New Proof of the $\nabla \mathbf{E_{nk}}$ formula

Our goal here is obtaining a reasonably short proof of the Haglund identity

$$\nabla E_{nk} \;=\; t^{n-k}(1-q^k)\Psi h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}] \qquad (*)$$

1. Basics on the q-Pochhammer Polynomials

It is customary to set

$$(z,q)_m \;=\; (1-z)(1-zq)\cdots(1-zq^{m-1}) \qquad 1.1$$

It is a classical result that we have

$$(z,q)_m \;=\; \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q q^{\binom{r}{2}}(-1)^r z^r \qquad 1.2$$

It is interesting to see how this identity can be established by symmetric function methods. To this end note that the "Cauchy" identity

$$\Omega[u\,(X-Y)] \;=\; \prod_i \frac{1-u\,y_i}{1-u\,x_i} \;=\; \sum_{m\geq 0} u^m h_m[X-Y] \qquad 1.3$$

for $X=\frac{1}{1-q}$ and $Y=\frac{z}{1-q}$ gives

$$\prod_{k\geq 0}\frac{1-uzq^k}{1-uq^k} \;=\; \Omega\big[u\,\tfrac{1-z}{1-q}\big] \;=\; \sum_{m\geq 0} u^m h_m\big[\tfrac{1-z}{1-q}\big]\,. \qquad 1.4$$

Since we obviously have

$$(1-u)\,\Omega\big[u\,\tfrac{1-z}{1-q}\big] \;=\; (1-zu)\,\Omega\big[u\,q\,\tfrac{1-z}{1-q}\big]\,,$$

equating coefficients of u^m in this equation, from the right hand side of 1.4 we derive that

$$h_m\big[\tfrac{1-z}{1-q}\big]-h_{m-1}\big[\tfrac{1-z}{1-q}\big] \;=\; q^m h_m\big[\tfrac{1-z}{1-q}\big]-z\,q^{m-1}h_{m-1}\big[\tfrac{1-z}{1-q}\big]\,.$$

This yields the recursion

$$h_m\big[\tfrac{1-z}{1-q}\big] \;=\; \tfrac{1-z\,q^{m-1}}{1-q^m}\,h_{m-1}\big[\tfrac{1-z}{1-q}\big]$$

and since $h_o\equiv 1$ we are finally led to the product formula

$$h_m\big[\tfrac{1-z}{1-q}\big] \;=\; \tfrac{1-z}{1-q}\,\tfrac{1-zq}{1-q^2}\cdots\tfrac{1-zq^{m-1}}{1-q^m} \;=\; \frac{(z;q)_m}{(q;q)_m}\,. \qquad 1.5$$

On the other hand, by equating coefficients of u^m, from 1.36 we can also derive the addition formula

$$h_m[X-Y] \;=\; \sum_{r=0}^{m} h_{m-r}[X](-1)^r e_r[Y] \qquad 1.6$$

Setting again $X=\frac{1}{1-q}$ and $Y=\frac{z}{1-q}$ gives

$$h_m\big[\tfrac{1-z}{1-q}\big] \;=\; \sum_{r=0}^{m} h_{m-r}\big[\tfrac{1}{1-q}\big](-1)^r e_r\big[\tfrac{z}{1-q}\big] \;=\; \sum_{r=0}^{m}\frac{1}{(q,q)_{n-r}}(-1)^r z^r\,\frac{q^{\binom{r}{2}}}{(q,q)_r} \qquad 1.7$$

and 1.35 is immediately derived by equating the right hand sides of 1.38 and 1.7.

It develops that the polynomial basis $\{(z;q)_m\}_{m\geq 0}$ has a "Taylor" formula that may be written in the form

$$P(z) \;=\; \sum_{r\geq 0} (z;q)_r \, \frac{q^r}{(q;q)_r} \, \big(\delta_q^r P(z)\,\big|_{z=1}\big)\,, \qquad 1.8$$

where δ_q is the q-difference operator defined by setting

$$\delta_q P(z) \;=\; \frac{P(z)-P(z/q)}{z}\,. \qquad 1.9$$

Formula 1.8 is an immediate consequence of the identities

$$\delta_q^k\,(z;q)_r\,\Big|_{z=1} \;=\; \begin{cases} 0 & \text{if } k\neq r\,, \\ \frac{(q;q)_r}{q^r} & \text{if } k=r \end{cases} \qquad 1.10$$

which in turn may be obtained by iterating the simple identity

$$\delta_q(z;q)_r \;=\; \frac{(z;q)_r \,-\, (z/q;q)_r}{z} \;=\; \frac{1-q^r}{q}\;(z;q)_{r-1}\,.$$

We will soon have the occasion to use the expansion in 1.8, thus to prepare us for this event we need to observe that if we denote by E the polynomial operator defined by setting

$$Ez^m \;=\; q^{-n}z^m \qquad\qquad \forall \quad m\geq 0$$

Then

$$\delta_q^n \;=\; \Big(\frac{1}{z}\big(1-E\big)\Big)^n \;=\; \frac{1}{z^n}(1-E)(1-qE)\cdots(1-q^{n-1}E)$$

and 1.2 gives

$$\delta_q^n \;=\; \frac{1}{z^n}\sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r E^r \qquad 1.11$$

Thus 1.8 can be rewritten as

$$P(z) \;=\; \sum_{k\geq 0}\frac{(z;q)_k}{(q;q)_k}\, q^k \,\sum_{r=0}^{k}\begin{bmatrix} k \\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r P(1/q^r). \qquad 1.12$$

This formula immediately gives us the following identity

Proposition 1.1

$$E_{nk}[X;q] \;=\; q^k\sum_{r=0}^{k}\begin{bmatrix} k \\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r e_n\big[X\tfrac{1-q^{-r}}{1-q}\big] \qquad 1.13$$

Proof

Recall that the E_{nk} are implicitly defined by the formula

$$e_n\big[X\tfrac{1-z}{1-q}\big] \;=\; \sum_{k=1}\frac{(z,q)_k}{(q,q)_k}\, E_{nk} \qquad 1.14$$

and we see that 1.13 is a direct consequence of 1.12 with

$$P(z) \;=\; e_n\big[X\tfrac{1-z}{1-q}\big]$$

2. The power of flip

We will denote by the symbol "$\downarrow$" the symmetric polynomial operator defined by setting for $F[X;q,t]\in\Lambda[q,t]$

$$\downarrow F[X;q,t] \;=\; \omega F[X;1/q,1/t]$$

We will refer to this operator a "*flip*". Note that flip is clearly an involution.

This given, have the following basic identities

Proposition 2.1

$$a)\quad \downarrow\nabla\downarrow \;=\; \nabla^{-1}\,, \qquad\qquad b)\quad \downarrow\Psi\downarrow \;=\; (-1)^{n-1}\nabla^{-1}\Psi \tag{2.1}$$

where the second identity holds when the action is on $\Lambda^{=n}$.

Proof

Recall that by definition we have for all $\mu\vdash n$

$$a)\quad \nabla\tilde H_\mu[X;q,t] \;=\; T_\mu\tilde H_\mu[X;q,t]\,, \qquad\qquad b)\quad \Psi\tilde H_\mu[X;q,t] \;=\; \Pi_\mu\tilde H_\mu[X;q,t] \tag{2.2}$$

Applying flip to both sides of 2.2 a) gives

$$\downarrow\nabla\downarrow\omega\tilde H_\mu[X;1/q,1/t] \;=\; \frac{1}{T_\mu}\,\omega\tilde H_\mu[X;1/q,1/t] \tag{2.3}$$

Recalling the Macdonald identity

$$T_\mu\,\omega\tilde H_\mu[X;1/q,1/t] \;=\; \tilde H_\mu[X;q,t] \tag{2.4}$$

we see that multiplying both sides of 2.3 by T_μ gives

$$\downarrow\nabla\downarrow\tilde H_\mu[X;q,t] \;=\; \frac{1}{T_\mu}\,\tilde H_\mu[X;q,t]$$

This proves 2.1 a).

Similarly 2.2 b) gives

$$\downarrow\Psi\downarrow\omega\tilde H_\mu[X;1/q,1/t] = \Pi_\mu(1/q,1/t)\,\omega\tilde H_\mu[X;1/q,1/t] \tag{2.5}$$

However, for $\mu\vdash n$ we have

$$\Pi_\mu(1/q,1/t) \;=\; \prod_{c\in\mu/(o,o)}(1-t^{-l'}q^{=a'}) \;=\; \frac{(-1)^{n-1}}{T_\mu}\Pi_\mu(q,t) \tag{2.6}$$

and another use of 2.4 gives

$$\downarrow\Psi\downarrow\tilde H_\mu[X;q,t] = \frac{(-1)^{n-1}}{T_\mu}\,\Pi_\mu\tilde H_\mu[X;q,t] \;=\; (-1)^{n-1}\Psi\nabla^{-1}\tilde H_\mu[X;q,t]$$

This proves 2.1 b).

We can now apply flip to both sides of the Haglund identity and obtain, using 2.1 a) and b)

$$\nabla^{-1} E_{nk}[X,1/q] \;=\; t^{k-n}(1-q^{-k})(-1)^{n-1}\nabla^{-1}\Psi\downarrow h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}] \qquad 2.7$$

Now we have

$$\begin{aligned}\downarrow h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}] &= h_k[\tfrac{X}{1-1/q}]h_{n-k}[\tfrac{X}{M/qt}]\\ &= h_k[\tfrac{qX}{q-1}]h_{n-k}[\tfrac{qtX}{M}]\\ &= q^n t^{n-k} h_k[\tfrac{X}{q-1}]h_{n-k}[\tfrac{X}{M}]\end{aligned}$$

and 2.7 becomes, canceling ∇^{-1} from both sides,

$$E_{nk}[X,1/q] \;=\; q^n(1-q^{-k})(-1)^{n-1}\Psi h_k[\tfrac{X}{q-1}]h_{n-k}[\tfrac{X}{M}] \qquad 2.8$$

Next we have the expansion

$$\begin{aligned}h_k[\tfrac{X}{q-1}]h_{n-k}[\tfrac{X}{M}] &= h_k[\tfrac{(t-1)X}{M}]h_{n-k}[\tfrac{X}{M}]\\ &= \sum_{\mu\vdash n}\frac{\tilde H_\mu[X;q,t]}{w_\mu}\big\langle \tilde H_\mu\,,\; h_k[\tfrac{(t-1)X}{M}]h_{n-k}[\tfrac{X}{M}]\big\rangle_*\\ &= \sum_{\mu\vdash n}\frac{\tilde H_\mu[X;q,t]}{w_\mu}\big\langle \tilde H_\mu\,,\; e_k[(t-1)X]e_{n-k}[X]\big\rangle\\ &= \sum_{\mu\vdash n}\frac{\tilde H_\mu[X;q,t]}{w_\mu}\big\langle e_k[(t-1)X]^\perp \tilde H_\mu\,,\; e_{n-k}[X]\big\rangle\\ &= \sum_{\mu\vdash n}\frac{\tilde H_\mu[X;q,t]}{w_\mu}\sum_{\nu\subseteq\mu} c_{\mu\nu}^{e_k[(t-1)X]^\perp}\end{aligned}$$

We will now make use of the summation formula given by the following

Theorem 2.1

For $f\in\Lambda^{=k}$ and $\mu\vdash n$ we have

$$\sum_{\substack{\nu\subseteq\mu\\ \nu\vdash k}} c_{\mu\nu}^{f^\perp}(q,t) \;=\; F[D_\mu] \qquad 2.9$$

with

$$F[X] \;=\; \nabla^{-1}\big((\omega f)\big[\tfrac{X-\epsilon}{M}\big]\big)\,. \qquad 2.10$$

For $f=e_k[(t-1)X]$ we get

$$F[X] \;=\; \nabla^{-1}h_k\big[\tfrac{X-\epsilon}{q-1}\big].$$

Taking the $*$-scalar product of both sides of (*) with $\tilde H_\mu$ gives

$$\big\langle \nabla E_{nk}\,,\ \tilde H_\mu\big\rangle_* \;=\; t^{n-k}(1-q^k)\big\langle \Psi h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}]\,,\ \tilde H_\mu\big\rangle_*$$

Thus

$$T_\mu\big\langle E_{nk}\,,\ \tilde H_\mu\big\rangle_* \;=\; t^{n-k}(1-q^k)\Pi_\mu\big\langle h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}]\,,\ \tilde H_\mu\big\rangle_*$$

Using 1.13 the left hand side becomes

$$\begin{aligned} LHS &= T_\mu q^k\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r\big\langle \tilde H_\mu\,,\ e_n\big[X\tfrac{1-q^{-r}}{1-q}\big]\big\rangle_* \\ &= T_\mu q^k\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r\big\langle \tilde H_\mu\,,\ e_n\big[\tfrac{X(1-t)(1-q^{-r})}{M}\big]\big\rangle_* \\ &= T_\mu q^k\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-1)^r\tilde H_\mu\big[M\tfrac{(1-q^{-r})}{(1-q)}\big] \end{aligned}$$

Note that

$$\begin{aligned} \downarrow\begin{bmatrix}k\\ r\end{bmatrix}_q &= \frac{(1-1/q)(1-1/q^2)\cdots(1-1/q^k)}{(1-1/q)(1-1/q^2)\cdots(1-1/q^r)(1-1/q)(1-1/q^2)\cdots(1-1/q^{k-r})} \\ &= \frac{(-1)^k(q,q)_k q^{-\binom{k}{2}-k}}{(-1)^r(q,q)_r q^{-\binom{r}{2}-r}(-1)^{k-r}(q,q)_{k-r}q^{-\binom{k-r}{2}-(k-r)}} \\ &= \begin{bmatrix}k\\ r\end{bmatrix}_q \frac{q^{-\binom{k}{2}}}{q^{-\binom{r}{2}}q^{-\binom{k-r}{2}}} \;=\; \begin{bmatrix}k\\ r\end{bmatrix}_q q^{r^2-kr} \end{aligned}$$

since

$$\begin{aligned} 2\tbinom{k-r}{2}+2\tbinom{r}{2}\,-\,2\tbinom{k}{2} &= (k-r)^2-(k-r)\,+\,r^2-r\,-\,k^2+k \\ &= k^2-2kr+r^2-k+r+r^2-r\,-\,k^2+k \\ &= -2kr+2r^2 \end{aligned}$$

Thus

$$\begin{aligned} \downarrow LHS &= \frac{q^{-k}}{T_\mu}\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{r^2-kr}q^{-r^2/2+r/2}(-1)^r\tilde H_\mu\big[M\tfrac{(1-q^r)}{tq(1-1/q)};1/q,1/t\big] \\ &= \frac{q^{-k}}{T_\mu}\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^r t^{-n}\tilde H_\mu\big[M\tfrac{(1-q^r)}{(q-1)};1/q,1/t\big] \\ &= \frac{q^{-k}}{T_\mu}\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{n-r} t^{-n}\omega\tilde H_\mu\big[M\tfrac{(1-q^r)}{(1-q)};1/q,1/t\big] \\ &= \frac{q^{-k}}{T_\mu^2}\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{n-r} t^{-n}\tilde H_\mu\big[MB_r(q,t);q,t\big] \\ &= \frac{q^{-k}t^{-n}}{T_\mu^2}\Pi_\mu\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{n-r}\frac{h_r\big[(1-t)B_\mu\big](q,q)_r}{(q,q)_{r-1}} \\ &= \frac{q^{-k}t^{-n}}{T_\mu^2}\Pi_\mu\sum_{r=0}^k \begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{n-r}h_r\big[(1-t)B_\mu\big](1-q^r) \end{aligned}$$

$$RHS \;=\; t^{n-k}(1-q^k)\Pi_\mu \big\langle h_k[\tfrac{X}{1-q}]h_{n-k}[\tfrac{X}{M}]\,,\;\tilde H_\mu\big\rangle_* \;=\; t^{n-k}(1-q^k)\Pi_\mu\big\langle \tilde H_\mu\,,\;e_k[(1-t)X]e_{n-k}[X]\big\rangle$$

Thus

$$\begin{aligned}\downarrow RHS \;&=\; t^{k-n}(1-q^{-k})\frac{(-1)^{n-1}\Pi_\mu}{T_\mu}\big\langle \tilde H_\mu[X;1/q,1/t]\,,\;e_k[(1-1/t)X]e_{n-k}[X]\big\rangle\\ &=\; t^{k-n}(1-q^{-k})\frac{(-1)^{n-1}\Pi_\mu}{T_\mu}\big\langle \omega\tilde H_\mu[X;1/q,1/t]\,,\;h_k[(1-1/t)X]h_{n-k}[X]\big\rangle\\ &=\; t^{k-n}(1-q^{-k})t^{-k}\frac{(-1)^{n-1}\Pi_\mu}{T_\mu^2}\big\langle \tilde H_\mu[X;q,t]\,,\;h_k[(t-1)X]h_{n-k}[X]\big\rangle\\ &=\; t^{-n}(1-q^{-k})\frac{(-1)^{n-1-k}\Pi_\mu}{T_\mu^2}\big\langle \tilde H_\mu[X;q,t]\,,\;e_k[(1-t)X]h_{n-k}[X]\big\rangle\end{aligned}$$

So we are reduced to show that

$$\begin{aligned}&\frac{q^{-k}t^{-n}}{T_\mu^2}\Pi_\mu\sum_{r=0}^{k}\begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{n-r}h_r\big[(1-t)B_\mu\big](1-q^r)\\ &\qquad\qquad =\; t^{-n}(1-q^{-k})\frac{(-1)^{n-1-k}\Pi_\mu}{T_\mu^2}\big\langle \tilde H_\mu[X;q,t]\,,\;e_k[(1-t)X]h_{n-k}[X]\big\rangle\end{aligned}$$

or better

$$\begin{aligned}&\sum_{r=0}^{k}\begin{bmatrix}k\\ r\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{r}h_r\big[(1-t)B_\mu\big](1-q^r)\\ &\qquad =\; (1-q^k)(-1)^k\big\langle \tilde H_\mu[X;q,t]\,,\;e_k[(1-t)X]h_{n-k}[X]\big\rangle\\ &\qquad =\; (1-q^k)(-1)^k\sum_{\substack{\nu\subseteq\mu\\ \nu\vdash n-k}} c_{\mu\nu}^{e_k[(1-t)X]^\perp}\end{aligned}$$

or better yet

$$\sum_{r=1}^{k}\begin{bmatrix}k-1\\ r-1\end{bmatrix}_q q^{\binom{r+1}{2}-kr}(-1)^{k-r}h_r\big[(1-t)B_\mu\big] \;=\; \sum_{\substack{\nu\subseteq\mu\\ \nu\vdash n-k}} c_{\mu\nu}^{e_k[(1-t)X]^\perp}$$

To show this we will evaluate the RHS by means of Theorem 2.1. That is

$$\sum_{\substack{\nu\subseteq\mu\\ \nu\vdash n-k}} c_{\mu\nu}^{e_k[(1-t)X]^\perp} \;=\; F[D_\mu] \tag{2.11}$$

with

$$F[X] \;=\; \nabla^{-1}\big((\omega f)\big[\tfrac{X-\epsilon}{M}\big]\big)\,. \tag{2.12}$$

and $f = e_k[(1-t)X]$. That gives

$$\begin{aligned} F[X] &= \nabla^{-1} h_k\big[\tfrac{X-\epsilon}{1-q}\big] \\ &= \nabla^{-1}\sum_{s=0}^{k} h_{k-s}\big[\tfrac{X}{1-q}\big] h_s\big[\tfrac{-\epsilon}{1-q}\big] \;=\; \sum_{s=0}^{k} q^{-\binom{k-s}{2}} h_{k-s}\big[\tfrac{X}{1-q}\big] e_s\big[\tfrac{1}{1-q}\big] \\ &= \sum_{s=0}^{k} q^{-\binom{k-s}{2}+\binom{s}{2}} h_{k-s}\big[\tfrac{X}{1-q}\big] \tfrac{1}{(q,q)_s} \end{aligned}$$

Since

$$-2\binom{k-s}{2} + 2\binom{s}{2} \;=\; -k^2+2ks-s^2+k-s+s^2-s \;=\; -k^2+2ks+k-2s$$

we may write

$$\begin{aligned} F[D_\mu] &= \sum_{s=0}^{k} q^{-\binom{k}{2}+ks-s} h_{k-s}\big[\tfrac{MB_\mu-1}{1-q}\big]\tfrac{1}{(q,q)_s} \\ &= \sum_{s=0}^{k} \frac{q^{-\binom{k}{2}+ks-s}}{(q,q)_s}\sum_{r=0}^{k-s} h_r\big[(1-t)B_\mu\big] h_{k-s-r}\big[\tfrac{-1}{1-q}\big] \\ &= \sum_{s=0}^{k} \frac{q^{-\binom{k}{2}+ks-s}}{(q,q)_s}\sum_{r=0}^{k-s} h_r\big[(1-t)B_\mu\big] (-1)^{k-s-r}\frac{q^{\binom{k-s-r}{2}}}{(q,q)_{k-s-r}} \end{aligned}$$

but

$$2\binom{k-s-r}{2} \;=\; (k-s-r)(k-s-r)-k+s+r \;=\; k^2-2sk-2rk+s^2+2sr+r^2-k+s+r$$

$$\binom{k-s-r}{2} \;=\; \binom{k}{2}+\binom{s+1}{2}-sk-rk+sr+\binom{r+1}{2}$$

So

$$\binom{k-s-r}{2}-\binom{k}{2}+ks-s \;=\; \binom{k}{2}+\binom{s+1}{2}-sk-rk+sr+\binom{r+1}{2}-\binom{k}{2}+ks-s$$

or

$$\binom{k-s-r}{2}-\binom{k}{2}+ks-s \;=\; \binom{s}{2}-rk+sr+\binom{r+1}{2}$$

and we get

$$F[D_\mu] \;=\; \sum_{r=0}^{k}\frac{(-1)^{k-r}q^{\binom{r+1}{2}-rk}}{(q,q)_{k-r}} h_r\big[(1-t)B_\mu\big]\sum_{s=0}^{k-r}\begin{bmatrix}k-r\\ s\end{bmatrix}_q q^{\binom{s}{2}}(-1)^s(q^r)^s$$