Multiplicativity of the $\mathbf{Q_{m,n}}$ operators at $\mathbf{t=1}$

Francois asked for a proof of the following result

Theorem 1

For all pairs m, n and all $F \in \Lambda$ we have

$$Q^{t=1}_{m,n} F \;=\; Q^{t=1}_{m,n}\, 1 \times F \tag{1}$$

The proof will follow from a few preliminary remarks and auxiliary facts.

To begin we will observe that since by definition

$$D_k F[X] \;=\; F[X + \tfrac{M}{z}] \sum_{i \geq 0} (-z)^i e_i[X]\Big|_{z^k} \qquad \text{(with } M = (1-t)(1-q)\text{)} \tag{2}$$

Setting $t = 1$ gives

$$D^{t=1}_k F[X] \;=\; (-1)^k \underline{e}_k F[X] \tag{3}$$

To use this fact in combination with the bracketing definition of the operators $Q_{m,n}$ runs into the messy problem of passing to the limit as $t \to 1$ of expressions that become $\frac{0}{0}$ at $t = 1$. To avoid this problem we will use Francois ' construction of the operators $Q_{m,n}$ whose validity was established in our joint paper "*Some remarkable new Plethystic Operators...*".

This construction was stated in our joint paper "*A Compositional (km,kn)-Shuffle Conjecture*". To avoid limiting problems, we will take the following version at $t = 1$ of the above mentioned construction.

Francois's recursive Algorithm at $\mathbf{t=1}$ to obtain $\mathbf{Q^{t=1}_{u,v} F[X]}$

Given a pair (u, v) of positive integers and $F[X] \in \Lambda$

If $u = 1$ *output* $= (-1)^v \underline{e}_v F[X]$.

else if $u < v$ *output* $= (S\mathbf{Q}^{t=1}_{u,v-u}) F[X]$,

else if $u > v$ *output* $= (N\mathbf{Q}^{t=1}_{u-v,v}) F[X]$,

else $u = v$ *output* $= \mathbf{Q}^{t=1}_{u,u} F[X]$,

To complete this construction observe first that the algebra $\mathcal{A}$ generated by the D_k operators at $t = 1$ is none other that the algebra of "*symmetric function multiplication operators*" . Using the elementary basis, every operator $U \in \mathcal{A}^{t=1}$ can be uniquely written in the form

$$\mathbf{U} \;=\; \sum_{\lambda} c_\lambda(q)\; \underline{e}_\lambda \tag{4}$$

This given we define

$$\begin{aligned}
a) \quad & S\mathbf{U} \;=\; \sum_{\lambda} c_\lambda(q)\; \underline{e}_\lambda \Big|_{\underline{e}_k \to \underline{e}_{k+1}} \\
b) \quad & N\mathbf{U} \;=\; \nabla^{t=1} \mathbf{U}\; (\nabla^{t=1})^{-1} \\
c) \quad & \mathbf{Q}_{u,u} \;=\; \tfrac{q}{q-1} \nabla^{t=1} \underline{h}_u[X(1/q-1)]\; (\nabla^{t=1})^{-1}
\end{aligned}$$

Keeping all this in mind, the only thing that is needed here to complete the proof of Theorem 1, accepting the above algorithm as definition of the operators $\mathbf{Q}^{t=1}_{u,v}$ is the following basic fact.

Proposition 1

The algebra $\mathcal{A}^{t=1}$ is invariant under conjugation by conjugation by $\nabla^{t=1}$

Proof

The idea is to note that $\downarrow\nabla$ defines an involution on the space of symmetric polynomials, (a fact which we proved in the 90's), which in the particular case $t = 1$ becomes "*muliplicative*". In the even more particular case $t, q = 1$ this involution is explicitely defined in Macdonald's book (page 34). In our case the action of operator $L = \downarrow\nabla\Big|_{t,q=1}$ is best obtained as follows.

Set

$$F(z) \;=\; \sum_{n\geq 0} z^n e_n \tag{5}$$

and let $f(z)$ be the compositional inverse of $F(z)$. Since the coefficients of $f(z)$ are polynomials in the $e'_k s$ they constitute a family of symmetric function we denote $\{e^*_k\}_k$, with $e^*_k \in \Lambda^{=k}$. This given, we can simply set, for $\mathbf{U}$ as in (4)

$$L\,\mathbf{U} \;=\; \sum_{\lambda} c_\lambda(q)\; \underline{e}_\lambda\Big|_{\underline{e}_k \to \underline{e}^*_k}$$

This gives the explicit form

$$\nabla^{t,q=1}\mathbf{U}\nabla(\nabla^{t,q=1})^{-1} \;=\; \omega L\mathbf{U}\omega L$$

In the case just $t = 1$ the result is a simple consequence of the well known multiplicativity of ∇ at $t = 1$, but for a more explicit version of its action, we need to use the q machinery I developed in my q-Lagrange paper. I will do this addendum shortly.