alternative must be excluded, because then $\Delta_1 \in \mathcal{J}_G$ and then the second inequality in 2.3 (i.e. $\mathcal{J}_G \subseteq gr\ \mathcal{J}_{[a]}$) would imply that $\Delta_1 \in gr\ \mathcal{J}_{[a]}$ and thus it would have to be orthogonal to itself. The remaining alternative yields that

$$\Delta_G \ \in \ \mathbf{H}_{[a]}\ .$$

Since $\mathbf{H}_{[a]}$ contains all the derivatives of everyone of its elements we deduce that

$$\mathcal{L}[\partial^p_x\ \Delta_G] \ \subseteq \ \mathbf{H}_{[a]}\ .$$

On the other hand part d) of Theorem 3.2 is equivalent to the statement that

$$\mathcal{L}[\partial^p_x\ \Delta_G]^{\perp} \ \subseteq \ \mathcal{J}_G\ ,$$

and this in turn is the same as

$$\mathbf{H}_G \ \subseteq \ \mathcal{L}[\partial^p_x\ \Delta_G]\ .$$

In summary, for any regular point a, we must have the string of inclusions

$$\mathbf{H}_G \ \subseteq \ \mathcal{L}[\partial^p_x\ \Delta_G] \ \subseteq \ \mathbf{H}_{[a]} \ \subseteq \ \mathbf{H}_G\ .$$

This forces $(i), (ii)$ and (iii). This given, part (iv) now follows from Theorem 2.4. Part (v) is a trivial consequence of part (iii). To complete the proof and verify (vi), we shall use formula 1.3.2 with a suitably chosen basis of the polynomial ring $\mathbf{R}$. To this end let $\mathcal{A}$ and $\mathcal{B}$ denote homogeneous bases for $\mathbf{R}^G$ and $\mathbf{H}_G$ respectively. The uniqueness part of (iv) then yields that the collection $\{\ ab:\ a \in \mathcal{A}\ ,\ b \in \mathcal{B}\ \}$ is a basis for $\mathbf{R}$. From 1.3.2 we then get that

$$\frac{1}{det(I - qA)} \ = \ \chi^{\mathbf{R}}(A;q) \ = \ \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} q^{degree(a)} q^{degree(b)}\ T_A\ ab|_{ab}\ .$$

But, since $T_A\ ab \ = \ a\ T_A b$ we see that this can be rewritten as

$$\frac{1}{det(I - qA)} \ = \ \Big(\sum_{a \in \mathcal{A}} q^{degree(a)}\Big)\ \Big(\sum_{b \in \mathcal{B}} q^{degree(b)}\ T_A\ b|_b\Big)\ ,$$

and 3.10 then follows by applying formula 1.2.5 to $\mathbf{R}^G$ and formula 1.3.2 to $\mathbf{H}_G$.

4. Cones.

A vector space $\mathbf{V}$ of polynomials or formal power series in $x_1, x_2, \ldots, x_n$, shall here and after be called a *cone* if it is the linear span of the partial derivatives of one of its elements. Extending the notation introduced in the last section, we shall express this by writing

$$\mathbf{V} \;=\; \mathcal{L}[\partial_x^p \Delta] \qquad\qquad (\text{for some } \Delta \in \mathbf{V}) \;. \tag{4.1}$$

The element Δ itself will be referred to as a *summit* of $\mathbf{V}$. In the same vein, given a finite collection $\{\Delta_1, \ldots, \Delta_m\}$, we shall write

$$\mathbf{V} \;=\; \mathcal{L}[\partial_x^p \Delta_i \;:\; i = 1 \ldots M\,] \tag{4.2}$$

to indicate that $\mathbf{V}$ is the linear span of the derivatives of the $\Delta_i\,'s$. When the $\Delta_i\,'s$ are polynomials, equation 4.2 has a useful characterization which is dual to that expressed by Proposition I.2.3. For convenience set

$$\mathbf{I}_\Delta \;=\; \mathbf{I}(\Delta_1, \ldots, \Delta_M) \;=\; \{\, f \in \mathbf{R} \;:\; f(\partial_x)\Delta_i = 0 \;\; for \;\; i = 1 \ldots M \,\} \;. \tag{4.3}$$

In words, $\mathbf{I}_\Delta$ denotes the ideal of polynomials which kill all of the $\Delta_i\,'s$. This given, we have

Proposition 4.1

For any finite collection of polynomials $\{\Delta_i\}_{i=1}^M$, the space $\mathbf{V}$ given by 4.2 is the orthogonal complement of the ideal $\mathbf{I}(\Delta_1, \ldots, \Delta_M)$. In particular $\mathbf{V}$ is a cone with summit Δ if and only if

$$\mathbf{V} = \mathbf{I}(\Delta)^\perp \;. \tag{4.4}$$

Proof

Note first that a polynomial Q is orthogonal to the space $\mathbf{V}$ defined by 4.2 if and only if we have

$$0 \;=\; \langle Q \,,\, \partial_x^p \Delta_i \rangle \;=\; L_o Q(\partial_x) \partial_x^p \Delta_i \;=\; L_o \partial_x^p Q(\partial_x) \Delta_i$$

for all $i = 1..M$ and all exponent vectors p. In other words, the orthogonality of Q to $\mathbf{V}$ holds true if and only if $Q(\partial_x)\Delta_i$ and all its derivatives vanish at the origin. By Taylor's theorem this is equivalent to each of the $Q(\partial_x)\Delta_i\,'s$ vanishing identically. This gives that

$$\mathbf{V}^\perp \;=\; \mathbf{I}(\Delta_1, \ldots, \Delta_M) \;, \tag{4.5}$$

and our assertion follows since for a finite dimensional space $\mathbf{V} \subseteq \mathbf{R}$ we always have

$$\mathbf{V}^{\perp\perp} \;=\; \mathbf{V} \;.$$

It is good to note that

Proposition 4.2

If a cone V is a homogeneous space of polynomials then it has a homogeneous summit which is unique up to a constant factor.

Proof

Let $\mathbf{V} = \mathcal{L}[\partial_x^p \Delta]$ and let Δ_1 denote the homogeneous component of highest degree in Δ. We aim to show that

$$\mathbf{V} \;=\; \mathcal{L}[\partial_x^p \Delta_1] \;. \tag{4.6}$$

By the homogeneity of $\mathbf{V}$ we must have $\Delta_1 \in \mathbf{V}$. This gives that

$$\Delta_1 \;=\; A(\partial_x)\Delta \qquad (\text{for some } A \in \mathbf{R}). \tag{4.7}$$

This enables us to define a linear map $\phi \,:\, \mathbf{V} \to \mathbf{V}$ by setting

$$\phi\, Q(\partial_x)\Delta \;=\; Q(\partial_x)\Delta_1 \;.$$

This given, to prove 4.6, we are left to show that ϕ is injective. Suppose then that for some homogeneous Q we have

$$Q(\partial_x)\Delta_1 \;=\; 0 \;.$$

Using 4.7, we then get that

$$Q(\partial_x)A(\partial_x)\Delta \;=\; 0 \;.$$

This gives that $QA \in \mathbf{I}(\Delta)$ and therefore, since $\mathbf{I}(\Delta)$ is homogeneous, that each of the homogeneous components of QA are in $\mathbf{I}(\Delta)$. Now note that the lowest homogeneous component of QA is given by the polynomial $A(0)Q$, where $A(0)$ denotes the constant term of A. In fact, $A(0) \neq 0$ since otherwise 4.7 would imply that $degree(\Delta_1) < degree(\Delta)$. This yields that $Q \in \mathbf{I}(\Delta)$, and that ϕ is injective as desired. The uniqueness of homogeneous summits is an immediate consequence of the fact that we cannot have $\Delta_2 = A(\partial_x)\Delta_1$ with Δ_1 and Δ_2 homogeneous (of the same degree) without $A = const.$

Remark 4.1

It may be worth pointing out that, by formal power series methods, we can quickly deduce that 4.7 implies 4.6. In fact, since 4.7 yields that $A(0) = 1$, the polynomial A has an inverse $1/A$ given by the formal power series

$$f \;=\; \sum_{m \geq 0} (1 - A)^m \;.$$

Applying, $f(\partial_x)$ to both sides of 4.7 then immediately yields that

$$\Delta \;=\; f(\partial_x)\Delta_1 \;\in\; \mathcal{L}[\partial_x^p \Delta_1] \;,$$

and 4.6 must then hold true as asserted.

The homogeneous summit of $\mathbf{V}$ whose leading coefficient is equal to one, will here and after be referred to as *the summit* of $\mathbf{V}$. These two propositions enable us to establish the following basic result.

Theorem 4.1

Let $\mathbf{V}$ *be a homogeneous cone, let* Δ *be its summit and let* $n_o = degree(\Delta)$ *then*

a) $maxdegree(\mathbf{V}) \;=\; n_o$.

b) If $\mathcal{B}$ *is a basis of* $\mathbf{R}/\mathbf{I}_\Delta$ *(or even if* $\mathcal{B}$ *is a homogeneous basis of* $\mathbf{V}$ *itself) then the collection*

$$\mathcal{B}^* \;=\; \{\, b(\partial_x)\Delta \;:\; b \in \mathcal{B} \,\} \tag{4.8}$$

is also a basis for $\mathbf{V}$.

c) The Hilbert series of $\mathbf{V}$ *is symmetric, that is*

$$F_{\mathbf{V}}(q) \;=\; F_{\mathbf{V}}(1/q)\, q^{n_o} \;=\; \sum_{k=0}^{n_o} d_k\, q^k \qquad (\; with\;\; d_k = d_{n_o-k}\;)\;. \tag{4.9}$$

Proof

Property a) is immediate since $\mathbf{V} = \mathcal{L}[\partial_x^p \Delta]$ implies that all the elements of $\mathbf{V}$ have degree at most n_o. Note further that that proposition 4.1 yields that $\mathbf{V}$ and $\mathbf{R}/\mathbf{I}_\Delta$ have the same dimension. Thus to prove part b) we need only show the independence of $\mathcal{B}^*$. However, this is an immediate consequence of the definition of $\mathbf{I}_\Delta$, since the relation

$$\sum_{b\in\mathcal{B}} c_b\; b(\partial_x)\Delta \;=\; 0$$

is equivalent to the statement that the polynomial

$$\sum_{b\in\mathcal{B}} c_b\; b$$

belongs to $\mathbf{I}_\Delta$. Thus the independence of $\mathcal{B}$ *mod* $\mathbf{I}_\Delta$ forces the vanishing of all coefficients c_p. In view of the first part of proposition 1.3.1 we see that the same conclusion must hold true when $\mathcal{B}$ is also a homogeneous basis of $\mathbf{V}$. This proves b). Finally, if $\mathcal{B}$ is such a basis, 1.2.5 and b) give

$$F_{\mathbf{V}}(q) \;=\; \sum_{b\in\mathcal{B}} q^{degree\left(b(\partial_x)\Delta\right)} \;=\; \sum_{b\in\mathcal{B}} q^{n_o - degree(b)} \;=\; F_{\mathbf{V}}(1/q)\, q^{n_o}\;.$$

and this proves c).

One of the main goals of these notes is the study and characterization of *conical orbits*, that is orbits $[a]$ whose corresponding space of harmonics $\mathbf{H}_{[a]}$ is a cone. We have seen in the previous section that, when G is GGR, all regular orbits are conical. However, as we shall see there are some remarkable, and fundamental examples of conical orbits even in cases when G does not have a single reflection. For this reason it will be quite important for us to develop a package of tools for the study of orbits under the weakest possible assumptions. We shall deal first with some results concerning coordinate rings of finite sets.

For a given finite set S let

$$\mathbf{J}_S \;=\; \{\, P \in \mathbf{R} \;:\; P(x) = 0\; \forall\; x \in S \,\}\;. \tag{4.10}$$

and, in analogy with what we did when $S = [a]$, set

$$\mathbf{R}_S \;=\; \mathbf{R}/\mathbf{J}_S \quad , \quad gr\,\mathbf{R}_S \;=\; \mathbf{R}/gr\,\mathbf{J}_S \quad , \quad \mathbf{H}_S \;=\; gr\,\mathbf{J}_S{}^{\perp} \;. \tag{4.11}$$

However, here we shall consider an additional ingredient, namely the space

$$\mathbf{\Gamma}_S \;=\; \mathbf{J}_S{}^{\perp} \;. \tag{4.12}$$

We should note that Proposition 1.2.3 yields that we may also define $\mathbf{H}_S$ and $\mathbf{\Gamma}_S$ by setting

$$\mathbf{H}_S = \big\{P : Q(\partial_x)P = 0 \;\; \forall \;\; Q \in gr\,\mathbf{J}_S\big\} \quad \text{and} \quad \mathbf{\Gamma}_S = \big\{f : Q(\partial_x)f = 0 \;\; \forall \;\; Q \in \mathbf{J}_S\big\} \;. \tag{4.13}$$

It is important to note at the onset that $\mathbf{\Gamma}_S$ is not a space of polynomials. Indeed, we have

Theorem 4.2

The space $\mathbf{\Gamma}_S$ is always a cone of exponentials,

(i) with the collection $\{\, e^{(x,b)} \;:\; b \in S \,\}$ as basis, and

(ii) with summit given by any linear combination

$$\Phi_\epsilon \;=\; \sum_{b\in S} \epsilon_b\, e^{(x,b)} \qquad (\; \epsilon_b \neq 0 \quad \forall\, b \in S\;) \tag{4.14}$$

Proof

Note that if b is a point of n-dimensional ambient space and P is any polynomial in $x_1, x_2, \ldots, x_n$ then

$$P(\partial_x)\, e^{(x,b)} \;=\; P(b)\, e^{(x,b)} \;. \tag{4.15}$$

This means that if $b \in S$ we shall have $P(\partial_x)\, e^{(x,b)} = 0$ for all $P \in \mathbf{J}_S$. This shows that all these exponentials lie in $\mathbf{\Gamma}_S$. It is not difficult to show that exponentials with different exponents (x, b) are independent. Thus to show (i) we need only show that the collection in (i) does span $\mathbf{\Gamma}_S$. To this end let $\{\phi_b(x)\}_{b\in S}$ be polynomials chosen to satisfy

$$\phi_b(x) \;=\; \begin{cases} 1 & \text{if } x = b \\ 0 & \text{if } x \in S \;\&\; x \neq b \end{cases} \;. \tag{4.16}$$

Now let f be an arbitrary element of $\mathbf{\Gamma}_S$. Since the difference

$$1 \;-\; \sum_{b\in S} \phi_b(x)$$

vanishes throughout S, 4.13 gives that

$$f \;=\; \sum_{b\in S} \phi_b(\partial_x)\, f \;. \tag{4.17}$$

Now it develops that the term $f_b(x) = \phi_b(\partial_x)f$ is a constant multiple of $e^{(x,b)}$. To see this observe that 4.16, for $b = (b_1, \ldots, b_n)$ also gives

$$x_i\, \phi_b(x) \;=\; \begin{cases} b_i & \text{if } x = b \\ 0 & \text{if } x \in S \;\&\; x \neq b \end{cases} \;\cong\; b_i\, \phi_b(x) \qquad (\; mod\, \mathbf{J}_S\;) \;.$$

But this implies that

$$\partial_{x_i} f_b(x) \;=\; b_i\, f_b(x)\ .$$

Since this must hold true for all $i = 1..n$, we inevitably conclude that

$$f_b(x) \;=\; f_b(0)\, e^{(x,b)}\ .$$

This gives (i). To show (ii) we need only observe that if

$$f \;=\; \sum_{b\in S} c_b\, e^{(x,b)}\ ,$$

and

$$Q(x) \;=\; \sum_{b\in S} \frac{c_b}{\epsilon_b}\, \phi_b(x)\ ,$$

then, when Φ_ϵ is given by 4.13, we must have

$$Q(\partial_x)\, \Phi_\epsilon \;=\; \sum_{b\in S} \frac{c_b}{\epsilon_b}\, \epsilon_b e^{(x,b)} \;=\; f\ .$$

Thus we see that $\mathbf{\Gamma}_S$ is a cone and the expression in 4.13 may be taken as a summit.

It is not always easy to find a good basis for the ideal $\mathbf{J}_S$ and even more difficult to find one for $gr\, \mathbf{J}_S$. The following criterion will turn out to be useful in the sequel.

Proposition 4.3

Let $Q_1, Q_2, \ldots, Q_N$ be polynomials vanishing on a finite set S and let $P_1, P_2, \ldots, P_M$ be homogeneous elements of the ideal $gr\,(Q_1, Q_2, \ldots, Q_N)$ with the property that

$$dim\, \mathbf{R}/(P_1, P_2, \ldots, P_M) \;\le\; |S| \tag{4.18}$$

Then we have

$$(1)\quad \mathbf{J}_S \;=\; (Q_1, Q_2, \ldots, Q_N) \qquad \textit{and} \qquad (2)\quad gr\, \mathbf{J}_S \;=\; (P_1, P_2, \ldots, P_M)\ .$$

Proof

Since the vanishing of the Q_i on S gives that

$$(Q_1, Q_2, \ldots, Q_N) \;\subseteq\; \mathbf{J}_S\ , \tag{4.19}$$

we deduce that

$$|S| \;=\; dim\, \mathbf{R}_S \;\le\; dim\, \mathbf{R}/(Q_1, Q_2, \ldots, Q_N)\ . \tag{4.20}$$

On the other hand the containment

$$(P_1, P_2, \ldots, P_M) \;\subseteq\; gr\,(Q_1, Q_2, \ldots, Q_N) \tag{4.21}$$

gives

$$dim\,\mathbf{R}/(Q_1,Q_2,\ldots,Q_N) \;=\; dim\,\mathbf{R}/gr\,(Q_1,Q_2,\ldots,Q_N) \;\leq\; dim\,\mathbf{R}/(P_1,P_2,\ldots,P_M)\ . \qquad 4.22$$

Combining this with 4.20 and 4.18 we see that equality must hold throughout in 4.22 , 4.20 and 4.18. But then 4.19 and 4.21 must be equalities as well. Q.E.D.

There is a close relation between the two spaces $\mathbf{H}_S$ and $\mathbf{\Gamma}_S$ which is in a sense dual to the relation between $\mathbf{J}_S$ and $gr\,\mathbf{J}_S$. To state it we need some definitions. For a given formal power series f in the variables $x_1,x_2,\ldots,x_n$ let $m(f)$ denote the homogeneous component of *smallest* degree in f and $lm(f)$ be the dlexicographically *least* monomial in $m(f)$.

Remark 4.2

The operation $f \;\rightarrow\; m(f)$ should be used with some care. To get better acquainted with it, some additional notation is required. For a formal power series (or polynomial)

$$f \;=\; \sum_{p_1\geq 0}\cdots\sum_{p_n\geq 0} c_{p_1,\ldots,p_n}\; x_1^{p_1}\cdots x_n^{p_n}\ .$$

we shall set

$$f\,|_k \;=\; \sum_{|p|=k} c_p\; x^p \;=\; \sum_{p_1+\cdots+p_n=k} c_{p_1,\ldots,p_n}\; x_1^{p_1}\cdots x_n^{p_n}$$

and let

$$f\,|_{\leq k} \;=\; \sum_{h=0}^{k} f\,|_h \quad , \quad f\,|_{<k} \;=\; \sum_{h=0}^{k-1} f\,|_h\ .$$

with an analogous meaning given to the symbols $f\,|_{\geq k}$ and $f\,|_{>k}$. When convenient these symbols may also be abreviated to

$$f_{=k}\ ,\ f_{\leq k}\ ,\ f_{<k}\ ,\ f_{\geq k}\ ,\ f_{>k}\ .$$

Clearly, all these operations are linear, and we shall refer to $f\,|_k$ as the *homogeneous component of degree k of f* even in those cases when $f\,|_k = 0$. In contrast, $m(f)$ is defined by setting

$$m(f) \;=\; f\,|_{k_o}$$

if and only if

$$f\,|_{<k_o} \;=\; 0 \quad \textbf{and} \quad f\,|_{k_o} \neq 0\ .$$

Thus we can see that $m(f)$ is *not* a linear operator on formal power series. It will be good to keep this in mind in the sequel.

Proposition 4.4

For any finite set S we have

$$\mathbf{H}_S \;=\; \mathcal{L}[\,m(f):\ f\in\mathbf{\Gamma}_S\,] \qquad 4.23$$

Proof

Let Q be any polynomial of degree d and let f be a formal power series. If *degree* $m(f) = k_o \geq d$ then

$$Q(\partial_x)\, f \;\; = \;\; Q_{=d}(\partial_x)\, f_{=k_o} \;\; + \;\; Q_{<d}(\partial_x)\, f_{=k_o} \;\; + \;\; Q_{=d}(\partial_x)\, f_{>k_o} \;\; + \;\; Q_{<d}(\partial_x)\, f_{>k_o} \;\; .$$

Note that all the monomials that may come out of the first term have degree precisely $k_o - d$, and since none of the three remaining terms can produce a monomial of that degree we must necessarily have

$$Q(\partial_x) f \,|_{k_o - d} \;\; = \;\; Q_{=d}(\partial_x)\, f_{=k_o} \;\; = \;\; h(Q)(\partial_x)\, m(f) \;\; .$$

Thus if $Q \in \mathbf{J}_S$ and $f \in \mathbf{\Gamma}_S$, then the second equality in 4.13 gives

$$h(Q)(\partial_x)\, m(f) \;\; = \;\; 0 \;\; .$$

In other words $m(f)$ is killed by every element of $gr\, \mathbf{J}_S$ and so, by the first equality in 4.13, we deduce that $m(f) \in \mathbf{H}_S$. We are left to show that these elements span $\mathbf{H}_S$. To this end let

$$\Phi_1 \;, \; \Phi_2 \;, \; \ldots \;, \; \Phi_s \tag{4.24}$$

be a basis for $\mathbf{\Gamma}_S$ obtained by the Gauss elimination process on the basis $\{\, e^{(x,b)} \;:\; b \in S \,\}$ arranged in an arbitrary order but eliminating the monomials x^p in the increasing dlex order. In other words, we are to produce this basis in such a manner that the sequence of *least* monomials

$$lm(\Phi_1) \;, \; lm(\Phi_2 \;, \;) \ldots \;, \; lm(\Phi_s) \tag{4.25}$$

is dlexicographically increasing. Clearly, this done, the resulting minimal degree homogeneous components

$$m(\Phi_1) \;, \; m(\Phi_2 \;, \;) \ldots \;, \; m(\Phi_s) \tag{4.26}$$

must be independent, for they will be triangularly related to the monomial basis of $\mathbf{R}$. However, since the elements in 4.24 are a basis for $\mathbf{\Gamma}_S$ and $dim\, \mathbf{\Gamma}_S = |S|$ we must have $s = |S|$ but then the independence of the polynomials in 4.26 together with the fact that also $\mathbf{H}_S$ has dimension $|S|$ yields that these polynomials must necessarily form a basis for $\mathbf{H}_S$. This completes our proof.

We are finally in a position to establish the basic result of this section.

Theorem 4.2

A G-regular orbit $[a]$ *is conical if and only if the Hilbert series of* $\mathbf{H}_{[a]}$ *is symmetric.*

Proof

In view of theorem 4.1 we need only show sufficiency here. So let a be a regular point and let

$$F_{\mathbf{H}_{[a]}} \;\; = \;\; \sum_{k=0}^{n_o} d_k \; q^k \;\; ,$$

with

$$d_k = d_{n_o - k} \quad . \tag{4.27}$$

Since $d_o = 1$ symmetry gives that $d_{n_o} = 1$ as well. So there must be a homogeneous polynomial Δ_o of degree n_o in $\mathbf{H}_{[a]}$ which is unique up to a multiplicative constant. We aim to show that $\mathbf{H}_{[a]}$ is a cone with summit Δ_o. To this end note that, since the action of G preserves degree, we must then have constants $\epsilon(A)$ such that

$$T_A \, \Delta_o \;\; = \;\; \epsilon(A) \, \Delta_o \qquad\qquad (\, \forall \, A \in G \,) \quad . \tag{4.28}$$

Clearly, the map $A \;\; \rightarrow \;\; \epsilon(A)$ is a representation of G, and as such the constants $\epsilon(A)$ cannot vanish. As we know, the regularity of a implies that $\mathbf{H}_{[a]}$ is a graded version of the left regular representation of G. Thus the (necessarily irreducible and non-trivial) representation ϵ can only occur with multiplicity one in $\mathbf{H}_{[a]}$. This means that any element of $\mathbf{H}_{[a]}$ that transforms as in 4.28 must be a constant multiple of Δ_o itself. This given, set

$$\Phi_{[a]} \;\; = \;\; \sum_{B \in G} \epsilon(B) \, e^{(x, aB)} \quad . \tag{4.29}$$

Note that for any $A \in G$ we have (recall that G is a group of orthogonal matrices)

$$T_A \, \Phi_{[a]} \;\; = \;\; \sum_{B \in G} \epsilon(B) \, e^{(x, aBA^{-1})} \;\; = \;\; \sum_{B \in G} \epsilon(BA) \, e^{(x, aB)} \;\; = \;\; \epsilon(A) \, \Phi_{[a]} \quad . \tag{4.30}$$

Set, for a moment,

$$\Delta_{[a]} \;\; = \;\; m(\Phi_{[a]}) \quad , \tag{4.31}$$

and note that since G preserves degree we must have $m(T_A f) = T_A m(f)$ for any element $f \in \mathbf{\Gamma}_{[a]}$. In particular, 4.30 yields that

$$T_A \, \Delta_{[a]} \;\; = \;\; \epsilon(A) \, \Delta_{[a]} \quad . \tag{4.32}$$

Since proposition 4.4 gives that $\Delta_{[a]} \in \mathbf{H}_{[a]}$ we must conclude that $\Delta_{[a]}$ must be a constant multiple of Δ_o. Since, Δ_o was originally determined only up to a multiplicative constant, we can assume here and after that $\Delta_{[a]} = \Delta_o$.

Our plan is to show that

$$\mathbf{H}_{[a]} \;\; = \;\; \mathcal{L}[\, \partial_x^p \, \Delta_{[a]} \,] \tag{4.34}$$

by an induction argument. More precisely, let $\mathcal{B}$ be a homogeneous basis for $\mathbf{H}_{[a]}$ and let $\mathcal{B}_k$ be the subset of elements of degree k in $\mathcal{B}$. We shall assume (in the notation of Remark 4.2) that

1_k $\quad P(\partial_x)\Phi_{[a]}) \, |_{<n_o - k} \;\; = \;\; 0$ implies that P is congruent $mod \, \mathbf{J}_{[a]}$ to a polynomial Q of degree $\leq k$.

and

2_k $\quad H_{n_o - k}(\mathbf{H}_{[a]}) \;\; = \;\; \mathcal{L}[\, b(\partial_x)\Delta_{[a]} \;:\; b \in \mathcal{B}_k \,]$

hold true for all $k < k_o$ then show 1_{k_o} and 2_{k_o} must hold as well.

Before we start our argument we must make two important observations which we shall here and after refer to as OB_1 and OB_2.

OB_1 : *To show that an $f \in \mathbf{\Gamma}_{[a]}$ is zero it is sufficient to show that all its homogeneous components of degree $\leq n_o$ vanish.*

This is simply due to the fact that by assumption $n_o = maxdegree(\mathbf{H}_{[a]})$ and thus for any $f \neq 0$, $m(f) \in \mathbf{H}_{[a]}$ implies $degree(m(f)) \leq n_o$. In particular this implies that, up to a constant factor, $\Phi_{[a]}$ is the only element of $\mathbf{\Gamma}_{[a]}$ which satisfies 4.30.

OB_2 : *To show that a polynomial Q belongs to $\mathbf{J}_{[a]}$ we need only show that it kills $\Phi_{[a]}$.*

In fact, the definition 4.29 and 4.15 yield that for any Q

$$Q(\partial_x)\,\Phi_{[a]} \;=\; \sum_{B\in G} \epsilon(B)\, Q(aB)\, e^{(x,aB)}$$

and thus when $Q(\partial_x)\Phi_{[a]} = 0$, the non-vanishing of the $\epsilon(A)'s$ together with the independence of the exponentials $e^{(x,aA)}$ (for a regular a) imply that Q must vanish throughout $[a]$.

This given, we can start by checking the validity of 1_0 and 2_0. Note that 1_0 says that any P such that

$$P(\partial_x)\Phi_{[a]}\,|_{<n_o} \;=\; 0 \tag{4.35}$$

must be congruent to a constant $mod\ \mathbf{J}_{[a]}$. Note that if $P(\partial_x)\Phi_{[a]}\,|_{n_o}$ also vanishes then OB_1 gives that Q kills $\Phi_{[a]}$ and thus OB_2 yields that Q is congruent to zero. On the other hand, if $P(\partial_x)\Phi_{[a]}\,|_{n_o}$ doesn't vanish then since the homogeneous elements of degree n_o in $\mathbf{H}_{[a]}$ are all multiples of $\Delta_{[a]}$, we must have

$$Q(\partial_x)\Phi_{[a]}\,|_{n_o} \;=\; c\,\Delta_{[a]}$$

for a suitable constant c. But then all the homogeneous components of degree n_o or less in

$$(Q(\partial_x) - c)\,\Phi_{[a]}$$

must vanish and OB_1 and OB_2 again yield that $Q - c \in \mathbf{J}_{[a]}$. This gives 1_0. Finally, since we may assume that $\mathcal{B}_0$ reduces to the constant 1, we see that 2_0 simply says that $H_{n_o}(\mathbf{H}_{[a]})$ consists of constant multiples of $\Delta_{[a]}$, which is indeed true by our construction of $\Delta_{[a]}$.

We are thus in a position to proceed with our induction, and we shall assume that 1_k and 2_k hold for all $k < k_o$. We start by proving 2_{k_o}. To this end note that by symmetry (4.27 for $k = k_o$) we need only show that the elements $\{b(\partial_x)\Delta_{[a]} : b \in \mathcal{B}_{k_o}\}$ are independent. So let there be constants c_b such that

$$\sum_{b\in\mathcal{B}_{k_o}} c_b\, b(\partial_x)\Delta_{[a]} \;=\; 0 \tag{4.36}$$

and set

$$P(x) \;=\; \sum_{b\in\mathcal{B}_{k_o}} c_b\, b(x) \tag{4.37}$$

Now, 4.36 implies that

$$P(\partial_x)\Phi_{[a]}\,|_{\leq n_o-k_o} \;=\; 0\ .$$

However, this brings us into 1_{k_o-1} and by induction we can find a polynomial Q of degree $< k_o$ congruent to P modulo $\mathbf{J}_{[a]}$. But now, since $degree\ P = k_o > degree\ Q$ and P is homogeneous, we deduce that $P \in gr\,\mathbf{J}_{[a]}$. Combining with the fact that by construction $P \in gr\,\mathbf{J}_{[a]}{}^{\perp}$ we finally get that P itself must vanish identically. The independence of $\mathcal{B}_{k_o}$ then yields that all the coefficients c_p must vanish as well. Thus 2_{k_o} must hold true as desired.

Next we show 1_{k_o}. So let

$$P(\partial_x)\Phi_{[a]}\,|_{<n_o-k_o} \;=\; 0\ . \tag{4.38}$$

If $P(\partial_x)\Phi_{[a]} = 0$ then by OB_2 we must have $P \in \mathbf{J}_{[a]}$ and we are done. If $P(\partial_x)\Phi_{[a]}$ doesn't vanish then 4.38 gives that

$$degree\ m(P(\partial_x)\Phi_{[a]}) \;=\; n_o - k_1 \qquad\qquad (\text{ with } k_1 \leq k_o\ )\ .$$

By 2_{k_1} which is now available up to and including k_o we can find a homogeneous polynomial Q_1 of degree k_1 such that

$$m(P(\partial_x)\Phi_{[a]}) \;=\; Q_1(\partial_x)\,\Delta_{[a]}\ . \tag{4.39}$$

This in turn implies that

$$(P(\partial_x) - Q_1(\partial_x))\Phi_{[a]}\,|_{\leq n_o-k_1} \;=\; 0\ .$$

However, since $k_1 \leq k_o$, this brings us down into the domain of 1_{k_o-1}, so we can use the induction hypothesis and conclude that $P - Q_1$ is congruent $mod\ \mathbf{J}_{[a]}$ to a polynomial Q_2 of degree at most $k_o - 1$. In other words we have shown that P is congruent $mod\ \mathbf{J}_{[a]}$ to the polynomial $Q = Q_1 + Q_2$ which is of degree at most k_o, which is precisely what we needed to show. This completes the induction and our proof.

5. Some classical examples.

To illustrate the power of the results we have gathered so far and motivate some of our later developments it will be good to work out some examples. We start with $G = S_n$ acting according to 1.1.3. It is well known and easy to show that this action of S_n is generated by reflections and we can take

$$\mathcal{N}(G) \;=\; \{\ u^{(i)} - u^{(j)}\ :\ 1 \leq i < j \leq n\ \} \tag{5.1}$$

with the $u^{(i)}{}'s$ the unit vectors given in 2.8. Thus Δ_G in this case is the Vandermonde determinant

$$\Delta_{S_n} \;=\; \prod_{i<j}(x_i - x_j) \;=\; det\,\|x_i^{n-j}\|\ . \tag{5.2}$$

The tools we have in our possession readily yield (as we shall see in more than one way) the following collection of basic properties:

Theorem 5.1

For the standard action of S_n by permutation matrices we have

(i) $\mathbf{H}_{S_n}$ *is a cone with summit* Δ_{S_n} *and dimension n!. In particular all regular orbits are conical and have the same harmonics.*

(ii) The collection of monomials

$$\mathcal{A} \;=\; \{\, x_1^{\epsilon_1}\cdots x_n^{\epsilon_n} \;:\; 0\le \epsilon_i \le i-1 \,\} \tag{5.3}$$

is a basis for $\mathbf{R}/\mathcal{J}_{S_n}$.

(iii) The harmonics

$$\{\, \partial_{x_1}^{\epsilon_1}\cdots\partial_{x_n}^{\epsilon_n}\,\Delta_{S_n} \;:\; 0\le \epsilon_i \le i-1 \,\} \tag{5.4}$$

form a basis for $\mathbf{H}_{S_n}$.

(iv) Every invariant $P\in \mathbf{R}^{S_n}$ *has a unique expression as a polynomial in the elementary symmetric functions*

$$e_k(x) \;=\; \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k} \tag{5.5}$$

(v) More generally, every $P\in\mathbf{R}$ *has an expansion of the form*

$$P \;=\; \sum_{x^\epsilon\in\mathcal{A}} x^\epsilon\, A_\epsilon(e_1,\dots e_n)\;. \tag{5.6}$$

with coefficients A_ϵ *polynomials in* $e_1,\dots e_n$ *uniquely determined by* P.

(vi)

$$(a)\;\; F_{\mathbf{R}^{S_n}}(q) \;=\; \frac{1}{(1-q)\cdots(1-q^n)} \quad,\quad (b)\;\; F_{\mathbf{H}_{S_n}}(q) \;=\; \prod_{i=1}^{n}(1+q+\cdots+q^{i-1}) \quad. \tag{5.7}$$

(vii) The graded character of $\mathbf{H}_{S_n}$ *has the expansion*

$$\chi^{\mathbf{H}_{S_n}}(\sigma;q) \;=\; \sum_{\lambda\vdash n}\chi^\lambda(\sigma)\sum_{T\in ST(\lambda)} q^{c(T)}\;. \tag{5.8}$$

Proof

Note first that these various properties are connected to each other and to our previous results as follows:

(i) is a specialization to S_n of part (iii) of theorem 3.3.
(vi) (b) is an immediate consequence of (ii).
(vi) (a) follows from (vi) (b) and formula 2.18.
(vii) follows by combining (vi) (a) with formulas 1.3.10 and 2.3.10.
(v) follows by combining (iv) with part (iv) of theorem 3.3.

Note further that

(iii) follows from (i) and (ii) using part b) of theorem 4.1

and conversely

(ii) follows from (iii) using part (c) of theorem 3.2.

Indeed, part (c) of theorem 3.2 yields that

$$\sum_{x^\epsilon \in \mathcal{A}} c_\epsilon \, x^\epsilon \;=\; 0 \qquad\qquad (mod\ \mathcal{J}_{S_n}\ )$$

implies that

$$\sum_{x^\epsilon \in \mathcal{A}} c_\epsilon \, \partial_x^\epsilon \, \Delta_{S_n} \;=\; 0\ .$$

Thus the independence of the polynomials in 5.4 implies the independence (mod $\mathcal{J}_{S_n}$)of the monomials in $\mathcal{A}$.

This leaves us with showing (ii) or (iii) and (iv). Moreover, since we know (by (i)) that $\mathbf{H}_{S_n}$ has dimension $n!$. We need only show that the monomials in $\mathcal{A}$ span or that the polynomials in 5.4 are independent. Before we reach the end of this section we shall encounter 4 different ways of establishing what remains to be proved. We start by giving what we adopt as the *official proof* of (ii) since it yields (iv) as a biproduct and thus enables us to complete the proof of the theorem. To this end, note that

$$(e_1, e_2, \ldots, e_n) \subseteq \mathcal{J}_{S_n}\ . \tag{5.9}$$

Moreover, since

$$(1-x_1t)(1-x_2t)\cdots(1-x_nt) \;\cong\; 1 \qquad\qquad (\ mod\ (e_1,..,e_n)\ )\ , \tag{5.10}$$

for every $k \geq 1$ we must have

$$(1-x_1t)\cdots(1-x_{k-1}t) \;\cong\; \frac{1}{(1-x_kt)\cdots(1-x_nt)} \qquad\qquad (\ mod\ (e_1,..,e_n)\ )\ . \tag{5.11}$$

Equating coefficients of t^k in this relation yields that the homogeneous symmetric function $h_k(x_k,..,x_n)$ lies in $(e_1,..,e_n)$. That is

$$h_k(x_k,..,x_n) \;=\; x_k^k \;+\; \sum_{s=0}^{k-1} x_k^s \, h_{k-s}(x_{k+1},..,x_n) \;\cong\; 0 \qquad\qquad (\ mod\ (e_1,..,e_n)\ )\ . \tag{5.12}$$

This immediately implies that, at the expense of an increase in the exponents of the variables $x_{k+1}, x_{k+2}, \ldots, x_n$ we can always reduce the exponent of the variable x_k to a value less than k. More precisely, using 5.12 for $k=1$, which simply reduces to

$$x_1 \cong -x_2 - x_3 - \ldots - x_n \qquad\qquad (\ mod\ (e_1,..,e_n)\ )$$

we can express (mod $(e_1,..,e_n)$) any monomial x^p as a linear combination of monomials in which x_1 doesn't appear. Recursively, having expressed every monomial x^p as a combination of monomials

in which each x_i appears only to a power less than i (for $i = 1, 2, \ldots, k-1$), by means of 5.12, we can complete the induction by expressing the latter monomials in terms of monomials in which x_k itself is also raised to a power less than k. This shows that the monomials in $\mathcal{A}$ span the quotient

$$\mathbf{R}/(e_1, .., e_n)$$

In other words we have shown (using 5.9) that

$$n! = dim\, \mathbf{R}/\mathcal{J}_{S_n} \leq dim \mathbf{R}/(e_1, .., e_n) \leq |\mathcal{A}| = n! \ . \tag{5.13}$$

Thus equality must hold throughout and therefore $\mathcal{A}$ spans $\mathbf{R}/\mathcal{J}_{S_n}$ This gives part (ii) of the theorem. However, since equality in 5.13 forces equality in 5.9 as well, proposition 2.2 yields also part (iv) as desired.

It will be instructive to go over alternate paths to the proof of this theorem, since this will build up our repertoire of tricks for the study of orbit harmonics. We shall present this material as a collection of remarks.

Remark 5.1 (Alternate proof of (ii))

Note that equation 5.12 yields that, under the appropriate dlex order of monomials, the monomials x_k^k belong to the ideal of monomials $\mathcal{M}(\mathcal{I})$ with $\mathcal{I} = (e_1, e_2, \ldots, e_n)$. This implies that no monomial in $\mathbf{B}_{\mathcal{I}}$ can contain x_k^k as a submonomial. In other words (using Remark 1.4.4) we must have

$$\mathbf{B}_{\mathcal{J}_{S_n}} \ \subseteq \ \mathbf{B}_{\mathcal{I}} \ \subseteq \ \mathcal{A} \ .$$

But if we know that $dim\, \mathbf{R}/\mathcal{J}_{S_n} = n!$ then equality must necessarily hold throughout. This not only shows (ii) but the remarkable fact that $\mathcal{A} = \mathbf{B}_{\mathcal{J}_{S_n}}$. A more general result of this nature is proved in [].

Remark 5.2 (Alternate proof of (iii))

We can obtain a direct derivation of (iii) by combining part (iii) of theorem 3.3 with the following general fact.

Proposition 5.1

If Δ is homogeneous and $x^q = x_1^{q_1} \cdots x_n^{q_n}$ is its leading monomial (in an appropriate dlex order) then the family of polynomials

$$\mathcal{B}_\Delta \ = \ \{ \, \partial_x^p \, \Delta \, : \, p_i \leq q_i \, \} \tag{5.14}$$

is an independent subset of the homogeneous cone with summit Δ. In particular, if in addition we have that

$$(1+q_1)(1+q_2)\cdots(1+q_n) \ = \ dim\, \mathcal{L}[\partial_x^p \Delta] \tag{5.15}$$

then $\mathcal{B}_\Delta$ must necessarily also be a basis of the cone.

Proof

Since the polynomial $\partial_x^p \Delta$ has leading monomial $x^{(q-p)}$, the family $\mathcal{B}_\Delta$ is triangularily related to the set of monomials $\{ x^p \ : \ p_i \le q_i \ \}$ and a-fortiori must also be independent. The last assertion follows since the product in 5.15 gives the cardinality of $\mathcal{B}_\Delta$.

Clearly, all the conditions of the proposition are satisfied for the Vandermonde determinant since its leading term is $x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$, and (iii) of theorem 3.3 yields 5.15. This type of argument can be carried out verbatim for other reflection groups. For instance for the hyperoctahedral group we have

$$\mathcal{N}(B_n) \ = \ \{ \, u^{(i)} \, , \, u^{(i)} \pm u^{(j)} \ : \ 1 \le i < j \le n \, \} \tag{5.16}$$

thus its discriminant is

$$\Delta_{(B_n)} \ = \ \prod_{i<j} x_i(x_i^2 - x_j^2) \ ,$$

whose leading monomial is

$$x_1^{2n-1}x_2^{2n-3}\cdots x_{n-1}^1 \ .$$

this gives 5.15 since the corresponding product evaluates to $2^n \, n!$ which equals the order of B_n. It should also be noted that the official proof of part (ii) can also be carried out almost verbatim for B_n, upon replacing $e_k(x_1, x_2, \ldots, x_n)$ by $e_k(x_1^2, x_2^2, \ldots, x_n^2)$ in the argument. Thus in particular, in more than one way we can conclude that the family of monomials

$$\{ \ x_1^{p_1}x_2^{p_2}\cdots x_{n-1}^{p_{n-1}} \ : \ 0 \le p_i \le 2n-2i+1 \ \} \tag{5.17}$$

is a basis for $\mathbf{R}/\mathcal{J}_{B_n}$.

Remark 5.3 (Alternate proof of (iv))

Let a be a regular point for S_n, that is any point with distinct coordinates. In particular we could take $a_i = i$. Since we must necessarily have

$$\big(\, e_1(x) - e_1(a), e_2(x) - e_2(a), \ldots, e_n(x) - e_n(a) \big) \ \subseteq \ \mathcal{J}_{[a]}$$

and the official proof of (ii) yields that

$$dim \ \mathbf{R}/(e_1, e_2, \ldots, e_n) \ \le \ n! \ ,$$

we can make use of proposition 4.3 with $Q_k = e_k(x) - e_k(a)$ and $P_k = e_k$ and conclude that

$$gr \ \mathcal{J}_{[a]} \ = \ (e_1, e_2, \ldots, e_n) \ .$$

This given, (iv) follows from theorem 3.3 (i) which gives $gr \ \mathcal{J}_{[a]} = \mathcal{J}_G$, and proposition 2.2.

We have already seen three different ways of completing the proof of Theorem 5.1. The fourth one, which we are about to present, is based on a remarkable result (theorem 5.2 below)

which will turn out to be a crucial tool in the study of orbit harmonics. We start by introducing some notation and establishing three auxiliary propositions.

For a given collection $\mathcal{B}$ of polynomials, we shall set

$$\mathcal{B}^{\leq i} \;=\; \{\, b \in \mathcal{B} \;:\; degree(b) \leq i \,\} \quad and \quad \mathcal{B}^{=i} \;=\; \{\, b \in \mathcal{B} \;:\; degree(b) = i \,\} \;.$$

Recall, that if F is a polynomial $h(F)$ denotes the homogeneous component of highest degree in F, here we shall also set $r(F) = F - h(F)$ and refer to it as the *remainder* of F. This given we have

Proposition 5.2

Let $\mathcal{B}$ be a basis for $\mathbf{R}/\mathcal{J}$, let $n_o = maxdegree\ \mathbf{R}/\mathcal{J}$ and set

$$F_{\mathbf{R}/gr\ \mathcal{J}}(q) \;=\; \sum_{i=0}^{n_o} d_i\, q^i \;. \tag{5.18}$$

Then the equalities

$$|\mathcal{B}^{\leq i}| \;=\; d_o + d_1 + \cdots + d_i \qquad (\, i = 0, 1, \ldots n_o \,) \tag{5.19}$$

imply that the collection of homogeneous polynomials

$$h(\mathcal{B}) \;=\; \{\, h(b) \;:\; b \in \mathcal{B} \,\} \tag{5.20}$$

is a basis for $\mathcal{R}/gr\mathcal{J}$.

Proof

We need only show that the collection $h(\mathcal{B})^{=k}$ is an independent set *mod* $gr\mathcal{J}$ (for any $k \leq n_o$). To this end let $F_1, F_2, \ldots, F_M$ be a standard basis for $\mathcal{J}$, that is one that satisfies the condition 1.4.2, to the effect that the corresponding set of leading monomials yield the minimal elements of the monomial ideal $\mathcal{M}(\mathcal{J})$. Suppose that for some non vanishing set of constants c_b we have

$$\sum_{b \in \mathcal{B}^{=k}} c_b\, h(b) \;\in\; gr\ \mathcal{J} \;. \tag{5.21}$$

Note that since, $\mathcal{M}(\mathcal{J}) = \mathcal{M}(gr\mathcal{J})$, the collection $\{h(F_1), h(F_2), \ldots, h(F_M)\}$ must give a standard basis for $gr\ \mathcal{J}$. We can thus use theorem 1.4.2 with $\mathcal{J}$ replaced by $gr\ \mathcal{J}$ and $f_i = h(F_i)$, and derive from 5.21 that there must be homogeneous polynomials A_i such that

$$\sum_{b \in \mathcal{B}^{=k}} c_b\, h(b) \;=\; \sum_{i=1}^{M} A_i h(F_i) \tag{5.22}$$

with

$$degree(A_i h(F_i)) \leq k \qquad (\, i = 1, 2, \ldots, M \,) \;. \tag{5.23}$$

However, we may also rewrite this as

$$\sum_{b\in\mathcal{B}^{=k}} c_b\, b \;-\; \sum_{b\in\mathcal{B}^{=k}} c_b\, r(b) \;=\; \sum_{i=1}^{M} A_i F_i \;-\; \sum_{i=1}^{M} A_i r(F_i)$$

and this in turn yields that

$$\sum_{b\in\mathcal{B}^{=k}} c_b\, b \cong \sum_{b\in\mathcal{B}^{=k}} c_b\, r(b) \;-\; \sum_{i=1}^{M} A_i r(F_i) \qquad (\,mod\ \mathcal{J}\,)\ , \tag{5.24}$$

Moreover, the definitions of $r(b)$, $r(F_i)$ and 5.23 guarantee that the polynomial on the right hand side of 5.24 is of degree $k-1$ at most. Note now that equation 1.4.20, combined with our assumption 5.19, yields that the subspace $\mathbf{R}^{\leq i}/\mathcal{J}$, linear span $(mod\ \mathcal{J})$ of monomials of degree $\leq i$ has dimension precisely

$$d_o + d_1 + \cdots + d_i \;=\; |\mathcal{B}^{\leq i}|\ .$$

This means that $\mathcal{B}^{\leq k-1}$ must necessarily be a basis for $\mathbf{R}^{\leq k-1}/\mathcal{J}$. In particular, the right hand side of 5.24 must be expressible as a linear combination of elements of $\mathcal{B}^{\leq k-1}$. However, this plainly contradicts the assumption that $\mathcal{B}$ is a basis for $\mathbf{R}/\mathcal{J}$. Thus 5.21 can only hold true when the coefficients c_b are all equal to zero. This completes our proof.

Proposition 5.3

Let a be a regular point, set

$$n_o = maxdegree\ \mathbf{R}_{[a]}\ . \tag{5.25}$$

Suppose that

$$dim\ \mathcal{H}_{n_o}(gr\ \mathbf{R}_{[a]}) \;=\; 1\ , \tag{5.26}$$

and let Δ_o be a polynomial of degree n_o which does not vanish (mod $gr\ \mathcal{J}_{[a]}$). Then

$$T_A\,\Delta_o \;=\; \epsilon(A)\Delta_o \qquad \forall\ A\in G \tag{5.27}$$

and the map $A \rightarrow \epsilon(A)$ is a non trivial representation of G. Moreover, for any polynomial P of degree strictly less than n_o we must have

$$\sum_{A\in G} \epsilon(A)\, T_A P \;=\; 0 \qquad (\,mod\ \mathcal{J}_{[a]}\,) \tag{5.28}$$

Proof

Recall that the regularity of a implies (Theorem 1.1) that $gr\ \mathbf{R}_{[a]}$ is a version of the left regular representation of G. In particular all its one dimensional constituents must occur with multiplicity one. Note further that, since each of the homogeneous components of $gr\ \mathbf{R}_{[a]}$ is G-invariant, 5.26 gives that, we must have constants $\epsilon(A)$ yielding 5.27. The map $A \rightarrow \epsilon(A)$ must be a non-trivial representation of G since otherwise $gr\ \mathbf{R}_{[a]}$ would have more that one occurrence of the

trivial. For the same reason, ϵ itself must also occur with multiplicity one in $gr\,\mathbf{R}_{[a]}$. Finally, note that whatever P may be, the polynomial

$$\pi_\epsilon\, P \;=\; \sum_{A\in G} \epsilon(A)\, T_A P \tag{5.29}$$

transforms by ϵ under the action of G. More precisely we must have

$$T_A\, \pi_\epsilon\, P \;=\; \epsilon(A)\, \pi_\epsilon\, P \qquad\qquad \forall\ A \in G\ . \tag{5.30}$$

If P is of degree less than n_o then $\pi_\epsilon\, P$ either vanishes mod $\mathcal{J}_{[a]}$ or yields an occurrence of ϵ in the submodule $\mathbf{R}_{[a]}^{\leq n_o-1}$. However, formula 1.4.22 for $k = n_o - 1$, which in our case specializes to

$$char\, \mathbf{R}_{[a]}^{\leq n_o-1} \;=\; \sum_{i=0}^{n_o-1} char\mathcal{H}_i(gr\, \mathbf{R}_{[a]})\ , \tag{5.31}$$

immediately yields that ϵ must also occur in one of the submodules $\mathcal{H}_i(gr\,\mathbf{R}_{[a]})$ (for $i < n_o$). But this would assure another occurrence of ϵ in $gr\,\mathbf{R}_{[a]}$ and the latter is excluded as we have already noted. Thus 5.28 must hold true precisely as asserted.

When a is a regular point and 5.25, 5.26 hold true, this proposition enables us to construct a remarkable involution on the module $\mathbf{R}_{[a]}$. This involution is easily described in terms of the family of interpolating polynomials used in section 1. More precisely let $\{\phi_A(x)\}_{A\in G}$ be a collection of polynomials satisfying

$$\phi_A(x) \;=\; \begin{cases} 1 & \text{if } x = aA \quad \text{and} \\ 0 & \text{if } x = aB \quad \text{with } B \in G \text{ and } B \neq A\ . \end{cases}$$

Then, for any polynomial P we have

$$P(x) \;=\; \sum_{A\in G} P(aA)\, \phi_A(x)\ . \qquad\qquad (\ mod\ \ \mathcal{J}_{[a]}\ )$$

This given, under the hypotheses of Proposition 5.3 we set

$$\theta_\epsilon\, P \;=\; \sum_{A\in G} \epsilon(A)\, P(aA)\, \phi_A(x)\ , \tag{5.32}$$

where ϵ is the representation yielding 5.27. Clearly, the map θ_ϵ is well defined since polynomials, as elements of $\mathbf{R}_{[a]}$ may be simply represented by their values at the points of $[a]$. Moreover, θ_ϵ is invertible since $\epsilon(A)\epsilon(A^{-1}) = 1$. Now let $\langle\ ,\ \rangle_{[a]}$ denote the scalar product on $\mathbf{R}_{[a]}$ defined by setting for any two polynomials P, Q

$$\langle P, Q\rangle_{[a]} \;=\; \sum_{A\in G} P(aA)Q(aA)\ . \tag{5.33}$$

This given, equation 5.28, can be translated into the following remarkable result.

Proposition 5.4

If a is a regular point, then under the hypotheses 5.25 and 5.26, the map θ_ϵ defined in 5.32 sends the subspace $\mathbf{R}_{[a]}^{\leq k}$ into the orthogonal complement of $\mathbf{R}_{[a]}^{\leq n_o-k-1}$ with respect to the scalar product in 5.33. In short we may write

$$\theta_\epsilon \, \mathbf{R}^{\leq k} \;\subseteq\; \left(\mathbf{R}^{\leq n_o-k-1} \right)^{\perp} \;. \tag{5.34}$$

and we must have

$$dim\, \mathbf{R}_{[a]}^{\leq k} \;+\; dim\, \mathbf{R}_{[a]}^{\leq n_o-k-1} \;\leq\; |G| \qquad (\, for \;\; k = 1, 2, \ldots, n_o - 1 \,) \;. \tag{5.35}$$

Proof

Note that if P is a polynomial of degree at most k and Q is a polynomial of degree at most $n_o - k - 1$ then the product is of degree at most $n_o - 1$. We can thus substitute PQ for P in equation 5.28 and obtain

$$\sum_{A \in G} \epsilon(A)\; P(xA) Q(xA) \;\cong\; 0 \qquad (\, mod \;\; \mathcal{J}_{[a]} \,)$$

But this means that the polynomial on the left hand side must vanish at all points of $[a]$, in particular, setting $x = a$ gives

$$\langle \theta_\epsilon P, Q \rangle_{[a]} \;=\; 0 \;.$$

This establishes 5.34. The inequality in 5.35 then immediately follows since θ_ϵ is a non singular and $dim\, \mathbf{R}_{[a]} = |[a]| = |G|$.

We are now in a position to derive the basic result of this section.

Theorem 5.2

Let a be a regular point. Set

$$n_o = maxdegree\, \mathbf{R}_{[a]} \;. \tag{5.36}$$

Suppose that $\mathcal{B}$ is a basis for $\mathbf{R}_{[a]}$ with

$$\mathcal{B}^{\leq n_o} = \mathcal{B} \quad and \quad |\mathcal{B}^{\leq i}| = d_o + d_1 + \cdots + d_i \qquad (\, i = 0, 1, \ldots, n_o \,) \;. \tag{5.37}$$

Then the equalities

$$d_i \;=\; d_{n_o - i} \qquad (\, i = 0, 1, \ldots, n_o \,) \tag{5.38}$$

imply

(i) $\quad h(\mathcal{B}) \;=\; \{\, h(b) \;:\; b \in \mathcal{B} \,\}$ *is a basis for* $gr\, \mathbf{R}_{[a]}$

(ii) $\quad F_{gr\mathbf{R}_{[a]}}(q) \;=\; F_{\mathbf{H}_{[a]}}(q) \;=\; \displaystyle\sum_{i=0}^{n_o} d_i \, q^i \;,$

(iii) $\quad \mathbf{H}_{[a]}$ *is a cone.*

Proof

Note that we need only show part (ii). For once this is established part (i) follows from Proposition 5.2. Moreover, part (ii) combined with the symmetry condition 5.38 places us in a position to use Theorem 4.2 which then yields part (iii) as well.

As we shall see part (ii) is an immediate consequence of Proposition 5.4. Thus we start by showing that our hypotheses yield us also condition 5.25. To this end note that we must have $d_{n_o} > 0$. For otherwise 5.37 forces $\mathcal{B}^{=n_o}$ to be empty. This in turn implies that $\mathbf{R}_{[a]}$ is spanned by $\mathcal{B}^{\leq n_o-1}$. Since the latter consists of polynomials of degree less than n_o, we would then be led to contradict 5.36. However, $d_{n_o} > 0$ and 5.38 give that $d_o > 0$, but since there can only be one constant in a basis, we must necessarily have

$$d_o = d_{n_o} = 1 \ . \tag{5.39}$$

So let ϕ be the unique element of $\mathcal{B}$ of degree n_o, and set

$$\Delta_o \ = \ h(\phi) \ .$$

Let $m = x^p$ be a monomial of degree n_o. Since $\mathcal{B}$ is a basis for $\mathbf{R}_{[a]}$ there must be constants c, c_b yielding

$$m - c\phi \ - \sum_{b \in \mathcal{B}^{\leq n_o-1}} c_b \, b \ \in \ \mathcal{J}_{[a]} \ .$$

This implies that $m - c\Delta_o$ (which is the highest homogeneous component of the left hand side) lies in $gr\ \mathcal{J}_{[a]}$. In other words

$$m \cong c\Delta_o \qquad (\ mod\ \ gr\ \mathcal{J}_{[a]}\ ) \ . \tag{5.40}$$

Since this must hold true for any monomial of degree n_o we must conclude that

$$dim\, \mathcal{H}_{n_o}(gr\, \mathbf{R}_{[a]}) \ = \ 1 \ . \tag{5.41}$$

To be precise, 5.40, in principle only shows that this dimension is at most one. However, if it were to be zero then $maxdegree\ gr\, \mathbf{R}_{[a]}$ would have to be less than n_o contradicting again 5.36, (recall that 1.4.12 gives that $gr\, \mathbf{R}_{[a]}$ and $\mathbf{R}_{[a]}$ have the same $maxdegree$). Thus we can be assured that $\Delta_o \neq 0$ (mod $gr\ \mathcal{J}_{[a]}$). With 5.41 we are in a position to use Proposition 5.4 which combined with 5.37 gives

$$dim\, \mathbf{R}_{[a]}^{\leq k} \ + \ dim\, \mathbf{R}_{[a]}^{\leq n_o-k-1} \ \leq \ d_o + d_1 + \cdots + d_{n_o} \ . \tag{5.42}$$

Note now that since $\mathcal{B}^{\leq i}$ is $(mod\ \mathcal{J}_{[a]})$ an independent set of polynomials of degree at most i, we must have

$$d_o + d_1 + \cdots + d_i \ \leq \ dim\, \mathbf{R}_{[a]}^{\leq i} \ . \tag{5.43}$$

Using this for $i = k$ and $i = n_o - k - 1$ in 5.42 gives

$$d_o + d_1 + \cdots + d_k \ + \ d_o + d_1 + \cdots + d_{n_o-k-1} \ \leq \ d_o + d_1 + \cdots + d_{n_o} \ ,$$

which simplifies to

$$d_o + d_1 + \cdots + d_k \ \leq \ d_{n_o-k} + d_{n_o-k+1} + \cdots + d_{n_o} \ .$$

However, the symmetry condition in 5.38 converts this into

$$d_o + d_1 + \cdots + d_k \;\leq\; d_k + d_{k-1} + \cdots + d_o$$

which forces 5.42 to be an equality. In particular we must have

$$d_o + d_1 + \cdots + d_k \;=\; dim\, \mathbf{R}_{[a]}^{\leq k} \quad . \tag{5.44}$$

But then formula 1.4.20 for $\mathbf{A} = \mathbf{R}_{[a]}$ gives that

$$d_o + d_1 + \cdots + d_k \;=\; \sum_{i=0}^{k} dim\, \mathcal{H}_i(gr\, \mathbf{R}_{[a]}) \quad .$$

Thus part (ii) of the theorem must hold true as desired and our proof is complete.

Remark 5.1

An interesting by-product of this proof is that under the assumptions 5.25 and 5.26 of Proposition 5.3, the condition

$$\theta_\epsilon\, \mathbf{R}^{\leq k} \;=\; \left(\mathbf{R}^{\leq n_o - k - 1} \right)^{\perp} \qquad (\, for\;\; k = 1, 2, \ldots, n_o - 1\,) \tag{5.45}$$

is necessary and sufficient for $\mathbf{H}_{[a]}$ to be a cone. In fact if we set $d_i = dim\, \mathcal{H}_i(\mathbf{H}_{[a]})$ then 1.4.20 gives

$$d_o + d_1 + \cdots + d_i \;=\; dim\, \mathbf{R}_{[a]}^{\leq i} \quad ,$$

and 5.45 is then equivalent to

$$\begin{aligned} d_o + d_1 + \cdots + d_k \;&=\; d_o + d_1 + \cdots + d_{n_o} \;-\; (d_o + d_1 + \cdots + d_{n_o - k - 1}) \\ &=\; d_{n_o - k} + d_{n_o - k + 1} + \cdots + d_{n_o} \end{aligned} \quad . \qquad (\, for\;\; k = 1, 2, \ldots, n_o - 1\,)$$

which is easily seen to hold if and only if $d_k = d_{n_o - k}$ for all k.

Remark 5.2

Note that when $\mathbf{H}_{[a]}$ is a cone, the standard monomial basis $\mathcal{B}_{\mathcal{J}_{[a]}}$ constructed in section 1.4 does satisfy 5.37 and 5.38. Indeed, 5.37 with $d_i = dim\, \mathcal{H}_i(gr\, \mathbf{R}_{[a]})$ must hold true because (Theorem 1.4.4) $\mathcal{B}_{\mathcal{J}_{[a]}}$ is also a basis for $gr\, \mathbf{R}_{[a]}$ and then 5.38 just expresses the symmetry of the Hilbert series of $gr\, \mathbf{R}_{[a]}$, which (by Theorem 4.2) is characteristic of conical orbits. Thus the mere existence of such a basis is not a further restriction. Nevertheless, as we shall see, Theorem 5.2 does lead to a practical mechanism for establishing orbit conicity. This is due to the fact that it provides a powerful bridge for transfering combinatorial properties of an orbit $[a]$ and its corresponding modules $\mathbf{R}_{[a]}$, $\Gamma_{[a]}$ into algebraic properties of the graded modules $\mathbf{H}_{[a]}$ and $gr\, \mathbf{R}_{[a]}$. This transfer is provided by the following heuristic process for constructing natural bases for our coordinate rings $\mathbf{R}_{[a]}$. This process consists in choosing a *natural* total order $\leq_n$ for the given orbit $[a]$ and then proceed to construct the desired basis as a collection of polynomials $\mathcal{B} = \{m_b(x)\}_{b \in [a]}$ statisfying the conditions

$$m_b(x) \;=\; \begin{cases} 1 & \text{if } x = b \quad \text{and} \\ 0 & \text{if } x = b' \text{ with } b' \in [a] \text{ and } b' <_n b \quad , \end{cases} \tag{5.46}$$

where, in the construction of $m_b(x)$ we try to assure that its total degree is as as small as possible. We shall see that this process may be carried out quite successfully even when $[a]$ is not regular.

To get across what we have in mind, it is best at this point to illustrate the process on a specific example. To this end, we shall show how, in this manner, we may use Theorem 5.2 to derive by yet another path, part (ii) of Theorem 5.1. We shall take $a = (1, 2, \ldots, n)$ and $G = S_n$ with the customary permutation action. Clearly, the elements of $[a]$ can be identified with permutations $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. It is then natural in this case to order the elements of $[a]$ *lexicographically*. Let us denote this order by $\leq_L$. Our goal is then to construct, for each $\sigma \in S_n$ a polynomial $m_\sigma(x)$ such that

$$m_\sigma(x) \;=\; \begin{cases} 1 & \text{if } x = \sigma \quad \text{and} \\ 0 & \text{if } x = \sigma' \text{ with } \sigma' \in S_n \text{ and } \sigma' <_L \sigma \quad . \end{cases} \tag{5.47}$$

We illustrate this construction in a special case. We take $n = 8$ and

$$\sigma \;=\; (4, 3, 1, 8, 2, 6, 7, 5) \quad .$$

We shall put together $m_\sigma(x)$ as a product of linear factors $x - c$ each of which is to vanish on a portion of the permutations lexicographically preceeding σ. To get across the basic idea we shall use a terminology which helps the visualization of the construction process. Note that the *natural* position of 1 in a permutation is as far to the left as possible. Thus, to find it in the third position as it is in the permutation σ given above we must *kick* it there. This is achieved by the factor $(x_1 - 1)(x_2 - 1)$. By this we mean that any polynomial $m(x)$ containing this factor can fail to vanish only on permutations in which 1 is at least in the third position. This given, the *natural* position of 2, is now the first. To kick it in the fifth position we may be tempted to use the factor $(x_1 - 2)(x_2 - 2)(x_3 - 2)(x_4 - 2)$. However, the term $(x_3 - 2)$ is superfluous. Crudely speaking, 1 has lexicographically first priority in occupying position 3, when it cannot be in any earlier position. Thus under these circumstances we do not need to kick 2 out of position 3 since 1 already achieves that end. Thus to save on total degree, the factor $(x_3 - 2)$ is not included. Similarly, to assure that 3 does not fall in the first position we kick it out of there by means of the factor $x_1 - 3$. Now 4 has first priority in position 1 and it is precisely there in σ, thus we need no extra factor for 4. As for 5, it only needs to be kicked out of positions $4, 6, 7$ since it can't occupy positions $1, 2, 3, 5$ because $1, 2, 3, 4$ have lexicographic priority for these positions. So to force 5 in the last position we throw in the factor $(x_4 - 5)(x_6 - 5)(x_7 - 5)$. Similarly, the positions of 6 and 7 are assured by the factors $(x_4 - 6)$ and $(x_4 - 7)$, since both 6 and 7 have priority over 8 to occupy position 4. Finally, no factor is needed for 8 since it will have to occupy the only remaining available position. In summary, this process leads to the choice

$$m_{43182675}(x) \;=\; \frac{(x_1-1)(x_2-1)(x_1-2)(x_2-2)(x_4-2)(x_1-3)(x_4-5)(x_6-5)(x_7-5)(x_4-6)(x_4-7)}{(4-1)(3-1)(4-2)(3-2)(8-2)(4-3)(8-5)(6-5)(7-5)(8-6)(8-7)} \quad .$$

It should not be difficult to see that the point $(4, 3, 1, 8, 2, 6, 7, 5)$ is lexicographically the first element of the orbit of $(1, 2, 3, 4, 5, 6, 7, 8)$ where this polynomial does not vanish. The choice of denominator is to assure that its value at this point is equal to 1.

The construction of $m_\sigma(x)$ in the general case should be quite clear now. To assure that σ_i falls in the i^{th} position we throw in the factor

$$\prod_{\{j<i\ ,\ \sigma_j>\sigma_i\}} (x_j - \sigma_i)\ ,$$

and the resulting choice should then be

$$m_\sigma(x) \ = \ \frac{\prod_{i=1}^{n-1} \prod_{\{j<i\ ,\ \sigma_j>\sigma_i\}} (x_j - \sigma_i)}{\prod_{i=1}^{n-1} \prod_{\{j<i\ ,\ \sigma_j>\sigma_i\}} (\sigma_j - \sigma_i)}\ . \tag{5.48}$$

The collection $\mathcal{B} = \{m_\sigma(x)\}_{\sigma\in S_n}$ will then satisfy 5.47 and therefore it will be a basis for $\mathbf{R}_{[1,2,..,n]}$. Moreover, we clearly see that the degree of $m_\sigma(x)$ is none other than the number of inversions of σ, thus

$$\sum_{\sigma\in S_n} q^{degree\ m_\sigma} \ = \ \sum_{\sigma\in S_n} q^{inv(\sigma)}\ .$$

Since reversing a permutation complements the number of inversions, it is also clear that if we set $d_i = dim(\mathcal{B}^{=i})$ then the condition 5.38 will also be satisfied. Thus Theorem 5.2 applies and we derive that the highest homogeneous components of the polynomials $m_\sigma(x)$ are a basis for $gr\,\mathbf{R}_{[1,2,..,n]}$. However, it is easy to see from 5.48 that the collection $h(\mathcal{B})$ in this case is none other than the basis $\mathcal{A}$ given by 5.3. Moreover, we also get that $\mathbf{H}_{[1,2,..,n]}$ is a cone with summit given by the polynomial Δ_o of degree $n_o = \binom{n}{2}$ which obeys 5.27, with ϵ the unique non trivial one dimensional character of S_n. Up to a constant, this forces Δ_o to be the Vandermonde determinant in $x_1, x_2, \ldots, x_n$. Property (i), which is assured by Theorem 3.3, then gives that $\mathbf{H}_{[1,2,..,n]} = \mathbf{H}_{S_n}$ or which is the same that $\mathcal{J}_{[1,2,..,n]} = \mathcal{J}_{S_n}$. Thus $\mathcal{A}$ must be a basis for $\mathbf{R}/\mathcal{J}_{S_n}$. This shows that the above construction does indeed lead to an alternate proof of part (ii) as asserted. Curiously, this argument yields a combinatorial byproduct namely that

$$\sum_{\sigma\in S_n} q^{inv(\sigma)} \ = \ \prod_{k=2}^{n} (1 + q + \cdots + q^{k-1})\ .$$

In the next chapter we shall see that there is a rich variety of results that can be obtained by the same process.

6. Characteristic properties of reflection groups

Some of the properties of S_n stated in theorem 5.1 are actually characteristic of finite reflection groups. We shall establish them in this section in full generality as further applications of the background we have acquired. We shall do so by following the developments in a classical paper of R. Steinberg [ ] and some UCSD lecture notes of I. G. Macdonald.

Our basic goal is to show that

Theorem 6.1

If G is a finite group of $n \times n$ matrices then any of the statements below implies the remaining ones.

(A) G *is generated by* $\mathcal{R}(G)$,

(B) $\mathbf{H}_G$ *is a cone,*

(C) $\mathbf{H}_{[a]} = \mathbf{H}_G$ *for all regular points,*

(D) $\dim \mathbf{H}_G = |G|$,

(E) If $\mathcal{B}$ *is a homogeneous basis for* $\mathbf{H}_G$ *then each* $P \in \mathbf{R}$ *has a unique expansion of the form*

$$P \;=\; \sum_{b \in \mathcal{B}} b\, A_b, \qquad\qquad A_b \in \mathbf{R}^G,$$

(F) If $F_1, F_2, \ldots, F_k \in \mathcal{J}_G$ *are homogeneous and* $p_1F_1 + p_2F_2 + \cdots + p_kF_k \equiv 0$ *then* $F_1 \notin (F_2, \ldots, F_k)$ *implies that* $p_1 \in \mathcal{J}_G$,

(G) If $I_1, I_2, \ldots, I_s$ *are a minimal set of homogeneous generators for* $\mathbf{R}^G$, *then*

(i) $I_1, I_2, \ldots, I_s$ *are algebraically independent, and*

(ii) $s = n$.

The proof of this theorem will be carried out in stages. In a sequence of propositions we shall show that each statement implies the immediately following one, then complete the picture by proving that the last statement implies the first. For simplicity we shall carry out all our arguments under the assumption that G is a group of real orthogonal matrices. The general case can be treated essentially in the same manner by suitably enlarging the ground field.

Proposition 6.1

$$(A) \Longrightarrow (B)$$

Proof.

This is simply part (iii) of theorem 3.3, which states that $\mathbf{H}_G$ is a cone with summit given by the discriminant Δ_G.

Proposition 6.2

$$(B) \Longrightarrow (C)$$

Proof.

Of course we have shown (part (i) of theorem 3.3) that $(A) \Rightarrow (C)$. However, since one of our goals is to show that $(C) \Rightarrow (A)$ we cannot use (A) in the proof of this proposition.

By proposition 4.2, we can assume that $\mathbf{H}_G$ has a homogeneous summit Δ. Let $n_0 =$ degree Δ. Then the assumption that

$$\mathbf{H}_G \;=\; \mathcal{L}[\partial_x^p \Delta] \tag{6.1}$$

immediately yields that

$$\dim \mathcal{H}_{n_0}(\mathbf{H}_G) \;=\; 1 \;. \tag{6.2}$$

Since each of the homogeneous components of $\mathbf{H}_G$ is G-invariant, 6.2 implies that the polynomial $T_A\Delta$ must be a constant multiple of Δ for all $A \in G$. Setting

$$T_A\, \Delta \;=\; \epsilon(A)\Delta$$

we can easily see that the map $A \rightarrow \epsilon(A)$ is a non-trivial, representation of G. For, we must have $\epsilon(AB) = \epsilon(A)\,\epsilon(B)$, and if $T_A\Delta = \Delta$ ($\forall\, A \in G$) then $\Delta \in \mathcal{J}_G \cap \mathcal{J}_G^{\perp}$ would imply that $\Delta = 0$ which is absurd.

Now suppose that a is a regular point. Since $\mathbf{H}_{[a]}$ is the left regular representation, $\mathbf{H}_{[a]}$ must contain the (irreducible) representation ϵ with multiplicity one. This implies that there must be a non-trivial polynomial $\Delta_1 \in \mathbf{H}_{[a]}$, which satisfies

$$T_A\Delta_1 \;=\; \epsilon(A)\Delta_1 \qquad\qquad (\forall A \in G) \;. \tag{6.3}$$

Clearly Δ_1 must be homogeneous since otherwise its homogeneous components would yield further copies of ϵ in $\mathbf{H}_{[a]}$. Moreover, $\Delta_1 \in \mathbf{H}_G$ since $\mathbf{H}_{[a]} \subseteq \mathbf{H}_G$. This gives

$$\Delta_1 \;=\; B(\partial_x)\,\Delta \tag{6.5}$$

for a suitable homogeneous polynomial B. Now (again 1.1.9) gives

$$\epsilon(A)\Delta \;=\; T_A\Delta_1 \;=\; (T_AB)(\partial_x)\,T_A\Delta \;=\; \epsilon(A)(T_AB)(\partial_x)\Delta$$

and thus

$$\Delta_1 \;=\; (T_AB)(\partial_x)\Delta$$

Summing over all $A \in G$ and dividing by $|G|$ yields

$$\Delta_1 \;=\; (\rho_G B)(\partial_x)\Delta$$

Since $(\rho_G B) \in \mathbf{R}^G$, if B were of positive degree then $\rho_G B$ would kill Δ, yielding $\Delta_1 = 0$. Since the latter is excluded by assumption, we must conclude that B must be a constant and then 6.5 yields that

$$\Delta \;\in\; \mathbf{H}_{[a]}.$$

Since $\mathbf{H}_{[a]}$ is closed under differentiation, (using 2.3 of theorem 2.1) we get

$$\mathcal{L}[\partial_x^P \Delta] \;\subseteq\; \mathbf{H}_{[a]} \;\subseteq\; \mathbf{H}_G \;=\; \mathcal{L}[\partial_x^P \Delta].$$

Thus all these inequalities must be equalities. In particular (C) must hold true as desired.

Proposition 6.3

$$(C) \Longrightarrow (D)$$

Proof.

We have proved (theorem 1.1) that if a is regular then $\mathbf{H}_{[a]}$ is the left regular representation and thus in particular we will have $\dim \mathbf{H}_{[a]} = |G|$. So, equality in (C) implies equality in (D).

Q. E. D

Proposition 6.4

$$(D) \Longrightarrow (E)$$

Proof.

This is simply 2.17 of theorem 2.4.

Proposition 6.5 (Steinberg [ ])

$$(E) \Longrightarrow (F)$$

Proof.

Let $F_1, F_2, \ldots, F_k \in \mathbf{R}^G$ be homogeneous and assume that

$$p_1 F_1 + p_2 F_2 + \cdots + p_k F_k \;=\; 0. \tag{6.6}$$

There is no loss in assuming that $p_1, p_2, \ldots, p_k$ are also homogeneous. Let then $\mathcal{B}$ be a homogeneous basis for $\mathbf{H}_G$, by (E) we can write

$$p_i(x) \;=\; \sum_{b \in \mathcal{B}} b(x) A_{b,i}(x), \qquad\qquad A_{b,i} \in \mathbf{R}^G. \tag{6.7}$$

Substituting in 6.6 gives

$$\sum_{b \in \mathcal{B}} b(x) \sum_{i=1}^{k} A_{b,i}(x)\, F_i(x) \;=\; 0$$

and uniqueness of expansions (guaranteed by (E)) yields

$$\sum_{i=1}^{k} A_{b,i}(x)\, F_i(x) \;=\; 0, \qquad\qquad (\forall b \in \mathcal{B}) \;.$$

In other words we must have

$$F_1(x)\, A_{b,1}(x) \;=\; -\sum_{i=2}^{k} F_i(x)\, A_{b,i}(x) \tag{6.8}$$

Note that since $\mathcal{B}$ is a homogeneous basis, the homogeneity of $p_1, p_2, \ldots, p_k$ yields that the coefficients $A_{b,i}$ in 6.7 must all be homogeneous as well. Note now that if for some $b_0 \in \mathcal{B}$ the coefficient $A_{b_o,1}$ happens to be a constant, then equation 6.8 for $b = b_0$ yields that

$$F_1 \ \in \ (F_2, \ldots, F_k) \ .$$

Thus, if the latter is excluded, all the coefficients $A_{b,1}$ must be homogeneous polynomials of positive degree. However, then equation 6.7 yields that $p_1 \in \mathcal{J}_G$ as desired.

Q.E.D.

Proposition 6.6 (Chevalley [ ])

$$(F) \Longrightarrow (G)$$

Proof.

Let $I_1, I_2, \ldots, I_s$ be homogeneous generators for $\mathbf{R}^G$ set

$$\text{degree}(I_s) = d_s \tag{6.9}$$

and suppose that s is minimal. Let there be, if possible, a polynomial $P(y_1, y_2, \ldots, y_s)$ such that

$$P(I_1, I_2, \ldots, I_s) \ = \ \sum_p c_{p_1 p_2 \cdots p_s} \, I_1^{p_1} I_2^{p_2} \cdots I_s^{p_s} \ = \ 0 \ . \tag{6.10}$$

Note that this holds true if and only if each of the separate summands

$$\sum_{p_1 d_1 + p_2 d_2 + \cdots + p_s d_s = M} c_{p_1 p_2 \cdots p_s} \, I_1^{p_1} I_2^{p_2} \cdots I_s^{p_s} \qquad (\ M = 1, 2, ..etc\ )$$

vanishes all by itself. So there is no loss in assuming this holds with P, a polynomial of the form

$$P(y_1, y_2, \ldots, y_s) \ = \ \sum_{p_1 d_1 + p_2 d_2 + \cdots + p_s d_s = M} c_{p_1 p_2 \cdots p_s} \, y_1^{p_1} y_2^{p_2} \cdots y_s^{p_s} \ .$$

Now let

$$P_i(y_1, y_2, \ldots, y_s) \ = \ \frac{\partial}{\partial y_i} P(y_1, y_2, \ldots, y_s)$$

and set

$$F_i(x) \ = \ P_i\big(I_1(x), I_2(x), \ldots, I_s(x)\big) \ .$$

Clearly, $F_i(x)$ is homogeneous as a polynomial in $x_1, \ldots, x_n$ and

$$\text{degree}\, F_i(x) \ = \ M - d_i \ . \tag{6.11}$$

Differentiating 6.10 with respect to x_k gives

$$F_1 \frac{\partial I_1}{\partial x_k} + F_2 \frac{\partial I_2}{\partial x_k} + \cdots + F_s \frac{\partial I_s}{\partial x_k} \ = \ 0, \qquad (\forall k = 1, 2, \ldots, n) \ . \tag{6.12}$$

It could very well be the case that a proper subset of $F_1, F_2, \ldots, F_s$ already generates the whole ideal $(F_1, F_2, \ldots, F_s)$. By relabeling the F's we can assume that (for some $m \leq s$)

$$(F_1, F_2, \ldots, F_m) \;=\; (F_1, F_2, \ldots, F_s) \tag{6.13}$$

and that no smaller subset of the F's generates $(F_1, F_2, \ldots, F_s)$. This given, we can write

$$F_j \;=\; \sum_{i=1}^{m} F_i\, A_{ij} \qquad\qquad (\, j = m+1, \ldots, s\,) \tag{6.14}$$

and note that from the homogeneity of the $F_i(x)$'s and 6.11 we deduce that there is no loss in assuming that each A_{ij} is homogeneous and

$$\begin{cases} (1)\; A_{ij} = 0 & if\; d_i < d_j \;\;, \\ (2)\; \text{degree} A_{ij} = d_i - d_j & if\; d_i \geq d_j \;\;. \end{cases} \tag{6.15}$$

Substituting 6.14 in 6.12 and grouping terms gives

$$\sum_{i=1}^{m} F_i \,(\frac{\partial I_i}{\partial x_k} + \sum_{j=m+1}^{s} \frac{\partial I_j}{\partial x_k} A_{ij}) \;=\; 0. \tag{6.16}$$

Since our choice of $F_1, F_2, \ldots, F_m$ excludes that

$$F_1 \;\in\; (F_2, \ldots F_m)$$

property (F)and 6.16 yields us that the coefficient

$$p_{1k} \;=\; \frac{\partial I_1}{\partial x_k} + \sum_{j=m+1}^{s} \frac{\partial I_j}{\partial x_k} A_{1j} \tag{6.17}$$

must lie in $\mathcal{J}_G$. Since $I_1, I_2, \ldots, I_s$ generate $\mathcal{J}_G$ as well we deduce that there must be homogeneous polynomials $B_1, B_2, \ldots, B_s$ giving

$$p_{1k} \;=\; \sum_{i=1}^{s} B_i(x)\, I_i(x).$$

However, note that the term in I_1 cannot occur in this expansion since 6.17 and 6.15 give that

$$\text{degree}\, p_{1k} \;=\; d_1 - 1.$$

So we must conclude that

$$p_{1k} \;\in\; (I_2, \ldots, I_s), \qquad\qquad (\forall k = 1, 2, \ldots, n). \tag{6.18}$$

Multiplying by x_k, summing and using 6.17 we finally get

$$\sum_{k=1}^{n} x_k \,\frac{\partial I_1}{\partial x_k} + \sum_{j=m+1}^{s} A_{1j} \sum_{k=1}^{n} x_k \,\frac{\partial I_j}{\partial x_k} \;\in\; (I_2, \ldots, I_s). \tag{6.19}$$

Since each I_j is homogeneous, Euler's theorem gives that

$$\sum_{k=1}^{n} x_k \frac{\partial I_j}{\partial x_k} \;=\; d_j\, I_j.$$

So 6.19 may be rewritten as

$$d_1\, I_1 + \sum_{j=m+1}^{s} A_{1j}\, d_j\, I_j \;\in\; (I_2, \ldots, I_s).$$

But this yields that

$$I_1 \;\in\; (I_2, \ldots, I_s)$$

which combined with proposition 2.2 would yield that $I_2, \ldots, I_s$ by themselves generate $\mathbf{R}^G$, contradicting the minimality of s. Since the only unwarranted assumption in our argument is equation 6.10, we must conclude that $I_1, I_2, \ldots, I_s$ must be algebraically independent.

What we have shown so far is more than is needed to complete the proof of the proposition. This is due to the following result which is worth a separate statement.

Lemma 6.1

Let G be a finite group of $n \times n$ matrices and let $I_1, I_2, \ldots, I_s$ be a set of algebraically independent homogeneous generators of $\mathbf{R}^G$. Then setting $d_k = \text{degree}\, I_k$ $(k = 1, 2, \ldots, s)$ we must have

(i) $s = n$,
(ii) $|G| = d_1 d_2 \cdots d_n$,
(iii) $|\mathcal{R}(G)| = d_1 - 1 + d_2 - 1 + \cdots + d_n - 1$

Proof.

The algebraic independence of $I_1, I_2, \ldots, I_s$ yields us that

$$F_{\mathbf{R}^G}(t) \;=\; \frac{1}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_s})}. \tag{6.20}$$

On the other hand, Molien's theorem (eq. 2.13) yields that

$$F_{\mathbf{R}^G}(t) \;=\; \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det |1 - tA|}. \tag{6.21}$$

Note that from 6.20 we see that $t = 1$ is a pole of order s for $F_{\mathbf{R}^G}(t)$. On the other hand, Moliens' theorem yields that $F_{\mathbf{R}^G}(t)$ must have a pole of order n at $t = 1$. In fact, the highest power of $1/(1-t)$ in the right hand side of 6.21 is given by the term corresponding to $A = \textit{identity}$, (since the latter matrix is the only element of G that has all its eigenvalues equal to 1). Thus $s = n$. This given, 6.20 gives that

$$\lim_{t \to 1} (1-t)^n\, F_{\mathbf{R}^G}(t) \;=\; \frac{1}{d_1 d_2 \cdots d_n}\;. \tag{6.22}$$

Doing the same with 6.21, yields

$$\lim_{t\to 1}(1-t)^n\, F_{\mathbf{R}^G}(t) \;=\; \frac{1}{|G|} \tag{6.23}$$

Thus (ii) must hold true as asserted.

To derive (iii) we need a slightly more refined argument. Note first that, using what we have shown so far, we can combine 6.20 and 6.21 into the identity

$$\frac{d_1 d_2\cdots d_n}{(1-t^{d_1})\cdots(1-t^{d_n})} \;=\; \frac{1}{(1-t)^n}+\frac{M}{(1+t)(1-t)^{n-1}}+O\left(\frac{1}{(1-t)^{n-2}}\right) \tag{6.24}$$

where for convenience

$$M=|\mathcal{R}(G)|$$

denotes the number of reflections of G. The special form of the right hand side of 6.24 is due to the fact that, apart from the identity term in 6.21 which contributes a pole of order n at $t=1$, only the terms corresponding to reflections can contribute a pole of order $n-1$ at $t=1$, (this is because any orthogonal matrix with $n-1$ eigenvalues equal to 1 is necessarily the matrix of a reflection).

Multiplying both sides of 6.24 by $(1-t)^n$ gives then

$$\prod_{i=1}^{n}\frac{d_i}{1+t+\cdots+t^{d_i-1}} \;=\; 1+M\frac{(1-t)}{(1+t)}+O((1-t)^2). \tag{6.25}$$

Now setting $t=1-\epsilon$ gives

$$\begin{aligned} 1+t+\cdots+t^{d_i-1} &= 1+1-\epsilon+1-2\epsilon+\cdots+1-(d_i-1)\epsilon+0(\epsilon^2)\\ &= d_i-\frac{d_i(d_i-1)}{2}\epsilon+O(\epsilon^2)\\ &= d_i\big(1-\frac{(d_i-1)}{2}\epsilon\big)+O(\epsilon^2)\ . \end{aligned}$$

Now clearly

$$\frac{1-t}{1+t} \;=\; \frac{\epsilon}{2-\epsilon} \;=\; \frac{\epsilon}{2}+O(\epsilon^2)\ .$$

Using these two relations in 6.25 gives

$$\prod_{i=1}^{n}\frac{1}{1-\frac{d_i-1}{2}\epsilon+0(\epsilon^2)} \;=\; 1+\frac{M}{2}\epsilon+O(\epsilon^2)\ .$$

Expanding each of the factors on the left hand side reduces this to

$$1+\left(\sum_{i=1}^{n}\frac{d_i-1}{2}\right)\epsilon+0(\epsilon^2) \;=\; 1+\frac{M}{2}\epsilon+O(\epsilon^2)\ ,$$

which yields (iii) by equating coefficients of ϵ.

To complete the proof of theorem 6.1 we need to establish two auxiliary results. It will be implicitly assumed here and in the rest of this section that G is a finite group of $n \times n$ real orthogonal matrices.

Lemma 6.2

If $\Theta_1, \Theta_2, \ldots, \Theta_n$ are algebraically independent and generate $\mathbf{R}^G$ then

$$\det \left\| \frac{\partial \Theta_i}{\partial x_j} \right\| \neq 0 . \tag{6.26}$$

Proof

The result can be established without the assumption that $\Theta_1, \ldots, \Theta_n$ generate G. However, for our purposes here we can throw in this additional condition and make our proof entirely elementary. The idea is to make use of the identity 2.8, established in the proof of proposition 2.1, to the effect that there are G-invariant polynomials F_{ks} giving

$$x_k^{|G|} = \sum_{s=0}^{|G|-1} x_k^s F_{ks} .$$

Letting $x_k^{p_k}$ be the least power of x_k that can be so expressed, and using our assumption we can write

$$x_k^{p_k} = \sum_{s=0}^{p_k-1} x_k^s G_{ks}(\Theta_1, \Theta_2, \ldots, \Theta_k) \tag{6.27}$$

with $G_{ks}(\Theta_1, \ldots, \Theta_n)$ polynomials in their arguments. Differentiating 6.27 with respect to x_j, after simple manipulation we get the identity.

$$\left[p_k x_k^{p_k-1} - \sum_{s=1}^{p_k-1} s\, x_k^{s-1} G_{ks}(\Theta) \right] \chi(k=j) = \sum_{i=1}^{n} B_{ki}(\Theta_1, \ldots, \Theta_n) \frac{\partial \Theta_i}{\partial x_j} \tag{6.28}$$

where, for convienence we have

$$B_{ki}(\Theta_1, \ldots, \Theta_n) = \frac{\partial}{\partial_{y_i}} \left(\sum_{s=0}^{p_k-1} x_k^s G_{ks}(y_1, y_2, \ldots, y_n) \right) \Bigg|_{y_j = \Theta_j}$$

Clearly, the polynomial

$$p_k\, x_k^{p_k-1} - \sum_{s=1}^{p_k-1} s\, x_k^{s-1} G_{ks}(\Theta)$$

cannot vanish since that would contradict the minimality of p_k. We can thus deduce from 6.27 that the product of the two matrices $\|B_{ki}(\Theta)\|$ and $\|\frac{\partial \Theta_i}{\partial x_j}\|$ is a non-singular diagonal matrix. Thus both matrices must have a non-vanishing determinant. This gives 6.26.

We can now combine the last two lemmas and obtain the following remarkable result

Theorem 6.2

If $\Theta_1, \Theta_2, \ldots, \Theta_n$ are algebraically independent homogeneous generators of $\mathbf{R}^G$ then we must have

$$\det\left\|\frac{\partial \Theta_i}{\partial x_j}\right\| \;=\; c\,\Delta_G(x) \;=\; c\prod_{\alpha\in\mathcal{R}(G)}(x,\alpha), \tag{6.29}$$

for some constant $c \neq 0$.

Proof

Note that if the differentials $dx_1, dx_2, \ldots, dx_n$ are "wedge" multiplied by the rule

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

then it immediately follows that for any polynomials $P_1, P_2, \ldots, P_n$

$$dP_1 \wedge dP_2 \wedge \cdots \wedge dP_n \;=\; \det\left\|\frac{\partial P_i}{\partial x_j}\right\|\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \;. \tag{6.30}$$

Let us now have G act on polynomials in $dx_1, dx_2, \ldots, dx_n$ by setting (as was done for $x_1, x_2, \ldots, x_n$)

$$T_A dx_j \;=\; \sum_{i=1}^{n} dx_i\, A_{ij}, \qquad \text{(for } A\in G \text{ and } j = 1,2,\ldots,n) \tag{6.31}$$

and extending this action multiplicatively. This given we can easily verify that for any polynomial $P(x_1,\ldots,x_n)$ we do have

$$T_A\, dP \;=\; d(T_A P)\,.$$

In particular, if $P \in \mathbf{R}^G$ then

$$T_A\, dP \;=\; dP, \qquad \forall A \in G\,.$$

This gives

$$T_A\,(d\Theta_1 \wedge d\Theta_2 \wedge \cdots, \wedge d\Theta_n) \;=\; d\Theta_1 \wedge d\Theta_2 \wedge \cdots \wedge d\Theta_n\;,$$

and using 6.30 with $P_i = \Theta_i$ we get

$$\left(T_A\,\det\left\|\frac{\partial \Theta_i}{\partial x_j}\right\|\right) T_A(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \;=\; \det\left\|\frac{\partial \Theta_i}{\partial x_j}\right\|\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\,. \tag{6.32}$$

However, 6.31 and 6.30 (with $P_i = T_A x_i$) give

$$T_A(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \;=\; \det A\;\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\,.$$

Substituting this in 6.32 yields

$$T_A\,\det\left\|\frac{\partial \Theta_i}{\partial x_j}\right\| \;=\; \det A\;\det\left\|\frac{\partial \Theta_i}{\partial x_j}\right\|, \qquad \forall A \in G\,.$$

We can thus use part *(b)* of theorem 3.2 and obtain that

$$\det\left\|\frac{\partial\Theta_i}{\partial x_j}\right\| \;=\; A(x)\,\Delta_G(x), \qquad\qquad (\text{with } A\in\mathbf{R}^G)\,. \tag{6.33}$$

However, note that if degree $(\Theta_i)=d_i$, then lemma 6.1 with $I_i=\Theta_i$ gives

$$\text{degree }\Delta_G \;=\; \sum_{i=1}^{n}(d_i-1) \;=\; \text{degree }\det\left\|\frac{\partial\Theta_i}{\partial x_j}\right\|$$

and this forces A in 6.33 to be a constant. Moreover, lemma 6.2 guarantees that this constant cannot vanish.

We are finally in a position to establish the last link in the proof of theorem 6.1.

Proposition 6.7 (*Shephard-Todd*)

$$(G)\Longrightarrow(A)$$

Proof

Assuming (G), lemma 6.1 yields that we must have n algebraically independent homogeneous generators $I_1,I_2,\dots,I_n$ for $\mathbf{R}^G$ such that if $d_i=\text{degree }(I_i)$ then

$$\begin{aligned}&(i)\quad d_1d_2\cdots d_n=|G|\\ &(ii)\quad d_1-1+d_2-1+\cdots+d_n-1=|\mathcal{R}(G)|\,.\end{aligned} \tag{6.34}$$

Let for a moment G^* denote the group generated by $\mathcal{R}(G)$. Clearly (ii) of 6.34 guarantees that G^* is not a trivial subgroup of G. We aim to show that $G^*=G$. To this end, note that since G^* is by definition generated by its reflections, then the implications

$$(A)\Rightarrow(B)\Rightarrow(C)\Rightarrow(D)\Rightarrow(E)\Rightarrow(F)\Rightarrow(G)\;\;,$$

(which we have already established) yield us that G^* itself satisfies (G). In particular we must have algebraically independent homogeneous generators $I_1^*,I_2^*,\dots,I_n^*$ for $\mathbf{R}^{G^*}$ such that if we set degree $(I_i^*)=d_i^*$ then

$$\begin{aligned}&(i)\quad d_1^*d_2^*\cdots d_n^*=|G^*|\\ &(ii)\quad d_1^*-1+d_2^*-1+\cdots+d_n^*-1=|\mathcal{R}(G^*)|\,.\end{aligned} \tag{6.35}$$

Note now that since G^* is a subgroup of G, the polynomials $I_1,\dots,I_n$ are also G^*-invariant. This implies that we must have expressions of the form

$$I_i \;=\; P_i(I_1^*,I_2^*,\dots,I_n^*) \tag{6.36}$$

with

$$P_i(y_1,y_2,\dots,y_n) \;=\; \sum_{p_1d_1^*+\cdots+p_nd_n^*=d_i} c^{(i)}_{p_1\cdots p_n}\,y_1^{p_1}\cdots y_n^{p_n}\;. \tag{6.37}$$

Differentiating 6.36 with respect to x_j gives

$$\frac{\partial I_i}{\partial x_j} \;=\; \sum_{k=1}^{n} \frac{\partial P_j}{\partial y_k}(I_1^*, I_2^*, \dots, I_n^*)\, \frac{\partial I_k^*}{\partial x_j}\,.$$

Thus, the non-vanishing of $\det \|\frac{\partial \Theta_i}{\partial x_j}\|$ (guaranteed by lemma 6.2) yields that

$$\det \left\| \frac{\partial P_i}{\partial y_k}(I_1^*, \dots, I_n^*) \right\| \;\neq\; 0\,.$$

Now this in turn implies that for at least one permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ we must have

$$\frac{\partial P_1}{\partial y_{\sigma_1}} \frac{\partial P_2}{\partial y_{\sigma_2}} \cdots \frac{\partial P_n}{\partial y_{\sigma_n}}\bigg|_{y_i = I_i^*(x)} \;\neq\; 0\,. \tag{6.38}$$

There is no loss in assuming that σ is the identity permutation since this can be achieved by a relabelling of $I_1^*, I_2^*, \dots, I_n^*$. But for $\sigma_i = i$, 6.38 gives

$$\frac{\partial P_i}{\partial y_i} \;\neq\; 0\,.$$

This implies that for at least one coefficient $c^{(i)}_{p_1 \cdots p_n}$ (in 6.37) we must have $p_i \geq 1$. But that can only happen if

$$d_i \geq d_i^*\,. \tag{6.38}$$

Combining all these inequalities with 6.34(ii) and 6.35(ii) we deduce that

$$|\mathcal{R}(G)| \;=\; d_1 - 1 + \cdots + d_n - 1 \;\geq\; d_1^* - 1 + \cdots d_n^* - 1 \;=\; \mathcal{R}(G^*)\,.$$

Since trivally $\mathcal{R}(G^*) = \mathcal{R}(G)$, we see that we must necessarily have $d_i = d_i^*$ (for $i = 1, 2, \dots, n$). But then, 6.34(i) and 6.35(i) give that

$$|G^*| \;=\; d_1^* \cdots d_n^* \;=\; d_1 \cdots d_n \;=\; |G|\,,$$

which forces $G^* = G$ as desired.