The New k-Pieri coefficients of Francois and Mark

Introduction

François and Mark set define the coefficients $c_{\mu,\nu}^{(k)}$ by setting

$$h_k^{\perp}\tilde{H}_\mu[X;q,t] \;=\; \sum_{\nu\subseteq_k\mu} c_{\mu,\nu}^{(k)}\tilde{H}_\nu[X;q,t] \qquad \text{I.1}$$

where "$\nu \subseteq_k \mu$" means that the partition ν is contained in μ and μ/ν has k cells. Remarkably they conjectured that these coefficients can be recursively obtained from the classical $c_{\mu,\nu}^{(1)}$ coefficients. They derived this recursion from a conjectured symmetric function identity identity which I noticed could be obtained from the Macdonald polynomial toolkit we developed in the 90's. François and Mark also conjectured some truly surprising summation formulas that may actually provide us with new tools in our attempts to prove the Shuffle Conjecture. Prompted by this possibility I explored various avenues for establishing and extending their formulas. In particular, to follow the classical development we need to consider the $c_{\mu,\nu}^{(k)}$ as the "dual Pieri" coefficients with the "Pieri" coefficients $d_{\mu,\nu}^{(k)}$ defined by setting

$$e_k[\tfrac{X}{M}]\tilde{H}_\nu[X;q,t] \;=\; \sum_{\mu\supset_k\nu} d_{\mu,\nu}^{(k)}\tilde{H}_\mu[X;q,t] \qquad \text{I.2}$$

where "$\mu \supset_k \nu$" means that the partition μ is contains in ν and μ/ν has k cells. In particular this reveals that the summation formulas of François and Mark really directly involve the $d_{\mu,\nu}^{(k)}$ rather than the $c_{\mu,\nu}^{(k)}$. Our main results are general summation formulas for the $c_{\mu,\nu}^{(k)}$ and the $d_{\mu,\nu}^{(k)}$ which naturally extend the classical results for the $k=1$ case. These notes cover most of the results I obtained in these efforts. For the benefit of the reader we will precede our presentation with a Macdonald Polynomial toolkit.

1. Auxiliary identities

The space of symmetric polynomials will be denoted Λ. The subspace of homogeneous symmetric polynomials of degree m will be denoted by $\Lambda^{=m}$. We will seldom work with symmetric polynomials expressed in terms of variables but rather express them in terms of one of the six classical symmetric function bases

(1) "*power*" $\{p_\mu\}_\mu$, (2) "*monomial*" $\{m_\mu\}_\mu$, (3) "*homogeneous*" $\{h_\mu\}_\mu$,
(4) "*elementary*" $\{e_\mu\}_\mu$, (5) "*forgotten*" $\{f_\mu\}_\mu$ and (6) "*Schur*" $\{s_\mu\}_\mu$.

We recall that the fundamental involution ω may be defined by setting for the power basis indexed by $\mu=(\mu_1,\mu_2,\ldots,\mu_k)\vdash n$

$$\omega p_\mu \;=\; (-1)^{n-k}p_\mu \;=\; (-1)^{|\mu|-l(\mu)}p_\mu \qquad 1.1$$

where for any vector $v=(v_1,v_2,\cdots,v_k)$ we set $|v|=\sum_{i=1}^k v_i$ and $l(v)=k$.

In dealing with symmetric function identities, specially with those arising in the Theory of Macdonald Polynomials, we find it convenient and often indispensable to use plethystic notation. This device has a straightforward definition which can be verbatim implemented in MAPLE or MATHEMATICA for computer experimentation. We simply set for any expression $E=E(t_1,t_2,\ldots)$ and any power symmetric function p_k

$$p_k[E] \;=\; E(\,t_1^k,t_2^k,\ldots). \qquad 1.2$$

This given, for any symmetric function F we set

$$F[E] \;=\; Q_F(p_1,p_2,\ldots)\Big|_{p_k\to E(\,t_1^k,t_2^k,\ldots)}, \qquad 1.3$$

where Q_F is the polynomial yielding the expansion of F in terms of the power basis. Note that in writing $E(t_1, t_2, \ldots)$ we are tacitly assuming that $t_1, t_2, t_3, \ldots$ are all the variables appearing in E and in writing $E(t_1^k, t_2^k, \ldots)$ we intend that all the variables appearing in E have been raised to their k^{th} power.

A paradoxical but necessary property of plethystic substitutions is that 1.2 requires

$$p_k[-E] \;=\; -p_k[E]. \tag{1.4}$$

This notwithstanding, we will still need to carry out ordinary changes of signs. To distinguish it from the "*plethystic* " minus sign, we will carry out the "*ordinary* " sign change by prepending our expressions with a superscripted minus sign, or as the case may be, by means of a new variables ϵ which outside of the plethystic bracket is simply replaced by -1. For instance, these conventions give for $X_n = x_1 + x_2 + \cdots + x_n$

$$p_k[-^{-}X_n] \;=\; (-1)^{k-1} \sum_{i=1}^{n} x_i^k, \tag{1.5}$$

or, equivalently

$$p_k[-\epsilon X_n] \;=\; -\epsilon^k \sum_{i=1}^{n} x_i^k \;=\; (-1)^{k-1} \sum_{i=1}^{n} x_i^k.$$

In particular we get for $X = x_1 + x_2 + x_3 + \cdots$

$$\omega p_k[X] \;=\; p_k[-^{-}X].$$

Thus for any symmetric function $F \in \Lambda$ and any expression E we have

$$\omega F[E] \;=\; F[-^{-}E] \;=\; F[-\epsilon E]. \tag{1.6}$$

In particular, if $F \in \Lambda^{=k}$ we may also rewrite this as

$$F[-E] \;=\; \omega F[^{-}E] \;=\; (-1)^k \omega F[E]. \tag{1.7}$$

The formal power series

$$\Omega \;=\; exp\Big(\sum_{k\geq 1} \frac{p_k}{k} \Big)$$

combined with plethystic substitutions will provide a powerful way of dealing with the many generating functions occurring in our manipulations.

Our developments crucially depend on the operators D_k and D_k^* defined by setting for every $F \in \Lambda$:

$$\begin{aligned} a) \quad & D_k\, F[X] \;=\; F\big[X + \tfrac{M}{z}\big]\, \Omega[-z\,X]\, \big|_{z^k} \\ b) \quad & D_k^*\, F[X] \;=\; F\big[X - \tfrac{\widetilde{M}}{z}\big]\, \Omega[z\,X]\, \big|_{z^k} \end{aligned} \quad (\dagger) \quad \text{for} \quad -\infty < k < +\infty \;, \tag{1.8}$$

where $M \;=\; (1-t)(1-q)$, $\widetilde{M} \;=\; (1-1/t)(1-1/q)$, and for convenience

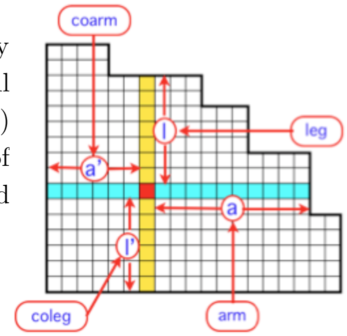
$$\Omega[-zX] \;=\; \sum_{m\geq 0} z^m h_m[-X] \;=\; \sum_{m\geq 0} (-1)^m z^m e_m[X]. \tag{1.9}$$

Here " $\big|_{z^k}$" denotes the operation of taking the coefficient of z^k in the preceding expression,

---

($\dagger$) The symbol " $\big|_{z^k}$" denotes taking the coefficient of z^k in the preceding expression.

To present our Macdonald polynomial kit, it is convenient to identify partitions with their (french) Ferrers diagram. Given a partition μ and a cell $c \in \mu$, Macdonald introduces four parameters $l = l_\mu(c)$, $l' = l'_\mu(c)$, $a = a_\mu(c)$ and $a' = a'_\mu(c)$ called *leg, coleg, arm* and *coarm* which give the number of lattice cells of μ strictly NORTH, SOUTH, EAST and WEST of c, (see attached figure). Following Macdonald we will set



$$n(\mu) \;=\; \sum_{c\in\mu} l_\mu(c) \;=\; \sum_{c\in\mu} l'_\mu(c) \;=\; \sum_{i=1}^{l(\mu)} (i-1)\mu_i .$$

Denoting by μ' the conjugate of μ, the basic ingredients playing a role in the theory of Macdonald polynomials are

$$\begin{aligned} &T_\mu = t^{n(\mu)} q^{n(\mu')} \,, \quad B_\mu(q,t) = \sum_{c\in\mu} t^{l'_\mu(c)} q^{a'_\mu(c)} \,, \quad \Pi_\mu(q,t) = \prod_{c\in\mu; c\neq(0,0)} (1 - t^{l'_\mu(c)} q^{a'_\mu(c)}), \\ &D_\mu(q,t) \;=\; MB_\mu(q,t) - 1 \,, \qquad w_\mu(q,t) = \prod_{c\in\mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}), \end{aligned} \tag{1.10}$$

together with a deformation of the Hall scalar product, which we call the "*star*" scalar product, defined by setting for the power basis

$$\langle p_\lambda \,,\, p_\mu \rangle_* \;=\; (-1)^{|\mu|-l(\mu)} \prod_i (1 - t^{\mu_i})(1 - q^{\mu_i})\, z_\mu \,\chi(\lambda = \mu),$$

where z_μ gives the order of the stabilizer of a permutation with cycle structure μ.

We also recall that we set

$$\nabla \tilde H_\mu(X;q,t) \;=\; T_\mu \tilde H_\mu(X;q,t)$$

The operators in 1.8 are connected to ∇ and the polynomials $\tilde H_\mu$ through the following basic identities:

$$\begin{array}{cc} (i) & D_0\, \tilde H_\mu = -D_\mu(q,t)\, \tilde H_\mu \\ (ii) & D_k\, \underline{e}_1 - \underline{e}_1\, D_k = M\, D_{k+1} \\ (iii) & \nabla\, \underline{e}_1 \nabla^{-1} = -D_1 \\ (iv) & \nabla^{-1} e_1^\perp \nabla = \frac{1}{M} D_{-1} \\ (v) & D_k\, \partial_1 \;-\; \partial_1 D_k \;=\; D_{k-1} \end{array} \tag{1.11}$$

where $\underline{e}_1$ is simply the operator "*multiplication by* e_1", and ∂_1 denotes its "*Hall*" scalar product adjoint.

Recall that for our version of the Macdonald polynomials the Macdonald Reciprocity formula states that

$$\frac{\tilde H_\alpha[1 + u\, D_\beta]}{\prod_{c\in\alpha}(1 - u\, t^{l'} q^{a'})} \;=\; \frac{\tilde H_\beta[1 + u\, D_\alpha]}{\prod_{c\in\beta}(1 - u\, t^{l'} q^{a'})} \qquad \text{(for all pairs } \alpha, \beta). \tag{1.12}$$

We will use here several special evaluations of 1.12. To begin, canceling the common factor $(1 - u)$ out of the denominators on both sides of 1.12 and then setting $u = 1$ gives

$$\frac{\tilde H_\alpha[MB_\beta]}{\Pi_\alpha} \;=\; \frac{\tilde H_\beta[MB_\alpha]}{\Pi_\beta} \qquad \text{(for all pairs } \alpha, \beta). \tag{1.13}$$

On the other hand replacing u by $1/u$ and letting $u=0$ in 1.12 gives

$$(-1)^{|\alpha|}\frac{\tilde H_\alpha[D_\beta]}{T_\alpha} \;=\; (-1)^{|\beta|}\frac{\tilde H_\beta[D_\alpha]}{T_\beta} \qquad \text{(for all pairs } \alpha,\beta). \tag{1.14}$$

Since for β the empty partition we can take $\tilde H_\beta = 1$ and $D_\beta=-1$, 1.12 in this case reduces to

$$\tilde H_\alpha[1-u\,] \;=\; \prod_{c\in\alpha}(1-ut^{l'}q^{a'}) \;=\; (1-u)\sum_{r=0}^{n-1}(-u)^r e_r[B_\mu-1]. \tag{1.15}$$

This identity yields the coefficients of hook Schur functions in the expansion.

$$\tilde H_\mu[X;q,t] \;=\; \sum_{\lambda\vdash|\mu|} s_\mu[X]\tilde K_{\lambda\mu}(q,t). \tag{1.16}$$

Recall that the addition formula for Schur functions gives

$$s_\mu[1-u] \;=\; \begin{cases} (-u)^r(1-u) & \text{if } \mu=(n-r,1^r) \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

Thus 1.16, with $X=1-u$, combined with 1.15 gives for $\mu\vdash n$

$$\big\langle \tilde H_\mu\,,\, s_{(n-r,1^r)}\big\rangle \;=\; e_r[B_\mu-1]$$

and the identity $e_rh_{n-r}=s_{(n-r,1^r)}+s_{(n-r-1,1^{r-1})}$ gives

$$\big\langle \tilde H_\mu\,,\, e_rh_{n-r}\big\rangle \;=\; e_r[B_\mu]. \tag{1.19}$$

Since for $\beta=(1)$ we have $\tilde H_\beta=1$ and $\Pi_\beta=1$, formula 1.13 reduces to the surprisingly simple identity

$$\tilde H_\alpha[M] \;=\; MB_\alpha\Pi_\alpha. \tag{1.20}$$

Last but not least we must also recall that our original Pieri formulas were defined by setting

$$a)\quad e_1\tilde H_\nu \;=\; \sum_{\mu\supset_1\nu} d_{\mu\nu}\tilde H_\mu\,, \qquad\qquad b)\quad e_1^\perp\tilde H_\mu \;=\; \sum_{\nu\subseteq_1\mu} c_{\mu\nu}\tilde H_\nu\,, \tag{1.21}$$

To be consistent with the notation in the introduction we need to set

$$c^{(1)}_{\mu,\nu} \;=\; c_{\mu,\nu} \qquad\text{and}\qquad d^{(1)}_{\mu,\nu} \;=\; \tfrac{1}{M}d_{\mu,\nu}$$

These Pieri coefficients are thus related by the identity

$$d^{(1)}_{\mu,\nu} \;=\; \tfrac{1}{M}d_{\mu\nu} \;=\; \frac{w_\nu}{w_\mu}c_{\mu\nu} \;=\; \frac{w_\nu}{w_\mu}c^{(1)}_{\mu,\nu}. \tag{1.24}$$

Recall that our Macdonald Polynomials satisfy the orthogonality condition

$$\big\langle \tilde H_\lambda\,,\, \tilde H_\mu\big\rangle_* \;=\; \chi(\lambda=\mu)w_\mu(q,t). \tag{1.25}$$

The $*$-scalar product, is simply related to the ordinary Hall scalar product by setting for all pairs of symmetric functions f, g

$$\big\langle f \ , \ g \big\rangle_* \ = \ \big\langle f \ , \ \omega\phi g \big\rangle, \tag{1.26}$$

where it has been customary to let ϕ be the operator defined by setting for any symmetric function f

$$\phi\, f[X] \ = \ f[MX]. \tag{1.27}$$

Note that the inverse of ϕ is usually written in the form

$$f^*[X] \ = \ f[X/M]. \tag{1.28}$$

In particular we also have for all symmetric functions f, g

$$\big\langle f \ , \ g \big\rangle \ = \ \big\langle f, \omega g^* \big\rangle_* \tag{1.29}$$

The orthogonality relations in 1.25 yield the "Cauchy" identity for our Macdonald polynomials in the form

$$\Omega\left[-\epsilon \tfrac{XY}{M}\right] \ = \ \sum_{\mu} \frac{\tilde H_\mu[X]\tilde H_\mu[Y]}{w_\mu}, \tag{1.30}$$

which restricted to its homogeneous component of degree n in X and Y reduces to

$$e_n\left[\tfrac{XY}{M}\right] \ = \ \sum_{\mu\vdash n} \frac{\tilde H_\mu[X]\tilde H_\mu[Y]}{w_\mu}. \tag{1.31}$$

Note that the orthogonality relations in 1.25 yield us the following Macdonald polynomial expansions

Proposition 1.1

For all $n \geq 1$ we have

$$a)\ \ e_n\left[\tfrac{X}{M}\right] = \sum_{\mu\vdash n}\frac{\tilde H_\mu[X]}{w_\mu}\ , \qquad b)\ \ h_k\left[\tfrac{X}{M}\right] e_{n-k}\left[\tfrac{X}{M}\right] = \sum_{\mu\vdash n}\frac{e_k[B_\mu]\tilde H_\mu[X]}{w_\mu}\ , \qquad c)\ \ h_n\left[\tfrac{X}{M}\right] = \sum_{\mu\vdash n}\frac{T_\mu \tilde H_\mu[X]}{w_\mu}$$

$$\begin{aligned} d)\ \ (-1)^{n-1}p_n \ &= \ (1-t^n)(1-q^n)\sum_{\mu\vdash n}\frac{\Pi_\mu \tilde H_\mu[X]}{w_\mu} \\ e)\qquad e_1[X/M]^n \ &= \ \sum_{\mu\vdash n}\frac{\tilde H_\mu[X]}{w_\mu}\big\langle \tilde H_\mu, e_1^n\big\rangle \\ f)\qquad e_n \ &= \ \sum_{\mu\vdash m}\frac{\tilde H_\mu[X] M B_\mu \Pi_\mu}{w_\mu} \end{aligned} \tag{1.32}$$

Finally it is good to keep in mind, for future use, that we have for all partitions μ

$$T_\mu \omega \tilde H_\mu[X; 1/q, 1/t] \ = \ \tilde H_\mu[X; q, t]. \tag{1.33}$$

Let us now recall that the Pieri Coefficients (for $k=1$) yielded the following summation formulas

$$\sum_{\nu\to\mu} c_{\mu\nu}^{(1)}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases} \frac{tq}{M}\, h_{k+1}\big[D_\mu(q,t)/tq\big] & \text{if } k\geq 1\ , \\ B_\mu(q,t) & \text{if } k=0\ . \end{cases} \tag{1.34}$$

$$\sum_{\mu\leftarrow\nu} d_{\mu\nu}^{(1)}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases} (-1)^{k-1}\, e_{k-1}\big[D_\nu(q,t)\big]/M & \text{if } k\geq 1\ , \\ 1/M & \text{if } k=0\ . \end{cases} \tag{1.35}$$

Note that if we set, for any symmetric function $F[X]$

$$\tau_u F[X] \;=\; F[X+u] \tag{1.36}$$

then we have the development

$$\tau_u F[X] \;=\; \sum_{k\geq 0} u^k h_k^\perp F[X] \tag{1.37}$$

Using the operator τ_u we can derive the following useful identities

Proposition 1.2

$$\begin{aligned} a)\quad & D_k\, h_m^\perp \;-\; h_m^\perp\, D_k \;=\; D_{k-1}\, h_{m-1}^\perp\ , \\ b)\quad & D_k^*\, h_m^\perp \;-\; h_m^\perp\, D_k^* \;=\; -D_{k-1}^*\, h_{m-1}^\perp\ . \end{aligned} \tag{1.38}$$

In particular we also have

$$\begin{aligned} a)\quad & D_k\, h_1^\perp \;-\; h_1^\perp\, D_k \;=\; D_{k-1}\ , \\ b)\quad & D_k^*\, h_1^\perp \;-\; h_1^\perp\, D_k^* \;=\; -D_{k-1}^*\ . \end{aligned} \tag{1.39}$$

Note that for $F[X]=e_1$, we derive from 1.11 (i) that the corresponding operator may be computed from the formula

$$\Delta_{e_1} \;=\; \tfrac{1}{M}(1-D_o) \tag{1.41}$$

Let us recall that the translation by u operator τ_u may simply defined by setting

$$\tau_u F[X] \;=\; F[X+u] \;=\; \sum_{k\geq 0} u^k h_k^\perp F[X] \tag{1.42}$$

Now we have the following basic identity

Theorem 1

$$\tfrac{1}{M} u D_{-1}\tau_u \;=\; \tau_u\Delta_{e_1} \;-\; \Delta_{e_1}\tau_u \tag{1.43}$$

Proof

The identity in 1.41 gives

$$\tau_u\Delta_{e_1} \;-\; \Delta_{e_1}\tau_u \;=\; \tfrac{1}{M}\tau_u(1-D_o) - \tfrac{1}{M}(1-D_o)\tau_u \;=\; \tfrac{1}{M}\big(D_o\tau_u - \tau_u D_o\big)$$

and 1.43 becomes (canceling the $\frac{1}{M}$)

$$u D_{-1}\tau_u \;=\; D_o\tau_u - \tau_u D_o \tag{1.44}$$

Now for any $F[X]$ we have

$$uD_{-1}\tau_u F[X \;=\; uF[X+\tfrac{M}{z}+u]\Omega[-zX]\Big|_{z^{-1}} \tag{1.45}$$

while

$$\big(D_o\tau_u-\tau_u D_o\big)F[X] \;=\; F[X+\tfrac{M}{z}+u]\Omega[-zX]\Big|_{z^0} - F[X+\tfrac{M}{z}+u]\Omega[-z(X+u)]\Big|_{z^0}. \tag{1.46}$$

But

$$\Omega[-z(X+u)] \;=\; \Omega[-zX]\Omega[-zu] \;=\; \Omega[-zX](1-zu)$$

and 1.46 becomes

$$\begin{aligned}\big(D_o\tau_u-\tau_u D_o\big)F[X] \;&=\; F[X+\tfrac{M}{z}+u]\Omega[-zX]\Big|_{z^0} - F[X+\tfrac{M}{z}+u]\Omega[-zX](1-zu)\Big|_{z^0}\\ &=\; zuF[X+\tfrac{M}{z}+u]\Omega[-z(X+u)]\Big|_{z^0} \;=\; uD_{-1}\tau_u F[X]\end{aligned}$$

In view of 1.45, his proves 1.44 and therefore also 1.43.

The following fundamental identities, shown in "explicit", play a crucial role in many later developments as well as here.

Proposition 1.3 (Glenn's Formula)

For any symmetric function f and partition μ we have

$$\big\langle f\,,\,\tau_1\tilde H_\mu\big\rangle_* \;=\; \nabla^{-1}\tau_{-\epsilon}f\Big|_{X\to D_\mu} \tag{1.40}$$

2. The François-Mark identities

Our first result is a proof of the François-Mark Conjecture

Theorem 2.1

For any $k\geq 1$ and $\mu\vdash n$ we have

$$c^{(k+1)}_{\mu,\nu} \;=\; \frac{1}{B_{\mu/\nu}}\sum_{\nu\subseteq_1\alpha\subseteq_k\mu} c^{(k)}_{\mu,\alpha}c^{(1)}_{\alpha,\nu}T_\alpha/T_\nu \tag{2.1}$$

or equivalently

$$\nabla^{-1}e_1^\perp\nabla h_k^\perp \;=\; h_{k+1}^\perp\Delta_{e_1} - \Delta_{e_1}h_{k+1}^\perp \tag{2.2}$$

Proof

Equating coefficients of u^{k+1} in 1.43 gives

$$\tfrac{1}{M}D_{-1}h_k^\perp \;=\; h_{k+1}^\perp\Delta_{e_1} - \Delta_{e_1}h_{k+1}^\perp$$

Using 1.11 (iv) gives

$$\nabla^{-1}\,e_1^\perp\nabla h_k = h_{k+1}^\perp\Delta_{e_1} - \Delta_{e_1}h_{k+1}^\perp$$

This proves 2.2. Next we apply the right hand side of 2.2 to $\tilde H_\mu[X;q,t]$ and get

$$\begin{aligned}\big(h_{k+1}^\perp\Delta_{e_1}\,-\,\Delta_{e_1}h_{k+1}^\perp\big)\tilde H_\mu &= B_\mu(q,t)\sum_{\nu\subseteq_{k+1}\mu}c_{\mu,\nu}^{(k+1)}\tilde H_\nu\,-\,\sum_{\nu\subseteq_{k+1}\mu}c_{\mu,\nu}^{(k+1)}B_\nu(q,t)\tilde H_\nu\\ &= \sum_{\nu\subseteq_{k+1}\mu}c_{\mu,\nu}^{(k+1)}B_{\mu/\nu}(q,t)\tilde H_\nu\end{aligned}\qquad 2.3$$

Now the left hand side applied to $\tilde H_\mu[X;q,t]$ gives

$$\begin{aligned}\nabla^{-1}\partial_{p_1}\nabla h_k^\perp\tilde H_\mu[X;q,t] &= \nabla^{-1}\partial_{p_1}\nabla\sum_{\alpha\subseteq_k\mu}c_{\mu,\alpha}^{(k)}\tilde H_\alpha[X;q,t]\\ &= \nabla^{-1}\partial_{p_1}\sum_{\alpha\subseteq_k\mu}c_{\mu,\alpha}^{(k)}T_\alpha\tilde H_\alpha[X;q,t]\\ &= \sum_{\alpha\subseteq_k\mu}c_{\mu,\alpha}^{(k)}\frac{T_\alpha}{T_\nu}\sum_{\nu\subseteq_1\alpha}c_{\alpha,\nu}^{(1)}\tilde H_\nu[X;q,t]\end{aligned}\qquad 2.4$$

Equating the right hand sides of 2.3 and 2.4 and equating the coefficients of $\tilde H_\nu$ on both sides gives 2.1.

The next, more general identity, of François and Mark we have not been able to prove and we will leave it as a

Conjecture 1.1 (François-Mark)

For all k,l $\nu\subseteq k_l\mu$

$$c_{\mu\nu}^{(k+l)}\;=\;\frac{1}{e_k[B_{\mu/\nu}]}\sum_{\nu\subseteq_k\alpha\subseteq_l\mu}c_{\mu,\alpha}^{(l)}\frac{T_\alpha}{T_\nu}\;c_{\alpha,\nu}^{(k)}\qquad 2.5$$

or equivalently

$$\nabla^{-1}h_k^\perp\nabla h_l^\perp\;=\;\sum_{r=0}^{k}(-1)^r\Delta_{h_r}h_{k+l}^\perp\Delta_{e_{k-r}}\qquad 2.6$$

Note that for $k=1$ 2.6 reduces to

$$\nabla^{-1}h_1^\perp\nabla h_l^\perp\;=\;h_{l+1}^\perp\Delta_{e_1}\,-\,\Delta_{h_1}h_{l+1}^\perp$$

which is easily seen to be none other than 2.2.

Remark 2.1

Following François the equivalence of 2.5 and 2.6 is shown as follows

$$\begin{aligned}\sum_{r=0}^{k}(-1)^r\Delta_{h_r}h_{k+l}^\perp\Delta_{e_{k-r}}\tilde H_\mu &= \sum_{r=0}^{k}(-1)^r e_{k-r}[B_\mu]\Delta_{h_r}h_{k+l}^\perp\tilde H_\mu\\ &= \sum_{r=0}^{k}(-1)^r e_{k-r}[B_\mu]\sum_{\nu\subseteq_{k+l}\mu}c_{\mu\nu}^{(k+l)}h_r[B_\nu]\tilde H_\nu\\ &= \sum_{\nu\subseteq_{k+l}\mu}c_{\mu\nu}^{(k+l)}\tilde H_\nu\sum_{r=0}^{k}e_{k-r}[B_\mu]e_r[-B_\nu]\;=\;\sum_{\nu\subseteq_{k+l}\mu}c_{\mu\nu}^{(k+l)}e_k[B_{\mu/\nu}]\tilde H_\nu\end{aligned}\qquad 2.7$$

On the other hand if we apply the LHS of 2.6 to $\tilde H_\mu$ we obtain

$$\begin{aligned}\nabla^{-1}h_k^\perp\nabla h_l^\perp\tilde H_\mu &= \nabla^{-1}h_k^\perp\sum_{\alpha\subseteq_l\mu}c^{(l)}_{\mu,\alpha}T_\alpha\tilde H_\alpha\\ &= \nabla^{-1}\sum_{\alpha\subseteq_l\mu}c^{(l)}_{\mu,\alpha}T_\alpha\sum_{\nu\subseteq_k\alpha}c^{(k)}_{\alpha,\nu}\tilde H_\nu\\ &= \sum_{\alpha\subseteq_l\mu}\sum_{\nu\subseteq_k\alpha}c^{(l)}_{\mu,\alpha}\frac{T_\alpha}{T_\nu}\,c^{(k)}_{\alpha,\nu}\tilde H_\nu\end{aligned}\qquad 2.8$$

Equating coefficients of $\tilde H_\nu$ on both 2.7 and 2.8 gives 2.5.

Remark 2.2

François attempts to prove 2.5 by inductive applications of 2.1. This suggests that 2.6 should be derivable by an inductive process from 2.2 but I failed to see how this could be done. Computer data confirm both 2.5 and 2.6.

We should point that the same relation we have seen in 1.24 holds true for any $k > 1$ as well,

Proposition 2.2

$$d^{(k)}_{\mu,\nu} = c^{(k)}_{\mu,\nu}\frac{w_\nu}{w_\mu}\qquad 2.9$$

Proof

Note that from the identity

$$h_k^\perp\tilde H_\mu[X;q,t] = \sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}\tilde H_\nu[X;q,t]$$

we derive that

$$\left\langle h_k^\perp\tilde H_\mu[X;q,t]\ ,\ \tilde H_\nu\right\rangle_* = c^{(k)}_{\mu,\nu}\,w_\nu$$

Thus from 2.2 it follows that

$$\begin{aligned}c^{(k)}_{\mu,\nu}\,w_\nu &= \left\langle h_k^\perp\tilde H_\mu[X;q,t]\ ,\ \omega\phi\tilde H_\nu\right\rangle\\ &= \left\langle\tilde H_\mu[X;q,t]\ ,\ h_k\omega\phi\tilde H_\nu\right\rangle\\ &= \left\langle\tilde H_\mu[X;q,t]\ ,\ \omega\phi e_k^*\tilde H_\nu\right\rangle = \left\langle\tilde H_\mu[X;q,t]\ ,\ e_k^*\tilde H_\nu\right\rangle_*\end{aligned}$$

and the definition in 2.1 gives

$$c^{(k)}_{\mu,\nu}\,w_\nu = d^{(k)}_{\mu,\nu}w_\mu$$

completing our proof.

Another quite remarkable identity of François and Mark can be stated as follows.

Conjecture 1.1 (François-Mark)

For all k and ν we have

$$\sum_{\mu\supset_k\nu}d^{(k)}_{\mu,\nu}\left(\tfrac{T_\mu}{T_\nu}\right)^2 = e_k\left[\tfrac{1}{M}-B_\nu\right]\qquad 2.10$$

or equivalently

$$\nabla^{-1}h_k^\perp\nabla h_n\left[\tfrac{X}{M}\right] = \sum_{r=0}^{k}(-1)^r e_{k-r}\left[\tfrac{1}{M}\right]\Delta_{h_r}h_{n-k}\left[\tfrac{X}{M}\right]\qquad 2.11$$

Remark 2.3

Following François the equivalence of 2.10 and 2.11 is shown as follows. Notice that the LHS of 2.11 is

$$\begin{aligned} \nabla^{-1}h_k^\perp \nabla h_n[\tfrac{X}{M}] &= \sum_{\mu\vdash n}\frac{T_\mu^2}{w_\mu}\sum_{\nu\subseteq_k\mu} c_{\mu,\nu}^{(k)}\frac{1}{T_\nu}\tilde H_\nu \\ &= \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu}{w_\nu}\sum_{\mu\supset_k\nu}\frac{w_\nu}{w_\mu}c_{\mu,\nu}^{(k)}(\tfrac{T_\mu}{T_\nu})^2 \\ (\text{by } 2.9) \quad &= \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu}{w_\nu}\sum_{\mu\supset_k\nu} d_{\mu,\nu}^{(k)}(\tfrac{T_\mu}{T_\nu})^2 \end{aligned} \qquad 2.12$$

while the RHS is

$$\sum_{r=0}^{k}(-1)^r e_{k-r}[\tfrac{1}{M}]\Delta_{h_r}h_{n-k}[\tfrac{X}{M}] \;=\; \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu}{w_\nu}\sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]e_r[-B_\nu]$$

and 2.10 follows since

$$\sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]e_r[-B_\nu] \;=\; e_k[\tfrac{1}{M}-B_\nu]$$

3. Summation formulas

The following two summatiomn formular may be viewed as the (present) $k>1$ analogues of the (former) $k=0$ and $k=1$ summation formula 1.35

Proposition 3.1

$$\begin{aligned} &a) \quad \sum_{\mu\supset_k\nu} d_{\mu,\nu}^{(k)} \;=\; e_k[\tfrac{1}{M}] \\ &b) \quad \sum_{\mu\supset_k\nu}\frac{T_\mu}{T_\nu}d_{\mu,\nu}^{(k)} \;=\; h_k[\tfrac{1}{M}] \qquad \text{(François-Mark)} \end{aligned} \qquad 3.1$$

Proof

We start with François-Mark proof of 3.1 b)

$$\begin{aligned} h_k^\perp h_n[\tfrac{X}{M}] &= \sum_{\mu\vdash n}\frac{T_\mu}{w_\mu}\sum_{\nu\subseteq_k\mu} c_{\mu,\nu}^{(k)}\tilde H_\nu[X;q,t] \\ &= \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu[X;q,t]}{w_\nu}\sum_{\mu\supset_k\nu}\frac{T_\mu w_\nu}{T_\nu w_\mu}c_{\mu,\nu}^{(k)} \\ (\text{by } 2.9\ ) \quad &= \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu[X;q,t]}{w_\nu}\sum_{\mu\supset_k\nu}\frac{T_\mu}{T_\nu}d_{\mu,\nu}^{(k)} \end{aligned} \qquad 3.2$$

On the other hand we have

$$h_k^\perp h_n[\tfrac{X}{M}] \;=\; \tau_u h_n[\tfrac{X}{M}]\Big|_{u^k} \;=\; h_n[\tfrac{X+u}{M}]\Big|_{u^k} \;=\; h_{n-k}[\tfrac{X}{M}]h_k[\tfrac{1}{M}] \qquad 3.3$$

and this, combined with 3.2 gives

$$\sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu[X;q,t]}{w_\nu}h_k[\tfrac{1}{M}] \;=\; \sum_{\nu\vdash n-k}\frac{T_\nu\tilde H_\nu[X;q,t]}{w_\nu}\sum_{\mu\supset_k\nu}\frac{T_\mu}{T_\nu}\,d^{(k)}_{\mu,\nu}$$

Equating coefficients of $\tilde H_\nu$ gives 3.1 b).

The companion identity is obtained in a similar way:

$$h_k^\perp e_n[\tfrac{X}{M}] \;=\; \sum_{\mu\vdash n}\frac{1}{w_\mu}\sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}\tilde H_\nu[X;q,t] \;=\; \sum_{\nu\vdash n-k}\frac{\tilde H_\nu[X;q,t]}{w_\nu}\sum_{\mu\supset_k\nu}d^{(k)}_{\mu,\nu} \qquad 3.3$$

On the other hand we have

$$h_k^\perp e_n[\tfrac{X}{M}] \;=\; \tau_u e_n[\tfrac{X}{M}]\Big|_{u^k} \;=\; e_n[\tfrac{X+u}{M}]\Big|_{u^k} \;=\; e_{n-k}[\tfrac{X}{M}]e_k[\tfrac{1}{M}]$$

and this combined with 3.3 gives

$$\sum_{\mu\supset_k\nu}d^{(k)}_{\mu,\nu} \;=\; e_k[\tfrac{1}{M}].$$

It turns out that Glenn's formula yields some truly general summation identities even for $k>1$. The challenge will be to see to what extent the following result will deliver explicit results and applications.

Theorem 3.1

$$\sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}P[D_\nu] \;=\; \sum_{r=0}^{k}\nabla^{-1}e^*_{k-r}\nabla P\Big|_{X=D_\mu} \qquad 3.4$$

$$\sum_{\mu\supset_k\nu}d^{(k)}_{\mu,\nu}P[D_\mu] \;=\; \sum_{r=0}^{k}e_{k-r}[\tfrac{1}{M}]\nabla^{-1}h_r^\perp\nabla P\Big|_{X=D_\nu} \qquad 3.5$$

Proof

Let us start with $f=\tau_\epsilon\nabla P$, so $P=\nabla\tau_{-\epsilon}P$ in the left hand side of 3.4, obtaining

$$\begin{aligned}\sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}P[D_\nu] \;&=\; \sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}\big\langle f\,,\,\tau_1\tilde H_\nu\big\rangle_* \;=\; \big\langle f\,,\,\tau_1 h_k^\perp\tilde H_\mu\big\rangle_*\\ &=\; \big\langle f\,,\,h_k^\perp\tau_1\tilde H_\mu\big\rangle_* \;=\; \big\langle \omega\phi f\,,\,h_k^\perp\tau_1\tilde H_\mu\big\rangle\\ &=\; \big\langle h_k\omega\phi f\,,\,\tau_1\tilde H_\mu\big\rangle \;=\; \big\langle \omega\phi e_k^* f\,,\,\tau_1\tilde H_\mu\big\rangle\\ &=\; \big\langle e_k^* f\,,\,\tau_1\tilde H_\mu\big\rangle_* \;=\; \nabla^{-1}\tau_{-\epsilon}e_k^*f\Big|_{X=D_\mu} \;=\; \nabla^{-1}\tau_{-\epsilon}e_k^*\tau_\epsilon\nabla P\Big|_{X=D_\mu}\end{aligned}$$

But

$$\tau_{-\epsilon}e_k^* \;=\; \sum_{r=0}^{k}e_{k-r}[\tfrac{X}{M}]e_r[-\epsilon] \;=\; \sum_{r=0}^{k}e_{k-r}[\tfrac{X}{M}]$$

This gives

$$\sum_{\nu\subseteq_k\mu}c^{(k)}_{\mu,\nu}P[D_\nu] \;=\; \sum_{r=0}^{k}\nabla^{-1}e^*_{k-r}\tau_{-\epsilon}\tau_\epsilon\nabla P\Big|_{X=D_\mu}$$

proving 3.4.

Let us now do the $d^{(k)}_{\mu,\nu}$ case. Proceeding as before with $f=\tau_\epsilon\nabla P$, $P=\nabla\tau_{-\epsilon}P$, the left hand side of 3.5 becomes

$$\begin{aligned}\sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}P[D_\mu] &= \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}\big\langle f\,,\,\tau_1\tilde H_\mu\big\rangle_* \;=\; \big\langle f\,,\,\tau_1 e_k^*\tilde H_\nu\big\rangle_*\\ &= \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\big\langle f\,,\,e_r[\tfrac{X}{M}]\tau_1\tilde H_\nu\big\rangle_* \;=\; \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\big\langle f\,,\,h_r[X]\omega\phi\tau_1\tilde H_\nu\big\rangle\\ &= \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\big\langle h_r^\perp f\,,\,\omega\phi\tau_1\tilde H_\nu\big\rangle \;=\; \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\big\langle h_r^\perp f\,,\,\tau_1\tilde H_\nu\big\rangle_*\\ &= \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\nabla^{-1}\tau_{-\epsilon}h_r^\perp f\Big|_{X=D_\nu} \;=\; \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\nabla^{-1}h_r^\perp\tau_{-\epsilon}\tau_\epsilon\nabla P\Big|_{X=D_\nu}\\ &= \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\nabla^{-1}h_r^\perp\nabla P\Big|_{X=D_\nu}\end{aligned}$$

proving 3.5.

Remark 3.1

The nature of the right hand side of 3.5 makes one wonder if Conjecture 1.1 can play a role here. In fact , 2.12 gives that

$$\nabla^{-1}h_r^\perp\nabla h_n[\tfrac{X}{M}] \;=\; \sum_{\alpha\vdash n-r}\frac{T_\alpha\tilde H_\alpha}{w_\alpha}\sum_{\mu\supset_k\alpha} d^{(k)}_{\mu,\alpha}(\tfrac{T_\mu}{T_\alpha})^2$$

and from 2.10 we derive

$$\nabla^{-1}h_r^\perp\nabla h_n[\tfrac{X}{M}] \;=\; \sum_{\alpha\vdash n-r}\frac{T_\alpha\tilde H_\alpha}{w_\alpha}e_r[\tfrac{1}{M}-B_\alpha]$$

Now setting $P=h_n[\frac{X}{M}]$ in 3.5 we obtain

$$\sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}h_n\big[\tfrac{MB_\mu-1}{M}\big] \;=\; \sum_{r=0}^{k} e_{k-r}[\tfrac{1}{M}]\sum_{\alpha\vdash n-r}\frac{T_\alpha\tilde H_\alpha[D_\nu]}{w_\alpha}e_r[\tfrac{1}{M}-B_\alpha]$$

Macdonald reciprocity gives

$$(-1)^{n-r}\frac{\tilde H_\alpha[D_\nu]}{T_\alpha} \;=\; (-1)^{n-k}\frac{\tilde H_\nu[D_\alpha]}{T_\nu}$$

Another tool that can be used to obtain summation formulas is "*flipping* ", that is the operation of making the replacements $(q,t)\to(\frac{1}{q},\frac{1}{t})$. The first step is obtained from the following identities.

Proposition 3.2

$$e_k^\perp\tilde H_\mu[X;q,t] \;=\; \sum_{\nu\subseteq_k\mu} c^{(k)}_{\mu,\nu}(\tfrac{1}{q},\tfrac{1}{t})\tfrac{T_\mu}{T_\nu}\tilde H_\nu[X;q,t] \qquad 3.6$$

$$h_k[\tfrac{X}{M}]\tilde H_\nu[X;q,t] \;=\; \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac{1}{q},\tfrac{1}{t})\tfrac{T_\nu}{T_\mu}\tilde H_\mu[X;q,t] \qquad 3.7$$

Proof

Note that flipping I.1 gives

$$h_k^\perp \tilde H_\mu[X;1/q,1/t] \;=\; \sum_{\nu\subseteq_k\mu} c^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tilde H_\nu[X;1/q,1/t]$$

applying ω to both sides we get

$$\omega h_k^\perp \omega\omega\tilde H_\mu[X;1/q,1/t] \;=\; \sum_{\nu\subseteq_k\mu} c^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\omega\tilde H_\nu[X;1/q,1/t]$$

and using the flip formula 1.33 for the Macdonalds we obtain

$$\omega h_k^\perp \omega\tilde H_\mu[X;q,t] \;=\; \sum_{\nu\subseteq_k\mu} c^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tfrac{T_\mu}{T_\nu}\tilde H_\nu[X;q,t] \qquad 3.7$$

Now for any two homogeneous symmetric functions F,G respectively of degrees $n-k$ and n we have

$$\big\langle F\,,\,(\omega h_k\omega)^\perp G\big\rangle \;=\; \big\langle \omega h_k\omega F\,,\,G\big\rangle \;=\; \big\langle e_kF\,,\,G\big\rangle \;=\; \big\langle F\,,\,e_k^\perp G\big\rangle$$

This yields that $\omega h_k^\perp\omega = e_k^\perp$ and thus 3.7 becomes

$$e_k^\perp \tilde H_\mu[X;q,t] \;=\; \sum_{\nu\subseteq_k\mu} c^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tfrac{T_\mu}{T_\nu}\tilde H_\nu[X;q,t]$$

proving 3.6.

Next note that flipping I.2 we obtain

$$e_k[\tfrac{tqX}{M}]\tilde H_\nu[X;\tfrac1q,\tfrac1t] \;=\; \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tilde H_\mu[X;\tfrac1q,\tfrac1t]$$

applying ω and using 1.33 again gives

$$t^kq^kh_k[\tfrac{X}{M}]\tilde H_\nu[X;q,t] \;=\; \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tfrac{T_\nu}{T_\mu}\tilde H_\mu[X;q,t]$$

or better

$$h_k[\tfrac{X}{M}]\tilde H_\nu[X;q,t] \;=\; \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)\tfrac{T_\nu}{T_\mu}\tilde H_\mu[X;q,t]$$

proving 3.7

Remark 3.2

Note that flipping the relations in 3.1 we obtain

$$a)\qquad \sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t) \;=\; e_k[\tfrac{qt}{M}]$$

$$b)\qquad \sum_{\mu\supset_k\nu} \frac{T_\nu}{T_\mu}\, d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t) \;=\; h_k[\tfrac{qt}{M}]$$

Note that flipping 2.10 we obtain

$$\sum_{\mu\supset_k\nu} d^{(k)}_{\mu,\nu}(\tfrac1q,\tfrac1t)(\tfrac{T_\nu}{T_\mu})^2 \;=\; e_k[\tfrac{tq}{M}-B_\nu(\tfrac1q,\tfrac1t)] \qquad 2.10$$

$$\nabla^{-1} h_k^{\perp} \nabla h_n^* \;=\; \sum_{r=0}^{k} (-1)^r e_{k-r}[\tfrac{1}{M}] \Delta_{h_r} h_{n-k}^*$$

by induction

$$\nabla^{-1} h_{k-1}^{\perp} \nabla h_n^* \;=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] \Delta_{h_r} h_{n-k+1}^*$$

apply on each side the operator $\nabla^{-1} h_1^{\perp} \nabla^2$

$$\nabla^{-1} h_1^{\perp} \nabla^2 \nabla^{-1} h_{k-1}^{\perp} \nabla h_n^* \;=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] \nabla^{-1} h_1^{\perp} \nabla^2 \Delta_{h_r} h_{n-k+1}^*$$

$$\nabla^{-1} h_1^{\perp} \nabla h_{k-1}^{\perp} \nabla h_n^* \;=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] \nabla^{-1} h_1^{\perp} \nabla^2 \Delta_{h_r} h_{n-k+1}^* \qquad 2.11$$

Using this identity

$$\nabla^{-1} e_1^{\perp} \nabla h_{k-1}^{\perp} \;=\; h_k^{\perp} \Delta_{e_1} \,-\, \Delta_{e_1} h_k^{\perp}$$

the LHS of 2.11 becomes for now

$$LHS = (h_k^{\perp} \Delta_{e_1} \,-\, \Delta_{e_1} h_k^{\perp}) \nabla h_n^*$$

Now for the RHS we use

$$h_1^{\perp} \Delta_{h_r} \;=\; \sum_{i=0}^{r} \nabla^{-i} \Delta_{h_{r-i}} h_1^{\perp} \nabla^i$$

and get

$$\begin{aligned}
RHS \;&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\nabla^{-1} h_1^{\perp} \Delta_{h_r} \nabla) \nabla h_{n-k+1}^* \\
&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\nabla^{-1} (\sum_{i=0}^{r} \nabla^{-i} \Delta_{h_{r-i}} h_1^{\perp} \nabla^i) \nabla) \nabla h_{n-k+1}^* \\
&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\sum_{i=0}^{r} \nabla^{-i-1} \Delta_{h_{r+1-i+1}} h_1^{\perp} \nabla^{i+1}) \nabla h_{n-k+1}^* \\
&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\sum_{i=0}^{r+1} \nabla^{-i} \Delta_{h_{r+1-i}} h_1^{\perp} \nabla^{i}) \nabla h_{n-k+1}^* \\
&\qquad\qquad - \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\Delta_{h_{r+1}} h_1^{\perp}) \nabla h_{n-k+1}^* \\
&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] h_1^{\perp} \Delta_{h_{r+1}} \nabla h_{n-k+1}^* - \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] (\Delta_{h_{r+1}} h_1^{\perp}) \nabla h_{n-k+1}^* \\
&=\; \sum_{r=0}^{k-1} (-1)^r e_{k-1-r}[\tfrac{1}{M}] \big(h_1^{\perp} \Delta_{h_{r+1}} \,-\, \Delta_{h_{r+1}} h_1^{\perp} \big) \nabla h_{n-k+1}^*
\end{aligned}$$

$$\nabla^{-1} h_k^{\perp} \nabla h_n^* \;=\; \sum_{r=0}^{k} (-1)^r e_{k-r}[\tfrac{1}{M}] \Delta_{h_r} h_{n-k}^*$$