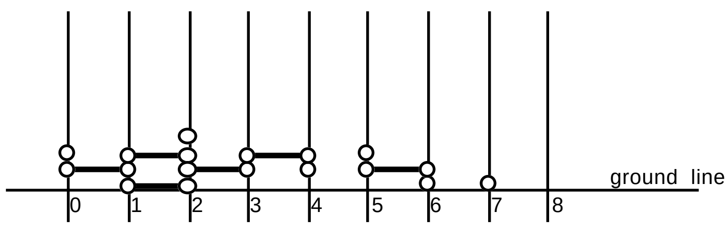
HEAPS, CONTINUED FRACTIONS
and
ORTHOGONAL POLYNOMIALS

Lecture Notes in Algebraic Combinatorics

ABSTRACT. These notes are an exposition of work of Françon-Viennot [5], Flajolet [4] and Viennot [8],[9] dealing with combinatorial aspects of the theory of continued fractions. We have added here some of the missing details, some of the proofs that where not given in the original works, and hopefully have corrected some of the errors without creating new ones. We also give a brief review of the classical connection between orthogonal polynomials and continued fractions.

1. Heaps of monomers and dimers

Before making definitions it is best to start with an example. In the figure below we have an instance of a *monomer-dimer heap*



1.1

Let us imagine that the vertical lines represent needles and that there are two basic pieces:

A *monomer*: Which is just a ring, and
A *dimer* which consists of two rings joined by a metal bar.

To put together a heap we simply pick a bunch of monomers and dimers and stack them on top of each other by threading them into the needles as indicated in figure 1.1 above. As it is indicated there the needles are perpendicular to the ground line at the points of coordinates $0, 1, 2, 3 \ldots$. Heaps of monomers and dimers will be represented here by words in the alphabet

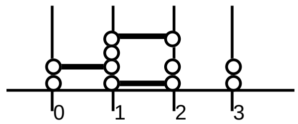
$$\mathcal{A} \;=\; \{\, m_o, m_1, m_2, \ldots \,,\, d_1, d_2, d_3, \ldots \}$$

by replacing each monomer of ground coordinate i by the letter m_i and each dimer projecting onto the interval $[i-1, i]$ by the letter d_i. The corresponding word is obtained by processing in this manner the successive pieces of the heap from left to right within a row, starting from the bottom row and proceeding upwards. For instance, this procedure applied to the heap of figure 1.1 yields the word

$$w \;=\; d_2 m_6 m_7 d_1 d_3 m_4 d_6 m_o d_2 d_4 m_5 m_2 \;\;.$$

Conversely, given any word $w \in \mathcal{A}^*$ we can construct a heap by reversing the process above. That is we read the letters of w from left to right and replace each m_i by a monomer of ground coordinate

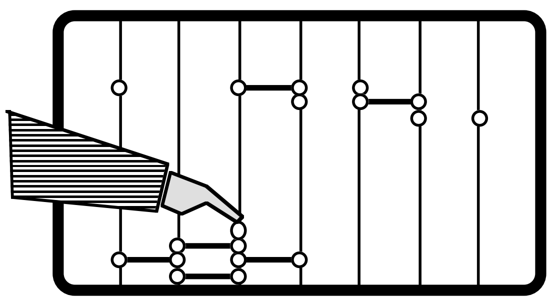
i and each d_i by a dimer spanning $[i-1, i]$. Of course we must also thread the corresponding monomers and dimers down the needles in the precise succession they are encountered as we read w. The final configuration is obtained by letting the pieces settle as far down as they can. This proceedure applied to the word $w_1 = m_o d_2 m_2 d_1 m_1 d_2 m_3 m_3$ produces the heap



1.2

Now it is easy to see that the word $w_2 = m_3 d_2 m_o d_1 m_3 m_1 m_2 d_2$ produces the same heap. At the same time the word which corresponds to this heap by the construction given above shoud be $w = m_o d_2 m_3 d_1 m_2 m_3 m_1 d_2$. Mathematically speaking a heap should be represented by an equivalence class of $\mathcal{A}$-words. Two words beeing equivalent if and only if they yield the same heap. Thus our procedure of constructing the word corresponding to a heap is just one of the ways of selecting a representative from each equivalence class of words.

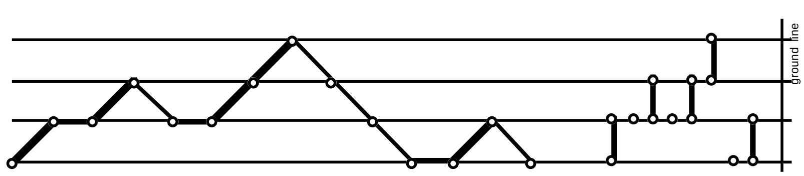
Imagine now that the needles of figure 1.1 are set into a top and bottom bar as in an abacus and we pull down the monomer m_2. This will result in the configuation



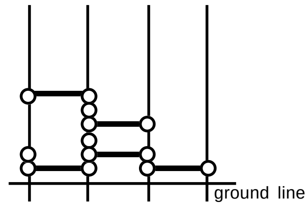
1.3

We can see that there are heaps that are brought down by pulling on a single piece. Such is also the case for the heap obtained by removing the monomers m_o and m_7 from the heap in 1.3 and adding on top the dimer d_4. Such a heap will be called a *pyramid* and the top piece the *summit* of the pyramid.

There is a very simple way of transforming Motzkin paths into heaps which, as we shall see, has remarkable mathematical consequences. This transformation is obtained by replacing each EAST step in the path by a monomer and each NORTH-EAST step by a dimer. We only need one example here to get accross what we have in mind. For instance, carrying out these replacements (from left to right) on the path on the left yields the configuration on the right.



the latter is then rotated 90^o clockwise so that the pieces settle down to the ground. This results in the heap given below



The transformation is even simpler at the word level. In fact, the a,b,c-word corresponding to the Motzkin path is $w = a_1c_1a_2b_1c_1a_2a_3b_2b_1b_0c_ob_1b_o$ while the word of the heap is $w' = d_1m_1d_2m_1d_2d_3m_0d_1$. We see that to go from the word w of the path to the word w' of the corresponding heap, we simply read the letters of w from right to left, replace each a by a d, each c by an m and remove all $b\,'s$. From our example we can easily extract that

Theorem 1.1

Our construction yields a bijection between Motzkin paths and heaps of monomers and dimers with the following properties.

1) *The image heap is always a pyramid with summit a monomer m_o or a dimer d_1.*
2) *Paths whose maximum height does not exceed n correspond to pyramids whose projection is in the interval $[0,n]$.*
3) *Dyck paths are sent into pyramids of dimers with summit d_1*
4) *If the image pyramid has d dimers and m monomers then the corresponding path has $2d+m$ steps*

2. The Cartier-Foata Languages

There is a purely algebraic way to define the equivalence classes of words yielding the same monomer-dimer heap. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be an alphabet and let $M = \|m_{ij}\|$ be a matrix with entries $0,1$ with the properties

$$\begin{aligned} &1) \quad m_{ii} = 1 \quad , \\ &2) \quad m_{ij} = m_{ji} \quad . \end{aligned} \qquad 2.1$$

Let us say that two letters a_i & a_j commute if and only if $m_{ij} = 0$. Given two words $w_1, w_2 \in \mathcal{A}^*$ let us write

$$w_1 \longleftrightarrow_M w_2$$

if and only if we have the two factorizations

$$w_1 = w'\, a_ia_j\, w'' \quad , \quad w_2 = w'\, a_ja_i\, w''$$

with a_i and a_j a pair of commuting letters. Finally, let us write $w_1 \approx_M w_2$ if and only if we have a sequence of words $\sigma^{(i)} \in \mathcal{A}^*$ $(1 = 1..n)$ such that

$$w_1 = \sigma^{(1)} \longleftrightarrow \sigma^{(2)} \longleftrightarrow \cdots \longleftrightarrow \sigma^{(n-1)} \longleftrightarrow \sigma^{(n)} = w_2$$

In words $w_1 \approx_M w_2$ if w_2 can be obtained from w_1 by a sequence of transpositions of pairs of adjacent commuting letters. The quotient of $\mathcal{A}^*$ by the equivalence "$\approx_M$" will be referred to as the *Cartier-Foata language* corresponding to the pair $(\mathcal{A}, M)$. Clearly the equivalence $\approx_M$ is compatible with concatenation of words. That is if $w_1 \approx_M w_1'$ and $w_2 \approx_M w_2'$ then $w_1 w_2 \approx_M w_1' w_2'$. Thus the concatenation product gives $\mathcal{A}^*/\approx_M$ the structure of a monoid.

Given a pair $(\mathcal{A}, M)$ it will be convenient to denote by $\mathcal{L}(\mathcal{A}, M)$ the formal sum of a suitably selected set of representatives for the equivalence classes of $\mathcal{A}^*$ modulo $\approx_M$. When dealing with a Cartier-Foata language, all identities between formal sums of words of $\mathcal{A}^*$ where the equal sign is subscripted by M should be understood to hold modulo the equivalence $\approx_M$. Clearly, to define a Cartier-Foata language we need only specify which are the commuting pairs. Indeed, this is the only information yielded by the matrix M. This given, we can easily see that if we take

$$\mathcal{A} \;=\; \{m_o, m_1, m_2, \ldots \,:\, d_o, d_1, d_2 \ldots\} \tag{2.1}$$

and declare that the commuting pairs are

$$\begin{aligned} &(1) \quad (m_i, m_j) \quad for \quad i \neq j \quad , \\ &(2) \quad (m_i, d_j) \quad for \quad \{i\} \cap \{j-1, j\} = \emptyset \quad , \\ &(3) \quad (d_i, d_j) \quad for \quad \{i-1, i\} \cap \{j-1, j\} = \emptyset \quad , \end{aligned} \tag{2.2}$$

then the corresponding Cartier-Foata language may be identified with the collection of heaps of monomers and dimers.

As we shall see we can associate to each Cartier-Foata language $\mathcal{L}(\mathcal{A}, M)$ a collection of heap-like objects which may be represented by the words of $\mathcal{L}(\mathcal{A}, M)$. Nevertheless, these languages are of independent interest and it will be good to start by establishing some of the basic identities concerning them. To this end we need to make some definitions.

For a given word $w \in \mathcal{L}(\mathcal{A}, M)$ set

$$I(w) \;=\; \{a \in \mathcal{A} \,:\, w \approx_M a w' \} \tag{2.3}$$

In words $I(w)$ represents the set of letters that may be brought to the front of w by a sequence of transpositions of pairs of adjacent commuting letters. For this reason we shall refer to $I(w)$ as the *initial part* of w. Similarly we define as the *end part* of w the set

$$E(w) \;=\; \{b \in \mathcal{A} \,:\, w \approx_M w' b \} \;. \tag{2.4}$$

In the monomer-dimer case (that is when the language is given by 2.1 and 2.2) a word w corresponds to a pyramid if and only if $E(w)$ has only one element. It will be convenient to extend our heap terminology to the general Cartier-Foata language and call *pyramid* any word whose $E(w)$ is a singleton. Note that if $E(w)$ is not a singleton the any two letters occurring in $E(w)$ must form a commuting pair. Of course the same holds true for $I(w)$. Subsets of $\mathcal{A}$ consisting of pairwise commuting letters shall be referred to as *free* subsets. their collection will be denoted by $\mathcal{F}(\mathcal{A},M)$. Given a free set F we shall also denote by F the word obtained by multiplying the letters of F in alphabetic order. Of course since all pairs of letters of F commute, whatever order we use produces the same element of the language so this abuse of notation cannot lead to ambiguities. With these conventions, the basic result of the theory of Cartier-Foata languages may be stated as follows as follows:

Theorem 2.1 (Cartier-Foata [3])

Modulo the equivalence $\approx_M$ the generating series of the language may be computed from the following identity

$$\mathcal{L}(\mathcal{A},M) \;\; =_M \;\; \frac{1}{\sum_{F\in\mathcal{F}(\mathcal{A},M)} (-1)^{|F|}\, F} \tag{2.5}$$

where $|F|$ denotes here the cardinality of F.

Proof

We only need to show that the product of the left hand side by the denominator of the right hand side evaluates to the empty word. To this end we write this product in the form

$$\sum_{w\in\mathcal{A}^*} w \sum_{F\in\mathcal{F}(\mathcal{A},M)} \sum_{v\in\mathcal{L}(\mathcal{A},M)} (-1)^{|F|}\, \chi(w \approx_M F\, v) \;\; . \tag{2.6}$$

Now, since F is a free set, the condition that $w = Fv$ is equivalent to the statement that $F \subseteq I(w)$. Thus we see that the inner sum may be simply rewritten as

$$\sum_{F\subseteq I(w)} (-1)^{|F|}$$

and this clearly evaluates to zero unless of course w itself reduces to the empty word. This establishes our identity.

This result has a number of interesting corollaries and applications. For instance let $c_n = c_n(\mathcal{A},M)$ denote the number of words of length n in $\mathcal{L}(\mathcal{A},M)$ and let $F_{\mathcal{A},M}(t)$ denote the generating functions of the sequence $\{c_n\}$ then from 2.6 we immediately deduce that

Proposition 2.1

$$F_{\mathcal{A},M}(t) \;\; = \;\; \frac{1}{\sum_{F\in\mathcal{F}(\mathcal{A},M)} (-t)^{|F|}} \tag{2.7}$$

In particular, for heaps of monomer and dimers on the interval $[0,2]$ there are twelve free sets. Namely, the empty set, the singletons

$$\{m_o\},\{m_1\},\{m_2\},\{d_1\},\{d_2\}$$

the pairs

$$\{m_o,m_1\},\{m_o,m_2\},\{m_1,m_2\},\{m_o,d_2\},\{d_1,m_2\}$$

and the triplet $\{m_o, m_1, m_2\}$. This gives, that in this case

$$F_{\mathcal{A},M}(t) \;=\; \frac{1}{1-6t+5t^2-t^3}\;. \tag{2.8}$$

A curious feature of the identity in 2.7 is that it provides a tool for proving that certain rational functions have Taylor series with non negative integral coefficients, a fact that, even in such a simple instance as 2.8, is not easily established.

Theorem 2.1 has a very useful refinement. Given a subalphabet $\mathcal{B} \subseteq \mathcal{A}$ we shall let $\mathcal{L}(\mathcal{B}, M)$ denote the sublanguage consisting of the words of $\mathcal{L}(\mathcal{A}, M)$ all of whose letters are in $\mathcal{B}$. At the same time we shall denote by $\mathcal{L}^{(\mathcal{B})}(\mathcal{A}, M)$ the subset of $\mathcal{L}(\mathcal{A}, M)$ consisting of all the words of $\mathcal{L}(\mathcal{A}, M)$ whose end part lies entirely in $\mathcal{B}$. This given, we have the following identity.

Theorem 2.2 (De Sainte-Catherine [9])

$$\mathcal{L}^{(\mathcal{B})}(\mathcal{A}, M) \;=_M\; \frac{\sum_{F\in\mathcal{F}(\mathcal{A}-\mathcal{B},M)} (-1)^{|F|}\,F}{\sum_{F\in\mathcal{F}(\mathcal{A},M)} (-1)^{|F|}\,F}\;. \tag{2.9}$$

Proof

Using theorem 2.1 (twice) this identity may be rewritten in the form

$$\mathcal{L}^{(\mathcal{B})}(\mathcal{A}, M)\;\times\;\mathcal{L}(\mathcal{A}-\mathcal{B}, M)\;=_M\;\mathcal{L}(\mathcal{A}, M)\;. \tag{2.10}$$

In turn this is equivalent to stating that every word of $\mathcal{L}(\mathcal{A}, M)$ may be uniquely factorized (modulo $\approx_M$) into the concatenation product of a word of $\mathcal{L}^{(\mathcal{B})}(\mathcal{A}, M)$ by a word of $\mathcal{L}(\mathcal{A}-\mathcal{B}, M)$. To see that this is indeed so, imagine that the letters of a word $w \in \mathcal{L}(\mathcal{A}, M)$ are strung as beads into a long string. Imagine further that pairs of letters that commute can *pass through each other*. This given the desired factorization can simply be obtained by pulling to the left all the *beads* of w that are in $\mathcal{B}$. This splits w into two words w_1 and w_2. Where w_1 consists of all the beads that had to move as a result of the pull and w_2 consists of all the beads to the right of the beads in $\mathcal{B}$ plus those that passed through them. We can also construct this factorization by a more standard mathematical argument. Let $w = a_1a_2\cdots a_ia_{i+1}\cdots a_n$ where a_i is the rightmost letter of w that is in $\mathcal{B}$. Then let $u = a_1a_2\cdots a_i$ and $v = a_{i+1}\cdots a_n$. We can now factorize the end part of u (modulo $\approx_M$) in the form $E(u) = \beta\rho$ where β consists of all the letters of $E(u)$ that are in $\mathcal{B}$ and ρ has all the remaining letters. Now, if $u \approx_M u'E(u)$ we can set $w_1 = u'\beta$ and $w_2 = \rho v$ and obtain that $w \approx_M w_1w_2$ with $E(w_1) \in \mathcal{B}^*$ and $w_2 \in (\mathcal{A}-\mathcal{B})^*$ as desired. The uniqueness is clear, since if $w \approx_M w_1'w_2'$ with $E(w_1') \in \mathcal{B}^*$ and $w_2' \in (\mathcal{A}-\mathcal{B})^*$ then we must necessarily have that $E(w_1') = \beta$, for (in passing from w to $w_1'w_2'$) no letter of u' can be moved past a letter of β. But this yields that $w_1' = u'\beta$ and $w_2' = \rho v$.

It will be convenient to denote by $\mathcal{P}(\mathcal{A}, M)$ the collection of pyramids of the Cartier-Foata language $\mathcal{L}(\mathcal{A}, M)$. It develops that pyramids are the building blocks of the language. The precise meaning of this assertion will emerge in the proof of the following result. To begin with let us use the symbol "$\equiv_c$" to denote equality of two formal sums of words, under the condition that the letters are allowed to unconditionally commute. In other words ordinary equality in the formal power series sense should hold if each word on either side is replaced by a monomial in which the letters appearing in the word are rearranged in increasing order. This given, we have the following remarkable identity

Theorem 2.3 (Viennot [9])

$$\sum_{\pi\in\mathcal{P}(\mathcal{A},M)} \frac{\pi}{|\pi|} \equiv_c log\big(\, \mathcal{L}(\mathcal{A},M)\,\big) \qquad 2.11$$

where again $|\pi|$ denotes the length of π.

Proof

This turns out to be a beautiful application of the theory of exponential structures. Given a word $w = a_1, a_2, \cdots a_n \in \mathcal{L}(\mathcal{A},M)$, and a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ we set

$$P(w,\sigma) \;=\; \binom{a_1}{\sigma_1}\binom{a_2}{\sigma_2}\cdots\binom{a_n}{\sigma_n} \qquad 2.12$$

and consider it as a word in the alphabet $P(\mathcal{A})$ whose letters are the pairs $\binom{a}{k}$ with $a \in \mathcal{A}$ and $k = 1, 2, ...$ We then define a new Cartier-Foata language $\mathcal{L}(P(\mathcal{A}),M)$ by letting two letters $\binom{a_i}{h}$ and $\binom{a_j}{k}$ of $P(\mathcal{A})$ commute if and only if a_i and a_j themselves commute (that is $m_{ij} = 0$). Using the *bead* imagery of the previous proof, we can factorize the word $P(w,\sigma)$ into a product of pyramids of the language $\mathcal{L}(P(\mathcal{A}),M)$. We simply thread the letters of $P(w,\sigma)$ on an infinite string, then pull to the left the letter $\binom{a_{j_1}}{1}$ of $P(w,\sigma)$. This will cause a number of letters of $P(w,\sigma)$ to move to the left as well. Again, a number of letters of $P(w,\sigma)$ that were originally to the left of $\binom{a_{j_1}}{1}$ will stay put. Those are the remaining letters of the end part of the left factor of $P(w,\sigma)$ terminanting with $\binom{a_{j_1}}{1}$. By our convention those *beads* allow $\binom{a_{j_1}}{1}$ to *pass through them.* The letters that move with $\binom{a_{j_1}}{1}$ form a word that we would like to denote by $P(w_1,E_1)$. To do this we only need to extend our notation by allowing σ in 2.12 to be any sequence of natural numbers of the same length as w. Next, we set $i_1 = 1$ and let i_2 equal to the smallest integer in $\{1, 2, \ldots, n\}$ that is not in E_1. Now we locate the bead $\binom{a_{j_2}}{i_2}$ of $P(w,\sigma)$ and pull to the left on it moving with it every bead that cant let $\binom{a_{j_2}}{i_2}$ go through. This produces another factor $P(w_2,E_2)$. We can then let i_3 be the smallest integer in $\{1, 2, \ldots, n\}$ that is not in $E_1 + E_2$. Then drag to the left the letter $\binom{a_{j_3}}{i_3}$ of $P(w,\sigma)$. In this manner we successively construct the factorization

$$P(w,\sigma) \;\approx_M\; P(w_1,E_1)P(w_2,E_2)P(w_3,E_3)\cdots P(w_k,E_k)\;\; . \qquad 2.13$$

Our proceedure guarantees the following properties

(1) Each factor $P(w_i,E_i)$ is a pyramid of $\mathcal{L}(\mathcal{A},M)$
(2) $i_1 = min\; E_1 \;<\; i_2 = min\; E_2 \;<\; \cdots i_k = min\; E_k$
(3) $\{1, 2, \ldots, n\} = E_1{+}E_2{+}\cdots{+}E_k$, that is $(E_1, E_2, \ldots, E_k)$ is a partition of the set $\{1, 2, \ldots, n\}$.

Let now $\Pi_k = \Pi_k(\mathcal{A},M)$ denote the collection of words $P(\pi,\tau)$ where

i) π is a pyramid of $\mathcal{L}(\mathcal{A},M)$ of length k.

ii) τ is a permutation in S_k with $\tau_k = 1$

This given we see that the factorization in 2.13 exhibits the collection

$$\Big\{ \; P(w,\sigma) \;\; : \;\; w \in \mathcal{L}(\mathcal{A}, M) \;\; , \;\; \sigma \in S_n \; \Big\}$$

as the exponential class constructed from the families Π_k ($k = 1, 2, ..$). From the theory of exponential classes we then immediately derive that

$$\sum_{w \in \mathcal{L}(\mathcal{A},M) \ \& \ \sigma \in S_n} \frac{P(w,\sigma)}{|w|!} \;\; \equiv_c \;\; exp\Big(\sum_{k \geq 1} \; \sum_{P(\pi,\tau) \in \Pi_k} \frac{P(\pi,\tau)}{k!} \;\Big) \qquad 2.14$$

To derive the identity in 2.12 we simply observe that the projection $P(w,\sigma) \rightarrow w$ removes the denominator $|w|!$ from 2.14, while the projection $P(\pi,\tau) \rightarrow \pi$, because of the restriction $\tau_k = 1$, only succeeds in reducing the denominator $k!$ to just k itself. Making these replacements in 2.14 yields

$$\sum_{w \in \mathcal{L}(\mathcal{A},M)} w \;\; \equiv_c \;\; exp\Big(\sum_{k \geq 1} \; \sum_{\pi \in \mathcal{P}(\mathcal{A},M) \ \& \ |\pi| = k} \frac{\pi}{k} \;\Big) \;\; ,$$

and 2.11 follows by equating the logarithms of both sides.

3. Orthogonal polynomials and continued fractions.

Let $d\mu$ be a unit measure on $(-\infty, +\infty)$, and set

$$\int_{-\infty}^{+\infty} x^n \, d\mu \;\; = \;\; \mu_n \qquad\qquad (\; \mu_o = 1 \;) \qquad 3.1$$

The $\{\mu_n\}\,'s$ are usually referred to as the *moments* of $d\mu$. They are well defined for instance in the case that $d\mu$ is supported by some finite interval $[a,b]$, that is when $\mu(x) = 0$ for $x \leq a$ and $\mu(x) = \mu(b)$ for $x \geq b$). This permits us to define a scalar product on the space of polynomials in x by setting for any two polynomials $P(x), Q(x)$

$$\langle \; P \; , \; Q \; \rangle_\mu \;\; = \;\; \int_{-\infty}^{+\infty} P(x) Q(x) \, d\mu \;\; . \qquad 3.2$$

It is easily seen that to compute this scalar product we only need to know the sequence of moments. Our object of study here is the sequence of polynomials $\{Q_n(x)\}_{n \geq 0}$ satisfying the following two conditions

$$\begin{aligned} &1) \quad Q_n(x) \;\; = \;\; x^n \; + \; \sum_{k=0}^{n-1} a_{n,k} \, x^k \\ &2) \quad \langle \; Q_n \; , \; Q_m \; \rangle_\mu \;\; = \;\; 0 \quad if \quad n \neq m \;\; . \end{aligned} \qquad 3.3$$

It develops that these polynomials have explicit expressions as ratios of determinants involving the $\mu_n\,'s$. More precisely we have

$$Q_n(x) \;\; = \;\; \frac{1}{d_{n-1}} \, det \begin{pmatrix} \mu_o & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix} \qquad 3.4$$

where

$$d_n \;=\; det \begin{pmatrix} \mu_o & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} \end{pmatrix} \tag{3.5}$$

To prove this, it is sufficient to note that 3.4 gives

$$\langle\, Q_n \,,\, x^i \,\rangle = \frac{1}{d_{n-1}}\, det \begin{pmatrix} \mu_o & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ \mu_i & \mu_{i+1} & \mu_{i+2} & \cdots & \mu_{i+n} \end{pmatrix} = \begin{cases} 0 & \text{if } i<n \\ \frac{d_n}{d_{n-1}} & \text{if } i=n \end{cases} \tag{3.6}$$

In particular for any polynomial $P(x)$ we have the expansion

$$P(x) \;=\; \sum_{i\geq 0} \frac{\langle\, P \,,\, Q_i \,\rangle_\mu}{\langle\, Q_i \,,\, Q_i \,\rangle_\mu}\, Q_i(x) \;\;, \tag{3.7}$$

We should note that, because of 3.3 1) and 2), the last non vanishing summand on the right hand side of 3.7, is given by $i = degree(P)$. Note further that we must have

$$\langle\, xQ_n \,,\, x^i \,\rangle_\mu \;=\; \langle\, Q_n \,,\, x^{i+1} \,\rangle_\mu \;=\; 0 \qquad\qquad (\;for\quad i=0,1,\ldots,n-2\;)\;.$$

and this implies that must also have

$$\langle\, xQ_n \,,\, Q_i \,\rangle_\mu \;=\; 0$$

for all $i \leq n-2$. Thus, when $P(x) = xQ_n(x)$ the expansion in 3.7 reduces to

$$\begin{aligned} xQ_n(x) \;=\; & \tfrac{\langle\, xQ_n \,,\, Q_{n-1} \,\rangle_\mu}{\langle\, Q_{n-1} \,,\, Q_{n-1} \,\rangle_\mu}\; Q_{n-1}(x) \;+ \\ & + \tfrac{\langle\, xQ_n \,,\, Q_n \,\rangle_\mu}{\langle\, Q_n \,,\, Q_n \,\rangle_\mu}\; Q_n(x) \;+ \\ & + \tfrac{\langle\, xQ_n \,,\, Q_{n+1} \,\rangle_\mu}{\langle\, Q_{n+1} \,,\, Q_{n+1} \,\rangle_\mu}\; Q_{n+1}(x) \;\;. \end{aligned} \tag{3.8}$$

Now, from 3.2 1) and 3.6 we get that

$$\langle\, xQ_n \,,\, Q_{n+1} \,\rangle_\mu = \langle\, x^{n+1} \,,\, Q_{n+1} \,\rangle_\mu = \langle\, Q_{n+1} \,,\, Q_{n+1} \,\rangle_\mu \;\;. \tag{3.9}$$

Thus, setting

$$\lambda_n \;=\; \frac{\int_{-\infty}^{+\infty}\; xQ_{n-1}(x)Q_n(x)\, d\mu}{\int_{-\infty}^{+\infty}\; Q_{n-1}^2(x)\, d\mu} \quad,\quad c_n \;=\; \frac{\int_{-\infty}^{+\infty}\; xQ_n^2(x)\, d\mu}{\int_{-\infty}^{+\infty}\; Q_n^2(x)\, d\mu} \;\;. \tag{3.10}$$

we can rewrite 3.9 in the form

$$xQ_n(x) = \lambda_n\, Q_{n-1}(x) + c_nQ_n(x) + Q_{n+1}(x) \;\;,$$

or better yet

$$Q_{n+1} = (x - c_n)Q_n - \lambda_n \, Q_{n-1} \ . \qquad 3.11$$

It is customary, in the theory of orthogonal polynomials to refer to this relation as the *three-term recursion* for the sequence $\{Q_n\}_{n\geq 0}$. Infact, starting from the trivial initial conditions

$$Q_{-1}(x) \equiv 0 \qquad , \qquad Q_o(x) \equiv 1 \qquad 3.12$$

we can recursively express each Q_n as a polynomial in x and the parameters $c_0, c_2, \ldots, c_{n-1}$; $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. It develops that for most classical sequences of polynomials the recursion in 3.12 was known and could be easily derived before any measure $d\mu$ or sequence of moments was associated with them. Now, there is a theorem, usually attributed to Favard [8], which asserts that a three term recursion is actually the necessary and sufficient condition for a sequence of polynomials $\{Q_n\}_{n\geq 0}$, satisfying 1) of 3.3, to be "orthogonal" with respect to "a sequence of moments". To be more precise, we should note that, given an arbitrary sequence $\{\mu_n\}_{n\geq 0}$, we can define a linear functional L_μ acting on polynomials in x by setting

$$L_\mu(x^n) \ = \ \mu_n \qquad (\ for \quad n = 0, 1, 2, \ldots \ ) \qquad 3.13$$

we can then define a "scalar product" by setting

$$\langle \ P \ , \ Q \ \rangle_\mu \ = \ L_\mu(PQ) \ . \qquad 3.14$$

This given, Favard's theorem says that if the $Q_n\,'s$ satisfy the recurrence 3.12 and the initial conditions 3.12 then they must satisfy the relations in 3.3 with a scalar product $\langle P, Q\rangle_\mu$ given by 3.14, 3.13 and a suitable sequence of $\mu_n\,'s$. Given the $\mu_n\,'s$, the existence of a "measure" $d\mu$ satisfying 3.1 is usually referred to as the *moment problem* and it may be of considerable analytical difficulty depending on what properties the measure $d\mu$ is required to satisfy. We shall not deal with this problem. Our interest here lies on the nature of the relations between the sequence of polynomials $\{Q_n\}_{n\geq 0}$, the sequence of moments $\{\mu_n\}_{n\geq 0}$ and the two sequences $\{c_n\}_{n\geq 0}$, $\{\lambda_n\}_{n\geq 1}$. It develops that these relations may be beautifully expressed by means of the theory of continued fractions. The corresponding identities are the contents of the following sequence of theorems which combine classical results of Jacobi, Rogers, Stieltjes and others. We start with a result that shows that each μ_n may actually be expressed as a polynomial in the $c\,'s$ and the $\lambda\,'s$.

Theorem 3.1

Let the sequence of polynomials $\{Q_n\}_{n\geq 0}$ be constructed by the recursions in 3.11 and the initial conditions in 3.12. Let μ_n be the coefficient of x^n in the the continued fraction

$$J(x; c, \lambda) \ = \ \cfrac{1}{1 - c_o x - \cfrac{\lambda_1 x^2}{1 - c_1 x - \cfrac{\lambda_2 x^2}{1 - c_2 x - \cfrac{\lambda_3 x^2}{1 - c_3 x - \cdots}}}} \qquad 3.15$$

More precisely let the Taylor expansion of $J(x;c,\lambda)$ at $x=0$ be

$$J(x;c,\lambda) \;=\; \sum_{n\geq 0} \mu_n\, x^n \;. \qquad 3.16$$

Then the polynomials Q_n do satisfy the orthogonality condition in 3.3 2) with respect to the scalar product given by 3.14, 3.13 and the μ_n's given by 3.16.

The continued fraction $J(x;c,\lambda)$ is usually referred to as the *Jacobi continued fraction*, it reduces to the so called *Stieltjes continued fraction* when all the $c_i's$ are equal to zero. The finite continued fraction

$$J^{(n)}(x;c,\lambda) \;=\; \cfrac{1}{1-c_o x - \cfrac{\lambda_1 x^2}{1-c_1 x - \cfrac{\lambda_2 x^2}{1-c_2 x - \cfrac{\cdots}{\cdots \cfrac{\cdots}{1-c_{n-1}x - \cfrac{\lambda_n x^2}{1-c_n x}}}}}}$$

is usually referred to as the n^{th} *convergent* of $J(x;c,\lambda)$. The next result relates $J^{(n)}(x;c,\lambda)$ to the polynomials Q_n's. To state it we need further notation. Given a polynomial $\phi(x;c_o,c_1,\ldots;\lambda_1,\lambda_2,\cdots)$ let $S\phi$ denote the polynomial obtained by replacing in ϕ each c_i by a c_{i+1} and each λ_i by a λ_{i+1}. We may refer to S itself as the *shift operator*. This given we have:

Theorem 3.2

$$J^{(n)}(x;c,\lambda) \;=\; \frac{SQ_n^*(x)}{Q_{n+1}^*(x)} \qquad 3.17$$

where

$$Q_n^*(x) = x^n Q_n(1/x) \qquad , \qquad Q_{n+1}^*(x) = x^{n+1} Q_{n+1}(1/x) \;. \qquad 3.18$$

The next result is an identity which is customarily used to establish the convergence of $J^{(n)}(x;c,\lambda)$ to $J(x;c,\lambda)$ as $n\to\infty$

Theorem 3.3

$$J^{(n)}(x;c,\lambda) \;-\; J^{(n-1)}(x;c,\lambda) \;=\; \frac{\lambda_1\lambda_2\cdots\lambda_n\, x^{2n}}{Q_n^*(x)Q_{n+1}^*(x)} \qquad 3.19$$

An immediate corollary of 3.19 is that the rational function $J^{(n)}(x;c,\lambda)$ does converge to $J(x;c,\lambda)$ at least in the formal power series sense. In fact we see that the coefficient of x^n in the Taylor series expansion of $J^{(m)}(x;c,\lambda)$ is the same as that of $J(x;c,\lambda)$ itself as soon as $2m>n$. This shows that μ_n may be directly computed from $J^{(\frac{n-1}{2})}(x;c,\lambda)$ if n is odd and from $J^{(\frac{n}{2})}(x;c,\lambda)$ if n is even. In any case, we see that 3.17 defines it to be a polynomial in $c_0,c_1,\ldots,c_m$ and $\lambda_1,\lambda_2,\ldots,\lambda_m$ where m is the largest integer in $n/2$. Nevertheless, the computation of μ_n by means of one of the convergents $J^{(m)}(x;c,\lambda)$, requires (in view of 3.17) the calculation of the Taylor series of the rational inverse of Q^*_{m+1}. To avoid this step Stieltjes devised an algorithm for the recursive computation of the moments μ_n. Stieltjes result may be stated as follows:

Theorem 3.4

Let $H = \|h_{ij}\|_{i,j\geq 0}$ be the lower triangular matrix determined by the double recursion

$$h_{n,j} \;=\; h_{n-1,j-1}\,\lambda_j + h_{n-1,j}\,c_j + h_{n-1,j+1} \qquad 3.20$$

and the following initial conditions

$$h_{oo} = 1 \qquad , \qquad h_{nj} = 0 \quad for \quad n < j \qquad 3.21$$

Then

$$h_{nk} \;=\; \int_{-\infty}^{+\infty} x^n\,Q_k(x)\,d\mu \;=\; J(x;c,\lambda)\,Q^*_k\,|_{x^{n+k}} \;\; . \qquad 3.22$$

In particular, the coefficient μ_n in 3.16 can be computed by means of the identity

$$\mu_n = h_{no} \qquad 3.23$$

Some comments are in order here about the first equality in 3.22. Let $\{\nu_n\}_{n\geq 0}$ be a sequence of constants (with $\nu_o = 0$) and let L_ν be a linear functional on polynomials defined by setting

$$L_\nu(x^n) \;=\; \nu_n \;\; .$$

Suppose that the sequence Q_n given by the three term recursion in 3.11 with the initial conditions 3.12 is orthogonal with respect to the scalar product

$$\langle\, P, Q \,\rangle_\nu \;=\; L_\nu(PQ) \;\; .$$

Set

$$f_{n,k} \;=\; L_\nu(x^n Q_k)$$

then $Q_o = 1$ and orthogonality give that

$$f_{oo} = 1 \qquad , \qquad f_{nk} = 0 \quad for \quad n < k \;\; .$$

Moreover, using 3.11 we derive that

$$\begin{aligned} f_{n+1,k} &= L_\nu(x^n(xQ_k)) \\ &= \lambda_k\,L_\nu(x^n Q_{k-1}) \;+\; c_k L_\nu(x^n Q_k) \;+\; L_\nu(x^n Q_{k+1}) \\ &= \lambda_k\,f_{n,k-1} \;+\; c_k\,f_{n,k} \;+\; f_{n,k+1} \;\; . \end{aligned}$$

Thus the two matrices $\|f_{n,k}\|$ and $\|h_{n,k}\|$ must be identical. In particular, for $k = 0$ we deduce that

$$\nu_n \;=\; L_\nu(x^n) \;=\; h_{n,o} \;\; .$$

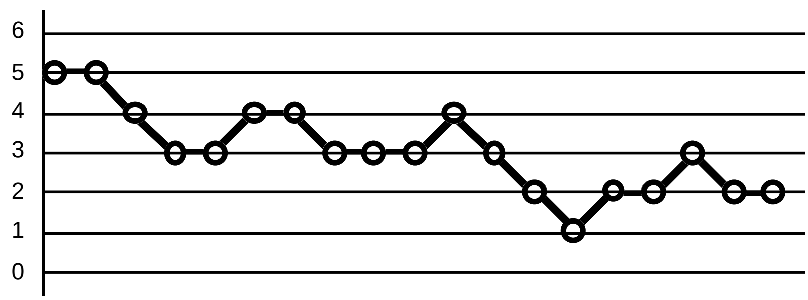
This shows that the three term recurrence uniquely determines the scalar product with respect to which the polynomials Q_k are to be orthogonal. In particular we see that the first equality in 3.22 is a immediate consequence of our definitions. Thus the essential part of Theorem 3.4 is given by the second equality in 3.22, which in particular states that the moment sequence μ_n is given by 3.16. In other words we see that Theorem 3.1 is a corollary of Theorem 3.3. Another useful consequence of 3.22 is the following fact

Theorem 3.5

The matrix $\|h_{n,k}/(\lambda_1\lambda_2\cdots\lambda_k)\|$ is the inverse of the matrix $\|a_{n,k}\|$ of the coefficients of the polynomials Q_n

4. Moments and Motzkin paths.

Our goal in this section is to show that all the constructs we have dealt with in the previous section have combinatorial interpretations that make the identities stated in theorems 3.1 to 3.5 visually self evident. To this end we shall deal here with a more general class of Motzkin paths. These are lattice paths that proceed by NORTHEAST, EAST and SOUTHEAST steps and remain throughout weakly above the x-axis without restrictions on the heights of the starting or ending points. For instance we give below a Motzkin path that starts at level 5 and ends at level 2



4.1

Here and after the collection of Motzkin paths that start at level a and end at level b will be denoted by $\Pi_{a,b}$. To such a path π we shall associate a word $w(\pi)$ by replacing (from left to right) each NORTHEAST edge by an a_i, each EAST edge by a c_i and each SOUTHEAST edge by a b_i, the subscript i giving the the starting height of the edge. For instance, carrying this out on the path in 4.1 yields the word

$$c_5b_5b_4c_3a_3c_4b_4a_3b_4c_3c_3a_3b_4b_3b_2a_1c_2a_2b_3c_2$$

Clearly, we can recover a path from its word. So in some contexts (see section 7) it is importand to regard a_i, b_i and c_i as sequences of non commuting variables. Nevertheless, in the present context it simplifies our notation to allow these letters to commute, and we shall do so in the following as long as there is no loss. With this convention, we let L denote the formal power series given by the following summation

$$L \;=\; \sum_{\pi\in\Pi_{0,0}} x^{n(\pi)}\, w(\pi)\ , \tag{4.2}$$

where we let $n(\pi)$ denote here the total number of edges of π. We shall also require that the empty path contributes a term equal to 1 to the sum in 4.2. This given, a moments's reflection should reveal that this formal power series must satisfy the following identity

$$L \;=\; 1 \;+\; c_o x\, L \;+\; x^2 a_o\,(SL)\, b_1\, L \;, \tag{4.3}$$

where S again denotes the "shift" operator that replaces each letter indexed by i by the same letter indexed by $i+1$. Successive iterations of 4.3 then yield that L must be given by the continued fraction

$$L = L(x;a,b,c) \;=\; \cfrac{1}{1 - c_o x - \cfrac{a_o b_1 x^2}{1 - c_1 x - \cfrac{a_1 b_2 x^2}{1 - c_2 x - \cfrac{a_2 b_3 x^2}{1 - c_3 x - \cdots}}}} \tag{4.4}$$

Comparing with 3.15 we see that any specialization of the sequences $\{a_i\}$, $\{b_i\}$ that makes $a_{i-1}b_i = \lambda_i$ reduces $L(x;a,b,c)$ to $J(x;c,\lambda)$. Here and after we shall assume that this equality holds true. We thus immediately get

Proposition 4.1

The sequence μ_n defined by 3.16 has the following combinatorial interpretation

$$\mu_n = \sum_{\pi \in \Pi_{o,o}(n)} w(\pi) \tag{4.5}$$

where $\Pi_{o,o}(n)$ denotes the collection of Moztkin paths of length n in $\Pi_{o,o}$

Note further that if we set $a_{i-1} = b_i = c_i = 0$ for all $i > n$ then $L(x;a,b,c)$ reduces to $J^{(n)}(x;c,\lambda)$. Clearly, this substitution, wipes out from 4.2 all the terms produced by paths whose maximum height exceeds n. Consequently we must have

Proposition 4.2 (Viennot [8])

$$J^{(n)}(x;c,\lambda) \;=\; \sum_{\pi \in \Pi_{o,o}^{\leq n}} x^{n(\pi)}\, w(\pi) \tag{4.6}$$

where $\Pi_{o,o}^{\leq n}$ denotes the paths in $\Pi_{o,o}$ of maximum height not exceeding n.

Recalling the statement of Theorem 1.1 (especially part 2)) we deduce that $J^{(n)}(x;c,\lambda)$ must be closely related to the language of pyramids of monomers and dimers projecting on $[0,n]$ and summits m_o or d_1. Note next that in the bijection giving Theorem 1.1 every monomer m_i comes from an EAST edge at height i and every dimer d_i may be made to correspond to a pair a_{i-1}, b_i of NORTHEAST and SOUTHEAST edges that respectively start at heights $i-1$ and i. This means that if, in the above mentioned language of pyramids, we replace each m_i by xc_i and each d_i by $x^2 a_{i-1} b_i$ the result should be the truncated continued fraction $J^{(n)}(x;c,\lambda)$. We are thus led to the following result

Proposition 4.3

We may write

$$J^{(n)}(x;c,\lambda) \;=\; \frac{N_n}{D_n} \tag{4.7}$$

with

$$i)\quad D_n = \sum_{F\in\mathcal{F}([0,n],M)} (-1)|F|\, w(F) \qquad and \qquad ii)\quad N_n = SD_{n-1} \tag{4.8}$$

where, $\mathcal{F}([0,n],M)$ denotes the collection of free sets of letters in the alphabet of monomer and dimers projecting on the interval $[0,n]$ and $w(F)$ is the word obtained from F by setting $m_i = xc_i$ and $d_i = x^2a_{i-1}b_i$.

Proof

Theorem 2.2 with A the alphabet of monomers and dimers projecting on $[0,n]$ and $B = \{m_o, d_1\}$ yields that we must have 4.7 with D_n as given by 4.8 i) and

$$N_n = \sum_{F\in\mathcal{F}([1,n],M)} (-1)|F|\, w(F) \tag{4.9}$$

where $\mathcal{F}([1,n],M)$ denotes the collection of free sets of monomers and dimers projecting on $[1,n]$. However, since every free set projecting on $[1,n]$ is an image by S of a free set projecting on $[0,n-1]$, we must also have 4.8 ii) as asserted.

We are now ready for

The Proof of Theorem 3.2.

Note that a free set F in 4.8 i) may either project on $[0,n-1]$ or may be written in the form $F = F_1 \cup \{m_n\}$ with F_1 projecting on $[0,n-1]$ or in the form $F = F_2 \cup \{d_n\}$ with F_2 projecting on $[0,n-2]$. This yields the recursion

$$D_n \;=\; D_{n-1}(1-w(m_n)) - D_{n-2}w(d_n) \tag{4.10}$$

and since we are supposed to let $w(m_i) = xc_i$ and $w(d_i) = x^2a_{i-1}b_i$, we see that 4.10 is none other than

$$D_n \;=\; D_{n-1}(1-xc_n) - D_{n-2}x^2\lambda_n \tag{4.11}$$

On the other hand the definitions in 3.18 combined with the three term recursion in 3.11 yield that

$$Q^*_{n+1} \;=\; (1-xc_n)Q^*_n - x^2\lambda_n Q^*_{n-1}\ .$$

Now by definition

$$J^{(0)}(x;c,\lambda) \;=\; \frac{1}{1-c_ox} \qquad and \quad J^{(1)}(x;c,\lambda) \;=\; \frac{1-c_1x}{(1-c_ox)(1-c_1x)-\lambda_1x^2}$$

which gives

$$D_o = 1-c_ox = Q^*_1 \qquad and \quad D_1 = (1-c_ox)(1-c_1x)-\lambda_1x^2 = Q^*_2$$

Thus the two sequences $\{D_n\}_{n\geq 0}$ and $\{Q^*_{n+1}\}_{n\geq 0}$ must be identical since they satisfy the same recursion and the same initial conditions. This completes the proof of 3.17

Before proceeding with the proof of the identities in 3.19 and 3.22 we should note that our Motzkin paths $\Pi_{o,k}$ provide us with an immediate combinatorial interpretation of the matrix entries $h_{n,k}$. To be precise we have

Proposition 4.4

$$h_{n,k} \;=\; \sum_{\pi\in\Pi_{o,k}(n)} w(\pi)\,\big|_{\substack{a_n=\lambda_{n+1}\\ b_n=1}} \tag{4.12}$$

where $\Pi_{o,k}(n)$ denotes the collection of paths that go from height 0 to height k in n steps.

Proof

Note that every path in $\Pi_{o,k}(n)$ must come either from height $k-1$ by a NORTHEAST step or from height k by an EAST step or from height $k+1$ by a SOUTHEAST step. Letting $\tilde h_{n,k}$ denote, for a moment, the right hand side of 4.12, this observation leads to the recursion

$$\tilde h_{n,k} \;=\; \tilde h_{n-1,k-1}a_{k-1} \;+\; \tilde h_{n-1,k}c_k \;+\; \tilde h_{n-1,k+1}b_{k+1}\;,$$

which reduces to 3.20 when we set $a_n=\lambda_{n+1}$ and $b_n=1$ for all $n\geq 0$. Note further that every path with n edges remains below height n and reaches that height only when it consists totally of NORTHEAST steps. Thus when $n<k$, the right hand side of 4.12 reduces to an empty sum, which forces $\tilde h_{n,k}=0$. Since trivially we have also $\tilde h_{o,o}=1$, we see that $h_{n,k}$ and $\tilde h_{n,k}$ do statisfy the same double recursion and the same initial conditions. Thus 4.12 must hold true as asserted.

To complete our combinatorial derivation of the identities of the last section we need an auxiliary result.

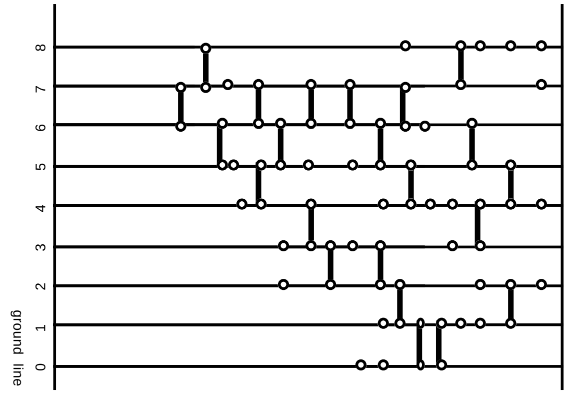
Proposition 4.5 (Viennot [8])

$$\frac{b_1b_2\cdots b_nx^n}{D_n} \;=\; \sum_{\pi\in\Pi^{\leq n}_{n,o}} x^{n(\pi)}\,w(\pi) \tag{4.13}$$

where $\Pi^{\leq n}_{n,o}$ denotes the set of Motzkin paths that go from height n to height 0 always remaining weakly below height n.

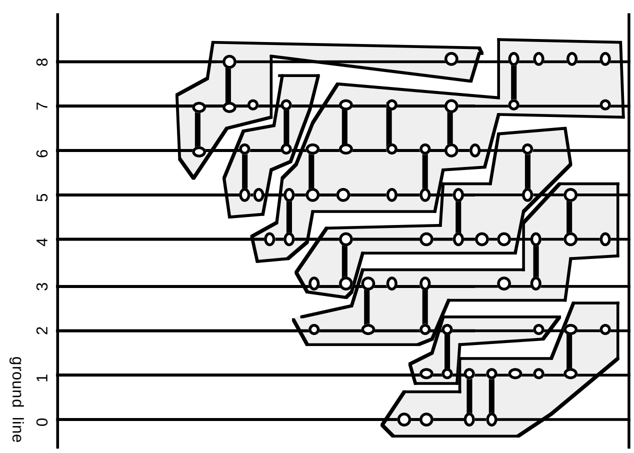
Proof

We start by observing that every monomer-dimer heap α on the interval $[0,n]$ may be uniquely decomposed into a product $\alpha=\alpha_o\alpha_1\cdots\alpha_{n-1}\alpha_n$ with α_i a pyramid projecting on $[i,n]$ with summit a monomer m_i or a dimer d_{i+1}. Of course, we do not exclude the possibility that some of the α_i may be empty here. To show this, a few pictures should suffice. For instance we give below a monomer dimer heap which projects on the interval $[0,8]$. Actually to properly see it we must turn the picture 90^o clockwise.

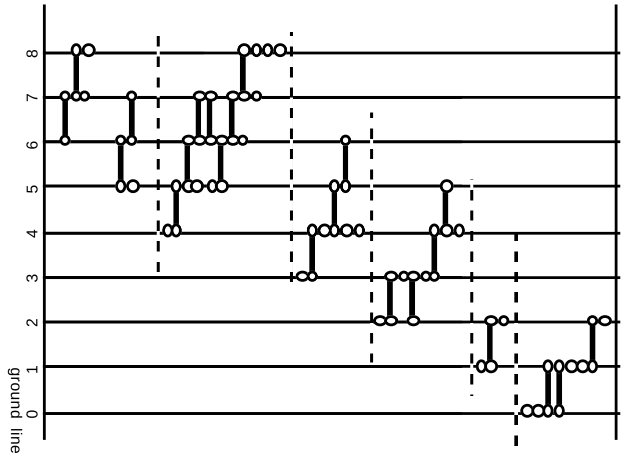


4.14

Starting from level zero we locate the leftmost piece (a monomer here). Then we enclose in a shaded area all the pieces that would have to move if we dragged that leftmost piece to the right. Imagine now that the pieces in the shaded area are moved away far to the right. Then repeat the process working on level 1. In the general case there may not be a single piece at level one, in this case there are several. The leftmost one is again a monomer. We then enclose in a second shaded area all the pieces that would have to move if we dragged that monomer way to the right. Imagine that we do drag this second batch to the right and out of sight. This done we work on level 2, etc. Proceeding in this manner row by row, we see that when we process level i (and find a piece at that level) the dragged heap is a pyramid α_i projecting on $[i, n]$ with summit a monomer m_i or a dimer d_{i+1} as desired. The process applied to the heap in 4.14 produces the successive shaded areas given below.



Notice that in this case, after we process level 6 we find no more pieces on the remaining levels 7 and 8. This results in $\alpha_7 = \alpha_8 = 1$. Bunching together the pieces in each of the 7 groups yields the final configuration

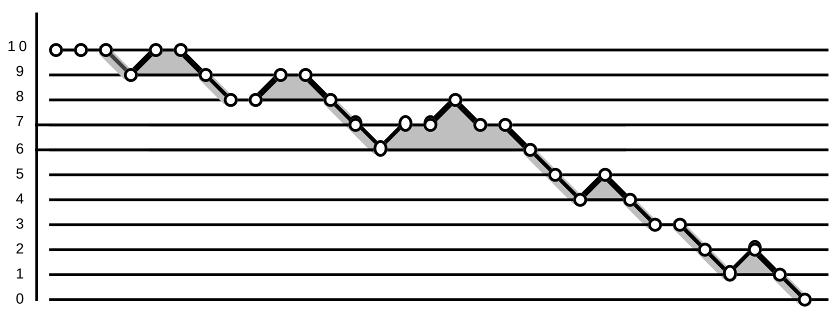


4.15

To proceed with our proof, we now use the bijection of Theorem 1.1 and convert each of these heaps back into a Motzkin path. We see that α_i will necessarily convert into a path π_i which starts at level i and ends at level i remaining weakly above that level throughout. Finally note that inserting between π_i and π_{i-1} (for $i = n, n-1, \ldots, 2, 1$) a SOUTHEAST step b_i we necessarily produce a Motzkin path π, which starts at level n ends at level 0 remaning throughout weakly below level n. We can see then that (with the substitutions $m_i = c_1 x$ and $d_i = a_{i-1} b_i x^2$) we shall have

$$\begin{aligned} x^{n(\pi)}\, w(\pi) &= x^{n(\pi_n)} w(\pi_n) b_n x x^{n(\pi_{n-1})} w(\pi_{n-1}) b_{n-1} x \cdots x^{n(\pi_2)} w(\pi_2) b_2 x x^{n(\pi_1)} w(\pi_1) b_1 x x^{n(\pi_o)} w(\pi_o) \\ &= w(\alpha_n) b_n x w(\alpha_{n-1}) b_{n-1} x \cdots w(\alpha_2) b_2 x w(\alpha_1) b_1 x w(\alpha_o) \ . \end{aligned} \tag{4.16}$$

Now, from the definition 4.8 i) and the Cartier-Foata Theorem 2.1, we see that the rational function $1/D_n$ gives the sum of the weights of the monomer dimer heaps projecting on $[0, n]$. Note further that the construction illustrated above maps this collection of heaps onto paths in $\Pi_{n,o}^{\leq n}$ bijectively. To see this we only need a single picture. We have illustrated below a path π in $\Pi_{10,0}^{\leq 10}$.



The shaded areas are precisely the regions left in the shade if the sun is shining from the point at $-\infty$ on the x-axis. The sparkled SOUTHEAST edges are those that are hit by sun light. We see that, removing the sparkled edge, π breaks up into smaller Motzkin paths $\pi_n, \pi_{n-1}, \ldots, \pi_1, \pi_o$ where π starts and ends at height i and remains above that height throughout. Of course, as we see in the example above, some of the $\pi_i\,'s$ could very well be empty. Nevertheless we can convert each π_i into a heap α_i by the bijection of Theorem 1.1. This produces a heap $\alpha = \alpha_n\alpha_{n-1}\cdots\alpha_1\alpha_o$ projecting on $[o,n]$. Now it is easy to see that α is precisely the heap that would produce π by the construction illustrated above. Thus we must conclude that 4.13 is simply obtained by summing the relation in 4.16 as α varies over all heaps projecting on $[0,n]$. This completes our proof.

Remark 4.1

Note that by reversing the order of the paths π_i in 4.16 and replacing the intermediate SOUTHEAST edges by NORTHEAST edges we obtain a Motzkin path π' which goes from level 0 to level n remaining throughout below level n. This construction replaces 4.16 by the identity

$$\begin{aligned} x^{n(\pi')}\,w(\pi') &= x^{n(\pi'_o)}w(\pi_o)a_oxx^{n(\pi'_1)}w(\pi_1)a_1xx^{n(\pi'_2)}w(\pi_2)a_2x\cdots x^{n(\pi'_{n-1})}w(\pi_{n-1})a_{n-1}xx^{n(\pi'_n)}w(\pi_n) \\ &= w(\alpha_o)a_oxw(\alpha_1)a_1xw(\alpha_2)a_2x\cdots w(\alpha_{n-1})a_{n-1}xw(\alpha_n) \\ &. \end{aligned} \qquad 4.17$$

It goes without saying that this observation shows that we also have a bijection between Motzkin paths that go from height 0 to height n remaining weakly below height n and monomer dimer heaps on $[o,n]$. Thus, denoting that collection of Motzkin paths by $\Pi^{\leq n}_{o,n}$, we see that summing 4.17 over all $\pi' \in \Pi^{\leq n}_{o,n}$ yields the companion identity

$$\frac{a_oa_1\cdots a_{n-1}x^n}{D_n} \;=\; \sum_{\pi\in\Pi^{\leq n}_{o,n}} x^{n(\pi)}\,w(\pi)\;. \qquad 4.18$$

We are now in a position to prove and 3.19 and 3.22. Note first that our Motzkin path interpretation of $J^{(n)}(x;c,\lambda)$ immediately yields that

$$J^{(n)}(x;c,\lambda) \;-\; J^{(n-1)}(x;c,\lambda) \;=\; \sum_{\pi\in\Pi^{(max=n)}_{o,o}} x^{n(\pi)}\,w(\pi)\;, \qquad 4.19$$

where $\Pi^{(max=n)}_{o,o}$ denotes the collection of Motzkin paths in $\Pi_{o,o}$ whose maximum height is equal to n. Now a path $\pi \in \Pi^{(max=n)}_{o,o}$ may be uniquely factorized in to a path $\pi_1 \in \Pi^{\leq n}_{o,n}$ followed by a SOUTHEAST step b_n and a path $\pi_2 \in \Pi^{\leq n-1}_{n-1,o}$. A moment's reflection leads us to conclude that this factorization yields the identity

$$J^{(n)}(x;c,\lambda) \;-\; J^{(n-1)}(x;c,\lambda) \;=\; \frac{a_oa_1\cdots a_{n-1}x^n}{D_n} \times b_nx \times \frac{b_{n-1}b_{n-2}\cdots b_2b_1x^{n-1}}{D_{n-1}}\;. \qquad 4.20$$

Indeed, summing over all possible π_1 and using 4.18 yields the first factor, and summing over all possible π_2 and using 4.13 (with n replaced by $n-1$) yields the last factor. Now we have shown in the proof of Theorem 3.2 that $D_n = Q^*_{n+1}$ and $D_{n-1} = Q^*_n$. Thus making these substitutions and setting $a_{i-1}b_i = \lambda_i$ we see that 4.20 reduces to 3.19.

To show 3.22 we start by observing that the difference $J(x;c,\lambda) - J^{(n-1)}(x;c,\lambda)$ may be given the combinatorial interpretation

$$J(x;c,\lambda) - J^{(n-1)}(x;c,\lambda) \;=\; \sum_{\pi\in\Pi^{max\geq n}_{o,o}} x^{n(\pi)}\,w(\pi)\;, \qquad 4.21$$

where $\Pi_{o,o}^{max\geq n}$ denotes the collection of all Motzkin paths in $\Pi_{o,o}$ which reach heights weakly above n. Now any $\pi\in\Pi_{o,o}^{max\geq n}$ may be uniquely factorized into a path $\pi_1\in\Pi_{o,n}$ followed by a SOUTHEAST edge and a path $\pi_2\in\Pi_{n-1,o}^{(\leq n-1)}$. The same reasonning that led us to 4.20 now yields

$$J(x;c,\lambda)-J^{n-1}(x;c,\lambda) \;=\; \Big(\sum_{\pi\in\Pi_{o,n}} x^{n(\pi)}\,w(\pi)\Big)\;\times\; b_n x\;\times\;\frac{b_{n-1}\cdots b_2 b_1 x^{n-1}}{D_{n-1}}\;.$$

Multiplying both sides by D_{n-1} and using 4.7 (with n repaced by $n-1$) we obtain the identity

$$b_n b_{n-1}\cdots b_1 x^n \sum_{\pi\in\Pi_{o,n}} x^{n(\pi)}\,w(\pi) \;=\; J(x;c,\lambda)D_{n-1}(x)-N_{n-1}(x)\;. \qquad 4.22$$

We now replace n by k, D_{n-1} and N_{n-1} respectively by Q_k^* and SQ_{k-1}^* and derive that

$$b_1 b_2\cdots b_k x^k \sum_{\pi\in\Pi_{o,k}} x^{n(\pi)}\,w(\pi) \;=\; J(x;c,\lambda)Q_k^*(x)-SQ_{k-1}^*(x)\;. \qquad 4.23$$

Equating coefficients of x^{k+n} and using our combinatorial interpretation 4.12 for the matrix elements $h_{n,k}$, we finally get that, with the specializations $a_n=\lambda_{n+1}$ and $b_n=1$ (for all $n\geq 0$) we must have

$$h_{n,k} \;=\; J(x;c,\lambda)Q_k^*(x)\,\big|_{x^{k+n}}\;. \qquad 4.24$$

The term $SQ_{k-1}^*(x)$ in 4.23 contributes nothing to 4.24 for the simple reason that it is a polynomial of degree $k-1$. This shows the equality of the first and last member of 3.22. To complete the proof of 3.22 we need only express the right hand side of 4.24 in terms of the polynomial Q_k. To this end note that if $Q_k=\sum_{s=0}^k a_{ks}\,x^s$ then from 3.16 we get that

$$J(x;c,\lambda)\,Q_k^*(x) \;=\; \Big(\sum_{m\geq 0}\mu_m\,x^m\Big)\Big(\sum_{s=0}^k a_{ks}\,x^{k-s}\Big)\;.$$

Equating coefficients of x^{k+n} gives

$$J(x;c,\lambda)\,Q_k^*(x)\,\big|_{x^{n+k}} \;=\; \sum_{s=o}^k a_{ks}\,\mu_{n+s}\;. \qquad 4.25$$

So if the $\mu_n\,'s$ are the moments of a measure μ we may rewrite this result in the form

$$J(x;c,\lambda)\,Q_k^*(x)\,\big|_{x^{n+k}} \;=\; \int_{-\infty}^{+\infty} Q_k(x)\,x^n\,d\mu\;. \qquad 4.26$$

This proves the last equality in 3.23.

If we do not want to make use of a measure, we can work with the bilinear form defined by 3.13 and 3.14 and rewrite 4.25 in the form

$$J(x;c,\lambda)\,Q_k^*(x)\,\big|_{x^{n+k}} \;=\; \langle\, Q_k\;,\;x^n\,\rangle_\mu \qquad 4.27$$

Since $h_{n,k} = 0$ for $n < k$, from 3.25 we deduce that

$$\langle\, Q_k \,,\, x^n \,\rangle_\mu \;=\; 0 \qquad\qquad (\,for \quad n < k\,)\ .$$

In particular, we must also have

$$\langle\, Q_k \,,\, Q_n \,\rangle_\mu \;=\; 0 \qquad\qquad (\,for \quad n < k\,)\ , \tag{4.28}$$

In other words the $Q_n\,'s$ are orthogonal precisely as asserted at the end of Theorem 3.1. This completes our proof of the identities of section 3.

The identity in 4.22 has an extension that is worth mentioning here.

Proposition 4.6 (Viennot [8])
For $0 \leq h \leq k$ we have

$$D_{h-1}\,\big(J(x;c,\lambda)D_{k-1} - N_{k-1}\,\big) \;=\; \lambda_1\lambda_2\cdots\lambda_h\, x^{h+k} \sum_{\pi\in\Pi_{h,k}} x^{n(\pi)}\, w(\pi)\ . \tag{4.29}$$

Proof

We start with 4.22 written for $n = k$

$$J(x;c,\lambda)D_{k-1}(x) - N_{k-1}(x) \;=\; b_k b_{k-1}\cdots b_1 x^k \sum_{\pi\in\Pi_{o,k}} x^{n(\pi)}\, w(\pi)\ . \tag{4.30}$$

Note that each path $\pi \in \Pi_{o,k}$ ha a unique decomposition into a path $\pi_1 \in \Pi_{0,h-1}^{\leq h-1}$ followed by a NORTHEAST step, followed by a path $\pi_2 \in \Pi_{h,k}$. Indeed, the cutting point for π_1 is precisely before the edge where π reaches for the first time height h. This gives

$$w(\pi) \;=\; w(\pi_1)\, a_{h-1}x\, w(\pi_2)\ .$$

Summing over all possible choices of π_1 and π_2 and using 4.18 (for $n = h-1$) gives the identity

$$\sum_{\pi\in\Pi_{o,k}} x^{n(\pi)}\, w(\pi) \;=\; \frac{a_o a_1\cdots a_{h-2}x^{h-1}}{D_{h-1}}\, a_{h-1}x \sum_{\pi\in\Pi_{h,k}} x^{n(\pi)}\, w(\pi)\ .$$

Substituting in 4.30 and setting $a_i = \lambda_i$ and $b_i = 1$ gives

$$J(x;c,\lambda)D_{k-1}(x) - N_{k-1}(x) \;=\; \frac{\lambda_1\lambda_2\cdots\lambda_h\, x^{h+k}}{D_{h-1}} \sum_{\pi\in\Pi_{h,k}} x^{n(\pi)}\, w(\pi)\ ,$$

and 4.29 follows upon multiplying both sides by D_{h-1}.

Note that since D_{h-1} and N_{k-1} are polynomials of degrees h and $k-1$ respectively, equating coefficients of x^{h+k+n} of both sides of 4.29 gives

$$D_{h-1}\, J(x;c,\lambda)D_{k-1}\,\big|_{x^{h+k+n}} \;=\; \lambda_1\lambda_2\cdots\lambda_h \sum_{\pi\in\Pi_{h,k}(n)} w(\pi)\ , \tag{4.31}$$

where $\Pi_{h,k}(n)$ denotes the collection of Motzkin paths that go from height h to height k in n steps. Recalling that $D_{h-1} = Q_h^*$ and $D_{k-1} = Q_k^*$ and imitating our derivation of 4.26 from 4.24 we are finally led to the identity

$$\int_{-\infty}^{+\infty} Q_h(x)Q_k(x)\, x^n \, d\mu \;=\; \lambda_1\lambda_2\cdots\lambda_h \sum_{\pi\in\Pi_{h,k}(n)} w(\pi) \;.$$

Our combinatorial setting can be used to establish further identities. Note that combining 3.22 and 4.12 we may write

$$\int_{-\infty}^{+\infty} x^n \, Q_k \, d\mu \;=\; \sum_{\pi\in\Pi_{o,k}(n)} w(\pi)\,\big|_{\substack{a_{i-1}=\lambda_i \\ b_i=1}} \;. \qquad 4.32$$

This identity yields us a combinatorial way of deriving explicit expressions for λ_n and c_n in terms of the moment sequence $\{\mu_n\}$.

Proposition 4.7

$$a)\quad \lambda_n \;=\; \frac{d_n d_{n-2}}{d_{n-1}^2} \qquad , \qquad b)\quad c_n \;=\; \frac{\chi_n}{d_n} \;-\; \frac{\chi_{n-1}}{d_{n-1}} \qquad 4.33$$

where

$$d_n = det \begin{pmatrix} \mu_o & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} \quad , \quad \chi_n = det \begin{pmatrix} \mu_o & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{pmatrix} \;. \qquad 4.34$$

Proof

Since there is only one way for a Motzkin path to reach height n from height 0 in n steps (NORTHEAST all the way), formula 4.32 for $k = n$ reduces to

$$\int_{-\infty}^{+\infty} x^n \, Q_n \, d\mu \;=\; a_o a_1 \cdots a_{n-1}\,|_{a_{i-1}=\lambda_i} \;=\; \lambda_1\lambda_2\cdots\lambda_n \;. \qquad 4.35$$

On the other hand, using 3.4 we get

$$\int_{-\infty}^{+\infty} x^n \, Q_n \, d\mu \;=\; \frac{d_n}{d_{n-1}} \;. \qquad 4.36$$

Combining with 4.35 we deduce that

$$\frac{d_n}{d_{n-1}} \;=\; \lambda_1\lambda_2\cdots\lambda_n \;, \qquad 4.37$$

and 4.33 a) immediately follows.

Note next that 3.4 and 4.34 yield that

$$\int_{-\infty}^{+\infty} x^{n+1}\, Q_n\, d\mu \;=\; \frac{1}{d_{n-1}}\, \chi_n \;. \tag{4.38}$$

On the other hand a Motzkin path can reach height n from height 0 in $n+1$ steps if and only if it takes i successive NORTHEAST steps followed by a single EAST step and then finish up with $n-i$ successive NORTHEAST steps (for $i = 0, 1, \ldots, n$). This gives

$$\sum_{\pi \in \Pi_{o,k}(n)} w(\pi)\,|_{\substack{a_{i-1}=\lambda_i \\ b_i=1}} \;=\; \big(c_o + c_1 + \cdots + c_n\big)\, \lambda_1 \lambda_2 \cdots \lambda_n \;. \tag{4.39}$$

Combining with 4.32, 4.37 and 4.38 we get

$$\chi_n \;=\; \big(c_o + c_1 + \cdots + c_n\big)\, d_n \tag{4.40}$$

which implies 4.33 b) as desired.

5. Tchebytcheff Polynomials

The simplest family of orthogonal polynomials, from the combinatorial point of view developped in the previous sections, should be one that corresponds to Dyck paths (that is $c_n = 0$) with constant λ_n. It happens that the Tchebytcheff polynomials of the first and second kind come pretty near to this. Let us recall that the latter are the polynomials $\{\alpha_n(x)\}_n$ defined by setting

$$\alpha_n(cos\theta) \;=\; cos\, n\theta \qquad\qquad (\, for \quad n \geq 0\,) \;. \tag{5.1}$$

while those of the second kind may be implicitely defined as the $\{\beta_n(x)\}_n$ satisfying

$$\beta_n(cos\theta) \;=\; \frac{sin\,(n+1)\,\theta}{sin\,\theta} \qquad\qquad (\, for \quad n \geq 0\,) \;. \tag{5.2}$$

Note that the trigonometric identities

$$\begin{aligned} &a) \quad cos(n+1)\theta \;+\; cos(n-1)\theta \;=\; 2\,cos\,\theta\,cos\,n\theta \\ &b) \quad sin(n+2)\theta \;+\; sin\,n\theta \;=\; 2\,cos\,\theta\,sin(n+1)\theta \end{aligned} \tag{5.3}$$

immediately yield the recursions

$$\begin{aligned} &a) \quad \alpha_{n+1}(x) \;=\; 2x\,\alpha_n(x) \;-\; \alpha_{n-1}(x) \;, \\ &b) \quad \beta_{n+1}(x) \;=\; 2x\,\beta_n(x) \;-\; \beta_{n-1}(x) \;. \end{aligned} \qquad (\, for \quad n \geq 1\,) \tag{5.4}$$

Computing directly from the definitions we also get that

$$\begin{aligned} &a) \quad \alpha_o(x) \;=\; 1 \quad , \quad \alpha_1(x) \;=\; x \\ &b) \quad \beta_o(x) \;=\; 1 \quad , \quad \beta_1(x) \;=\; 2x \end{aligned} \tag{5.5}$$

We see then that 5.4 a) and 5.5 a) determine $\alpha_n(x)$ has a polynomial in x of degree n and leading coefficient 2^{n-1}. Similarly, 5.4 b) and 5.5 b) yield that $\beta_n(x)$ is a polynomial degree n and leading coefficient 2^n. To get sequences of polynomials with leading term x^n as required by 3.3 1) we must rescale. We shall thus set

$$\begin{array}{lrcl} a) & A_n(x) & = & \alpha_n(x)/2^{n-1} \ , \\ b) & B_n(x) & = & \beta_n(x)/2^n \ . \end{array} \qquad (\,for \quad n\geq 1\,) \tag{5.6}$$

This has the effect of changing 5.4 a) to

$$\begin{aligned} A_2(x) &= x\,A_1(x) - \tfrac{1}{2}A_o(x) \quad \text{and} \\ A_{n+1}(x) &= x\,A_n(x) - \tfrac{1}{4}A_{n-1}(x) \ ,(\,for \quad n\geq 2\,) \end{aligned} \tag{5.7}$$

with

$$A_0(x)=1 \qquad \text{and} \qquad A_1(x)=x \ . \tag{5.8}$$

While 5.4 b) becomes

$$B_{n+1}(x) = x\,B_n(x) - \tfrac{1}{4}B_{n-1}(x) \ , \qquad (\,for \quad n\geq 1\,) \tag{5.9}$$

with

$$B_0(x)=1 \qquad \text{and} \qquad B_1(x)=x \ . \tag{5.10}$$

It will also be convenient to set

$$A_n^*(x) = x^nA_n(1/x) \qquad \text{and} \qquad B_n^*(x) = x^nB_n(1/x) \ . \tag{5.11}$$

This given, comparing with 3.11 we immediately derive from Theorems 3.1 , 3.2 that

Theorem 5.1

$$lim_{n\to\infty}\frac{B_n^*(x)}{A_{n+1}^*(x)} = \cfrac{1}{1-\cfrac{x^2/2}{1-\cfrac{x^2/4}{1-\cfrac{x^2/4}{1-\cfrac{x^2/4}{1-\cdots}}}}} \tag{5.12}$$

and

$$lim_{n\to\infty}\frac{B_n^*(x)}{B_{n+1}^*(x)} = \cfrac{1}{1-\cfrac{x^2/4}{1-\cfrac{x^2/4}{1-\cfrac{x^2/4}{1-\cfrac{x^2/4}{1-\cdots}}}}} \tag{5.13}$$

Moreover, the α_n 's and β_n 's are orthogonal with respect to measures μ^a and μ^b with moments given by

$$\mu_{2n}^a = \frac{1}{2^{2n-1}}\frac{1}{n+1}\binom{2n}{n} \qquad , \qquad \mu_{2n-1}^a = 0 \tag{5.14}$$

and

$$\mu^b_{2n} \;=\; \frac{1}{2^{2n}}\,\frac{1}{n+1}\binom{2n}{n} \quad , \quad \mu^b_{2n-1} \;=\; 0 \tag{5.15}$$

Proof

We only need to explain the numerator $B^*_n(x)$ on the left hand side of 5.12. To this end note that our combinatorial setting gives us that

$$A^*_{n+1}(x) \;=\; \sum_{F\in\mathcal{F}([0,n],M)} (-1)^{|F|}\, w(F)\ \big|_{d_i=\lambda_i x^2} \ . \tag{5.16}$$

with $\lambda_1 = 1/2$ and $\lambda_n = 1/4$ for all $n \geq 2$. On the other hand 4.8 ii) gives that the numerator in the left hand side of 5.12 is given by

$$SD_{n-1} \;=\; \sum_{F\in\mathcal{F}([1,n],M)} (-1)^{|F|}\, w(F)\ \big|_{d_i=\lambda_i x^2} \ . \tag{5.17}$$

and this is equal to $B^*_n(x)$ since in the interval $[1,n]$ we can't place the dimer d_1 and we can set $d_i = x^2/4$ ($\forall\ i$) on the right hand side of 5.17.

Remark 5.1

Note that 5.16 gives that the polynomials $A^*_n(x)$ are all functions of x^2, clearly the same holds true for all the $B^*_n(x)$. The relations in 5.11 then imply that $\alpha_n(x)$ and $\beta_n(x)$ are even functions of x for even n and odd functions for n odd. This can also be easily deduced from the recursions in 5.4. It is also seen that the combinatorial interpretation

$$B^*_n(x) \;=\; \sum_{F\in\mathcal{F}([0,n-1],M)} (-1)^{|F|}\, w(F)\ \big|_{d_i=\lambda_i x^2} \ .$$

yields us that

$$B^*_n(x) \;=\; \sum_{k=0}^{n} \binom{n-k}{k}\ (-1)^k (x^2/4)^k \ .$$

This is because there are precisely $\binom{n-k}{k}$ ways of placing k non overlapping dimers on $[0, n-1]$. Using 5.11 and 5.6 we are thus led to the explicit formula

$$\beta_n(x) \;=\; \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k\ (2x)^{n-2k} \ . \tag{5.18}$$

Now this is neater than the formula yielded by the trigonometric identity

$$cos\ (n+1)\theta + i\ sin\ (n+1)\theta \;=\; (cos\ \theta + i\ sin\ \theta)^{n+1}$$

In fact, equating imaginary parts we get

$$sin\ (n+1)\theta \;=\; \sum_{m=0}^{n+1} \binom{n+1}{2m+1} (-1)^m\ cos^{n-2m}\theta\ sin^{2m+1}\theta$$

which using 5.2 translates into

$$\beta_n(x) \;=\; \sum_{m=0}^{n+1} \binom{n+1}{2m+1} (-1)^m x^{n-2m} \,(1-x^2)^m \;. \qquad 5.19$$

We should note that equating coefficients of x^{n-2k} in 5.18 and 5.19 leads to the binomial identity

$$\binom{n-k}{k} 2^{n-2k} \;=\; \sum_s \binom{n+1}{2k+2s+1}\binom{k+s}{s} \;. \qquad 5.20$$

The measures μ^a and μ^b corresponding to the two sequences $\{\alpha_n(x)\}_n$ and $\{\beta_n(x)\}_n$ are easily obtained in this case. Indeed, since

$$\frac{1}{\pi}\int_o^\pi cos\,(n\theta)\,cos\,(m\theta)\,d\theta \;=\; \begin{cases} 1 & \text{if } n=m=0 \\ 1/2 & \text{if } n=m\geq 1 \\ 0 & \text{otherwise} \end{cases}$$

we derive from 5.1 (by the substitution $x=cos\theta$) that

$$\frac{1}{\pi}\int_{-1}^1 \alpha_n(x)\,\alpha_m(x)\frac{dx}{\sqrt{1-x^2}} \;=\; \begin{cases} 1 & \text{if } n=m=0 \\ 1/2 & \text{if } n=m\geq 1 \\ 0 & \text{otherwise} \end{cases} \;.$$

Similarly, from

$$\frac{2}{\pi}\int_o^\pi sin\,(n+1)\theta\;sin\,(m+1)\theta\;d\theta \;=\; \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise} \end{cases}$$

making the substitution $x=cos\,\theta$ and noting that $sin^2\,\theta\;d\theta=-\sqrt{1-x^2}\;dx$ we get

$$\frac{2}{\pi}\int_{-1}^1 \beta_n(x)\,\beta_m(x)\sqrt{1-x^2}dx \;=\; \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise} \end{cases} \;.$$

We thus obtain that μ^a and μ^b are both unit measures concentrated on the interval $[-1,1]$ with

$$d\mu^a \;=\; \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\;dx \qquad \text{and} \qquad d\mu^b \;=\; \frac{2}{\pi}\sqrt{1-x^2}\;dx \;.$$

In particular, comparing with 5.14 and 5.15 we obtain a combinatorial interpretation for the identities

$$\frac{1}{\pi}\int_{-1}^1 x^{2n}\frac{dx}{\sqrt{1-x^2}} \;=\; \frac{1}{2^{2n-1}}\;\frac{1}{n+1}\binom{2n}{n}$$

and

$$\frac{2}{\pi}\int_{-1}^1 x^{2n}\;\sqrt{1-x^2}dx \;=\; \frac{1}{2^{2n}}\;\frac{1}{n+1}\binom{2n}{n}$$

6. The Rogers-Ramanujan Continued Fraction.

The next simplest example is obtained by setting $a_n = q^{n+1}$, $b_n = 1$ and $c_n = 0$. With this choice we are again left with summing only over Dyck paths. It develops that both the moment sequence and the corresponding polynomials D_n and N_n have interesting combinatorial interpretations.

Theorem 6.1

For a given Dyck path π let $a(\pi)$ denote number of lattice squares weakly below π and weakly above the x-axis, and let $\mathcal{D}_n$ denote the collection of Dyck paths of length $2n$. This given, we have

$$J(x;q) \;=\; \cfrac{1}{1 - \cfrac{qx^2}{1 - \cfrac{q^2x^2}{1 - \cfrac{q^3x^2}{1 - \cdots}}}} \;=\; \sum_{n\geq 0} c_n(q)\, q^n\, x^{2n} \quad . \tag{6.1}$$

where

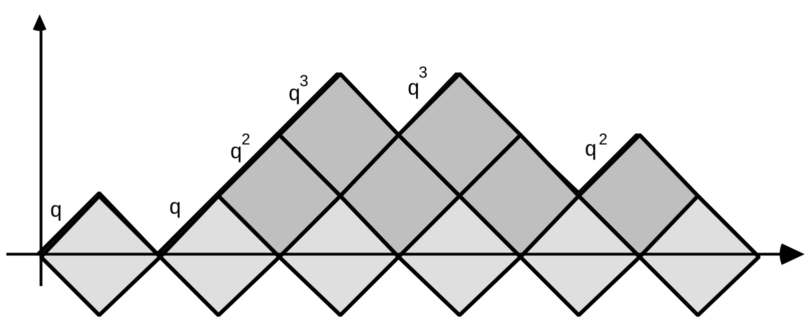
$$c_n(q) \;=\; \sum_{\pi \in \mathcal{D}_n} q^{a(\pi)} \quad . \tag{6.2}$$

Proof

The moments of the measure corresponding to the continued fraction on the left hand side of 6.1, according to 4.5, are given by the identity

$$\mu_n = \sum_{\pi \in \Pi_{o,o}} w(\pi)\ \big|_{\lambda_i = q^i} \quad . \tag{6.3}$$

A look at the figure below should make it obvious that the contribution of a NORTHEAST edge e of a Dyck path π to $w(\pi)\ \big|_{\substack{c_i=0 \\ \lambda_i=q^i}}$ is the factor q^{m+1}, where m is the number of lattice squares southeast of e that are weakly below π and weakly above the x-axis. Since setting $c_i = 0$ kills all but the summands corresponding to Dyck paths we see that the sum on the right hand side of 6.3 must evaluate to $q^n c_n(q)$ and thus 6.1 must hold true as asserted.



Let now $D_n(x;q)$ denote the denominator of the n^{th} convergent of the continued fraction $J(x;q)$ in 6.1. Note that the recursion in 4.11 in this case reduces to

$$D_n(x;q) \;=\; D_{n-1}(x;q) \;-\; x^2\, q^n\, D_{n-2}(x;q) \quad . \tag{6.4}$$

Note further that the dimers $d_{i_1}, d_{i_2}, \ldots, d_{i_k}$ do not overlap if and only if

$$i_1 + 2 \leq i_2 \ , \ i_2 + 2 \leq i_3 \ , \ \cdots \ , \ i_{k-1} + 2 \leq i_k$$

In other words there is an obvious bijection between k-sets of non overlapping dimers and k-part partitions with part differences greater or equal to 2. For convenience let $\Delta_2(k)$ denote this collection of partitions and Δ_2 denote the union of all the $\Delta_2(k)$. Now it is well known and easy to show that we have

$$\sum_{\delta \in \Delta_2(k)} q^{|\delta|} \;=\; \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{6.5}$$

where for a given partition δ we let $|\delta|$ denote the sum of its parts. Tus if $k(\delta)$ denotes the number of parts of δ we may also write

$$\sum_{\delta \in \Delta_2} x^{2k(\delta)} q^{|\delta|} \;=\; \sum_{k \geq 0} \frac{x^k \, q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \ . \tag{6.6}$$

We may then translate 3.17 into the following remarkable identity:

Theorem 6.2

$$J(x;q) \;=\; \cfrac{1}{1 \;-\; \cfrac{qx^2}{1 \;-\; \cfrac{q^2x^2}{1 \;-\; \cfrac{q^3x^2}{1 \;-}}}} \;=\; \frac{\displaystyle\sum_{k\geq 0} \frac{(-1)^k x^{2k}\, q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}}{\displaystyle\sum_{k\geq 0} \frac{(-1)^k x^{2k}\, q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}} \tag{6.7}$$

Proof

Since the coefficient of $(-1)^k x^{2k}$ in $D_n(x;q)$ is given by the sum

$$\sum_{\substack{\delta \in \Delta_2(k) \\ max\ \delta\ \leq n}} q^{|\delta|}$$

and the latter converges to the righthand side of 6.5 (in the formal power series sense), we see that we must have

$$lim_{n\to\infty}\, D_n(x;q) \;=\; \sum_{k\geq 0} \frac{(-1)^k x^{2k}\, q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{6.8}$$

Now, as we have seen, the n^{th} convergent of $J(x;q)$ is given by the expression

$$J^{(n)}(x;q) \;=\; \frac{N_n(x;q)}{D_n(x;q)} \tag{6.9}$$

with

$$N_n(x;q) \;=\; \sum_{F\in\mathcal{F}([1,n],M)} (-1)^{|F|}\; w(F)\; \Big|_{\substack{d_i=\lambda_i x^2\\ c_i=0}} \;.$$

Since the latter is the image of $D_{n-1}(x;q)$ by the shift operator S, we immediately deduce that, we can obtain $N_n(x;q)$ by replacing all the q^i contributing to $D_{n-1}(x;q)$ by q^{i+1} and this gives that the coefficient of $(-1)^k x^{2k}$ in $N_n(x;q)$ must be equal to

$$\sum_{\substack{\delta\in\Delta_2(k)\\ max\ \delta\ \le n-1}} q^{|\delta|+k(\delta)}$$

Thus we must have

$$lim_{n\to\infty}\, N_n(x;q) \;=\; \sum_{k\ge 0} \frac{(-1)^k x^{2k}\, q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \;. \qquad 6.10$$

Now, 4.20 in this case becomes

$$J^{(n)}(x;q) \;-\; J^{(n-1)}(x;q) \;=\; \frac{q^{\binom{n+1}{2}}\, x^{2n}}{D_n(x;q)D_{n-1}(x;q)}$$

and this implies that $J^{(n)}(x;q)\to J(x;q)$ as a formal power series in x or q as we desire. Combining this with 6.8 and 6.10 yields 6.7 and completes our proof.

Note that since this proof shows that we can interpret 6.7 as a limit theorem involving formal power series in q we can set $x=i$ (the square root of -1) in 6.7 and derive the Rogers-Ramanujan continued fraction identity. More precisely we have

Theorem 6.3

Let $D_n(q)$ and $N_n(q)$ denote the sequences defined by the recursions

$$\begin{aligned} D_n(q) &= D_{n-1}(q) + q^n\, D_{n-2}(q)\;, \\ N_n(q) &= N_{n-1}(q) + q^n\, N_{n-2}(q) \end{aligned} \qquad (\,for \quad n\ge 2\,) \qquad 6.11$$

and the initial conditions

$$D_o(q)=1\;,\; D_1(q)=1+q \qquad , \qquad N_o(q)=1\;,\; N_1(q)=1 \qquad 6.12$$

Then

$$a) \quad lim_{n\to\infty}\, D_n(q) \;=\; \sum_{k\ge 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \;,$$

$$b) \quad lim_{n\to\infty}\, N_n(q) \;=\; \sum_{k\ge 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \;.$$

In particular we must have

$$\cfrac{1}{1 \ + \ \cfrac{q}{1 \ + \ \cfrac{q^2}{1 \ + \ \cfrac{q^3}{1 \ +}}}} \ = \ \frac{\sum_{k\geq 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}}{\sum_{k\geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}} \tag{6.13}$$

We should mention that Rogers and Ramanujan (see [1]) showed that

$$\prod_{n\geq 1}(1-q^n) \ \sum_{k\geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \ = \ \sum_{m=-\infty}^{+\infty} (-1)^m \ q^{(5m^2-m)/2} \ . \tag{6.14}$$

This combined with the Jacobi triple product identity

$$\sum_{m=-\infty}^{+\infty} (-1)^m \ q^{\binom{n}{2}} \ z^m \ = \ \prod_{n\geq 0} (1-q^{n+1})(1-zq^n)(1-q^{n+1}/z) \tag{6.15}$$

with q replaced by q^5 and $z = q^2$ yields that

$$\begin{aligned}\prod_{n\geq 1}(1-q^n) \ \sum_{k\geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \ &= \ \sum_{m=-\infty}^{+\infty} (-1)^m \ q^{(5m^2-m)/2} \\ &= \ \prod_{n\geq 0} (1-q^{5n+5})(1-q^{5n+2})(1-q^{5n+3})\end{aligned}$$

Now this, upon division by $\prod_{n\geq 1}(1-q^n)$, yields

$$\sum_{k\geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} \ = \ \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \tag{6.16}$$

which is usually referred to as the first Rogers-Ramanujan identity. The identity is actually due to Rogers who was the first to formulate it and **prove** it in 1894 (about 20 years before Ramanujan formulated it without proof). The misnomer is due to Hardy, a mathematician apt to tabloid sensationalism, quite minor compared to Rogers, whose life work Hardy was very poorly competent to judge. From the combinatorial point of view the best proof of 6.14 was given by Schur who in 1917 rediscovered 6.16 and proved it in two different ways. By Hardy's criterion it should be referred to as the *Rogers-Ramanujan-Schur identity* if we proceed alphabetically and the *Rogers-Schur-Ramanujan identity* if we proceed by merit! There are several combinatorial proofs of the Jacobi identity. One of the can be found in [6], Schur proof of 6.14 can also be found in [6].

Now in the same manner it can be shown that

$$\sum_{k\geq 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \ = \ \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \tag{6.17}$$

Using 6.16 and 6.17 the identity in 6.13 can be rewriteen in the truly remarkable form

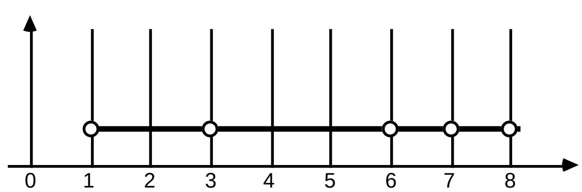
$$\cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 +}}}} = \prod_{n\geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})} \ . \qquad 6.18$$

We should mention that a direct proof of 6.17 which avoids 6.14 but not the triple product identity can be found in [2].

7. Partitions and Hermite Polynomials.

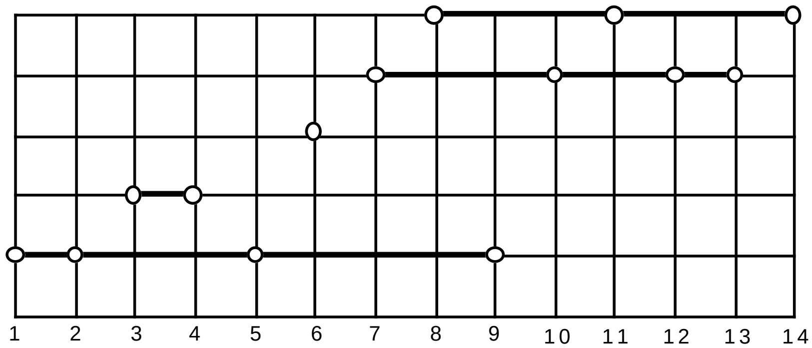
Our goal here is to present a beautiful bijection due to Flajolet [4] between set partitions and labelled Motzkin paths and as a biproduct obtain the continued fraction corresponding to the Hermite polynomials.

To this end we need some auxiliary constructions and notations. We shall begin with representing a subset $S = \{i_1 < i_2 < \ldots < i_s\} \subseteq \{1, 2, \ldots, n\}$ as the line diagram obtained by placing k successive tiny circles $c_1, c_2, \ldots, c_s$ joined by line segments, with the center of circle c_r at a point of x-coordinate i_r. For instance, this construction applied to the set $S = \{1, 3, 6, 7, 8\}$ gives the diagram below:



It will be conveniett to refer to *min* S and *max* S (1 and 8 here) as the *opening* and *closing* elements of the diagram and refer to the others as *inner* elements.

It is customary to write the parts of a set partition in order of increasing minimal elements, we shall follow this convention here. This given, we can represent a set partition $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ of $\{1, 2, \ldots, n\}$ by a juxtaposition of line diagrams with the k^{th} diagram representing part A_k and drawn on the line $y = k$. The resulting composite diagram will be denoted by $\mathcal{D}(\mathcal{A})$. For instance, in this manner, the partition $\mathcal{A} = \big\{\{1, 2, 5, 9\}, \{3, 4\}, \{6\}, \{7, 10, 12, 13\}, \{8, 11, 14\}\big\}$ is represented by the figure below.



7.1

We see that if an element i of $\{1,2,\ldots,n\}$ belongs to part A_h then the circle whose x-coordinate is i will lie at height h. We shall refer to the latter as the *height* of i and denote it by $h(i)$. We shall also say that an element i of $\{1,2,\ldots,n\}$ is *covered* by a part A_r if and only if $min\ A_r \leq i \leq max\ A_r$. Note that although the partition above has a total of 5 parts, the element 11 is covered only by two parts. It will be convenient to order the parts covering a given element i according to their heights in the diagram. For instance, the second part covering 8 in the picture above is $\{7,10,12,13\}$.

The next step in Flajolet's construction is to convert $\mathcal{D}(\mathcal{A})$ into a *labelled* Motzkin path $L\pi(\mathcal{A})$. The corresponding unlabelled path $\pi(\mathcal{A})$ is easiest to describe in terms of its Motzkin word $w(\mathcal{D}(\mathcal{A}))$. We simply let

$$w(\mathcal{D}(\mathcal{A})) \;=\; d_1 d_2 \cdots d_n \tag{7.2}$$

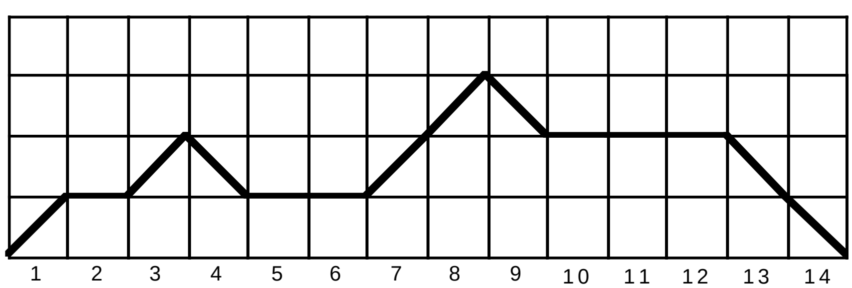
where, given that i is covered by k parts of $\mathcal{D}(\mathcal{A})$, we set

$$d_i \;=\; \begin{cases} a_k & \text{if } i \text{ is an } \underline{opening} \text{ element,} \\ b_k & \text{if } i \text{ is a } \underline{closing} \text{ element,} \\ c_k & \text{if } i \text{ is a } \underline{inner} \text{ element,} \\ c_k & \text{if } i \text{ is a } \underline{singleton} \text{ part of } \mathcal{A}\ . \end{cases} \tag{7.3}$$

Carrying this out for the partition in 7.1 yields the word

$$a_o c_1 a_1 b_2 c_1 c_1 a_1 a_2 b_3 c_2 c_2 c_2 b_2 b_1 \ .$$

Recalling that the letters a_k, b_k, c_k represent respectively NORTHEAST , SOUTHEAST and EAST steps of the path, we see that the partition diagram in 7.1 can thus be converted into the following Motzkin path

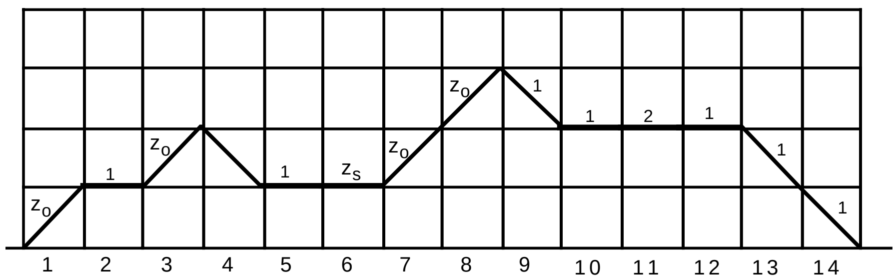


7.4

The final object of this construction, that is the labelled Motzkin path $L\pi(\mathcal{A})$ is simply obtained by labeling the edges of $\pi(\mathcal{A})$ according to the following rules. If e_i is the i^{th} edge of $\pi(\mathcal{A})$ then the label of e_i is

$$L(e_i) \;=\; \begin{cases} z_o & \text{if } i \text{ is an } \underline{opening} \text{ element,} \\ k & \text{if } i \ \underline{closes} \text{ the } k^{th} \text{ part covering } i, \\ k & \text{if } i \text{ is an } \underline{inner} \text{ element of the } k^{th} \text{ part covering } i, \\ z_s & \text{if } i \text{ is a } \underline{singleton} \text{ part of } \mathcal{A}\ . \end{cases} \tag{7.5}$$

This procedure converts the path in 7.4 into the labelled path given below.



7.6

It is not difficult to see that this construction yields a bijection between partitions of the set $\{1,2,\ldots,n\}$ and labelled Motzkin paths in $\Pi_{oo}(n)$ where the label $L(e)$ of an edge starting at height k is

$$L(e) \;=\; \begin{cases} z_o & \text{if } e \text{ is a NORTHEAST edge,} \\ \in \{1,2,\ldots,k\} & \text{if } e \text{ is a SOUTHEAST edge,} \\ \in \{1,2,\ldots,k\} & \text{if } e \text{ is an EAST edge,} \\ z_s & \text{if } i \text{ is a } \underline{singleton} \text{ part of } \mathcal{A}\ . \end{cases} \tag{7.7}$$

This observation can be translated into the following remarkable result

Theorem 7.1(Flajolet)

If μ_{n,n_o,n_i,n_s} denotes the number of partitions of the set $\{1,2,\ldots,n\}$ with n_o non-singleton parts, n_i inner elements and n_s singletons then we have the following continuous fraction expansion

$$\cfrac{1}{1 - z_s x - \cfrac{1\cdot z_o x^2}{1-(z_s+z_i)x - \cfrac{2\cdot z_o x^2}{1-(z_s+2z_i)x - \cfrac{3\cdot z_o x^2}{1-(z_s+3z_i)x-\cdots}}}} = \sum_{n=2n_o+n_i+n_s} \mu_{n,n_o,n_i,n_s}\, x^n z_o^{n_o} z_i^{n_i} z_s^{n_s} \tag{7.8}$$

Proof

Denoting by $\mathcal{P}_n$ the collection of all partitions of $\{1,2,\ldots,n\}$, we see that the right hand side of 7.8 is none other than

$$\sum_{n\geq 0}\sum_{A\in\mathcal{P}_n} z_o^{n_o(A)} z_i^{n(i(A)} x_s^{n_s(A)} \tag{7.9}$$

where $n_o(A)$, $n_i(A)$ and $n_s(A)$ respectively denote the number of non-singleton parts, the number of inner elements and the number of singleton parts of A. The Flajolet correspondence allows us to rewrite 7.9 as a sum over labelled Motzkin parts. That is

$$\text{RHS of} \quad 7.8 \;=\; \sum_{n\geq 0} x^n \sum_{L\pi\in L\Pi_{oo}(n)} p(L\pi)\ , \tag{7.10}$$

where $L\Pi_{oo}(n)$ denotes the collection of all labelled paths obtained by taking each path in $\Pi_{oo}(n)$ and labeling it in all possible ways consistent with the requirements in 7.7. Here of course we must define $p(L\pi)$ so that if $L\pi$ is the labelled path corresponding to a partition A then

$$p(L\pi) \;=\; z_o^{n_o(A)} z_i^{n(i(A)} x_s^{n_s(A)}\ .$$

Let $L\pi$ denote the generic labelled path obtained by labeling π according to 7.7 and set

$$P(\pi) \;=\; \sum_{L\pi} p(L\pi)$$

It is not difficult to see that the contribution of an edge e of π to $P(\pi)$ is a factor $f(e)$ which is given by

$$f(e) \;=\; \begin{cases} z_o & \text{if } e \text{ is a NORTHEAST edge,}\\ k & \text{if } e \text{ is a SOUTHEAST edge,}\\ z_s+kz_i & \text{if } e \text{ is an EAST edge,}\end{cases}$$

This yields that

$$\sum_{L\pi\in L\Pi_{oo}(n)} p(L\pi) \;=\; \sum_{\pi\in\Pi_{oo}(n)} w(\pi)\ |_{a_k=z_o,b_k=k,c_k=z_s+kz_i}\ .$$

In other words, the right hand side of 7.8 is simply equal to the formal series obtained by setting $a_k=z_o, b_k=k, c_k=z_s+kz_i$ in the right hand side of 4.2. But then equation 4.4 gives that we can also obtain the same result by making these replacements in the continued fraction

$$\cfrac{1}{1-c_ox-\cfrac{a_ob_1x^2}{1-c_1x-\cfrac{a_1b_2x^2}{1-c_2x-\cfrac{a_2b_3x^2}{1-c_3x-\cdots}}}}\ .$$

However doing this gives precisely the right hand side of 7.8 as desired.

Theorem 7.1 yields the continued fraction expansion for the generating functions of the Bell numbers B_n, the Stirling numbers S_{nk} and the numbers I_n and J_n of involutions of S_n with and without fixed points. These are the contents of the four theorems given below.

Theorem 7.2

Denoting by B_n the number of partitions of $\{1,2,\ldots,n\}$ we have

$$\cfrac{1}{1-x-\cfrac{1\cdot x^2}{1-2\cdot x-\cfrac{2\cdot x^2}{1-3\cdot x-\cfrac{3\cdot x^2}{1-4\cdot x-\cdots}}}} \;=\; \sum_{n\geq 0} B_n\, x^n \;. \qquad 7.11$$

Proof

Just set $z_o = z_i = z_s = 1$ in 7.8.

Theorem 7.3

If S_{nk} denotes the number of partitions of $\{1,2,\ldots,n\}$ into k parts, then

$$\cfrac{1}{1-tx-\cfrac{1\cdot tx^2}{1-(t+1)x-\cfrac{2\cdot tx^2}{1-(t+2)x-\cfrac{3\cdot tx^2}{1-(t+3)x-\cdots}}}} \;=\; \sum_{n\geq 0} x^n \sum_{k=1}^{n} S_{nk} t^k \;. \qquad 7.12$$

Proof

Just set $z_i = 1$ and $z_o = z_i = t$ in 7.8.

Theorem 7.4

If I_n denotes the number of involutions in the symmetric group S_n then

$$\cfrac{1}{1-x-\cfrac{1\cdot x^2}{1-x-\cfrac{2\cdot x^2}{1-x-\cfrac{3\cdot x^2}{1-x-\cdots}}}} \;=\; \sum_{n\geq 0} I_n\, x^n \;. \qquad 7.13$$

Proof

Note that if we set $z_i = 0$ and $z_o = z_i = 1$ in 7.8 then the right hand side of 7.8 yields the generating function of the number of partitions of $\{1,2,\ldots,n\}$ with no parts of cardinality greater than 2. But the latter partitions are clearly in bijection with involutions.

Theorem 7.5

If J_n denotes the number of involutions in the symmetric group S_n which have no fixed points then $J_n = 0$ for n odd and

$$\cfrac{1}{1-\cfrac{1\cdot x^2}{1-\cfrac{2\cdot x^2}{1-\cfrac{3\cdot x^2}{1-\cdots}}}} \;=\; \sum_{n\geq 0} J_{2n}\, x^{2n} \;. \qquad 7.14$$

Proof

By the same reasons we gave in the previous proof we can obtain this result by setting $z_s = z_i = 0$ and $z_o = 1$ in 7.8.

Remark 7.1

It is well known that the number of permutations of S_n with α_i cycles of length i is given by the expression

$$\frac{n!}{\prod_i i^{\alpha_i} \alpha_i!}$$

This gives that the number of involutions of S_n with k 2-cycles is given by

$$\frac{n!}{(n-2k)!\, 2^k k!}$$

Thus setting $z_s = 1$, $z_i = 0$ and $z_o = t$ in 7.8 gives the identity

$$\cfrac{1}{1 - x - \cfrac{1 \cdot tx^2}{1 - x - \cfrac{2 \cdot tx^2}{1 - x - \cfrac{3 \cdot tx^2}{1 - x - \cdots}}}} \;=\; \sum_{n \geq 0} x^n \sum_{k \leq n/2} \frac{n!}{(n-2k)!\, 2^k k!}\, t^k \;. \qquad 7.15$$

Similarly we get that the number J_{2n} of involutions without fixed points in S_{2n} is

$$\frac{(2n)!}{2^n n!} \;=\; 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

Thus formula 7.14 may be read as

$$\cfrac{1}{1 - \cfrac{1 \cdot x^2}{1 - \cfrac{2 \cdot x^2}{1 - \cfrac{3 \cdot x^2}{1 - \cdots}}}} \;=\; \sum_{n \geq 0} 1 \cdot 3 \cdot 5 \cdots (2n-1)\, x^{2n} \;. \qquad 7.16$$

This last observation yields us the connection between involutions and the Hermite polynomials $H_n(x)$. The latter may be defined by letting

$$H_n(x) \;=\; (-1)^n\, e^{x^2/2}\, D^n\, e^{-x^2/2} \qquad 7.17$$

Note that a single integration by parts gives that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^m\, H_n(x)\, e^{-x^2/2}\, dx \;=\; \frac{m}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^{m-1}\, H_{n-1}(x)\, e^{-x^2/2}\, dx$$

If $m \leq n$ we can apply this relation m times in succession and obtain that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^m\, H_n(x)\, e^{-x^2/2}\, dx \;=\; \begin{cases} n! & \text{if } m = n \text{ and} \\ 0 & \text{if } m \leq n \end{cases} \;. \qquad 7.18$$

This implies that the Hermite polynomials are orthogonal with respect to the measure μ on $(-\infty,+\infty)$ given by setting

$$d\mu \;=\; \frac{1}{\sqrt{\pi}}\, e^{-x^2/2}\, dx \qquad 7.19$$

Since the leading term in $H_n(x)$ is x^n we can apply the theory directly here and obtain

Theorem 7.6

Let $H_n^*(x) = x^n H_n(1/x)$ and let $K_n^*(x)$ be the sequence of polynomials defined by the recursion $K_{n+1}^*(x) = K_n^*(x) - nx^2 K_{n-1}^*(x)$ with initial conditions $K_o^*(x) = K_1^*(x) = 1$ then

$$lim_{n\to\infty}\, \frac{K_n^*(x)}{H_{n+1}^*(x)} \;=\; \cfrac{1}{1-\cfrac{1\cdot x^2}{1-\cfrac{2\cdot x^2}{1-\cfrac{3\cdot x^2}{1-\cdots}}}} \qquad 7.20$$

In other words this is the continued fraction corresponding to the Hermite polynomials. In particular, they must satisfy the recursion

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}^*(x) \qquad 7.21$$

Proof

From the continued fraction expansion in 7.16 and theorem 3.1 it follows that the sequence of polynomials $\{Q_n(x)\}$ orthogonal with respect to a measure with moments μ_n given by

$$\begin{cases} a) & \mu_{2n-1} \;=\; 0 \qquad\qquad \text{and} \\ b) & \mu_{2n} \;=\; 1\cdot 3\cdots(2n-1) \end{cases} \qquad 7.22$$

and satisfying the requirement in 3.3 a) must satisfy the recurrence $Q_{n+1} = xQ_n - nQ_{n-1}$. So to prove our assertions we need only verify the measure μ given in 7.19 has precisely these moments. Now 7.22 a) is trivial since μ is symmetrically distributed around the origin. As for 7.22 b) we only need to observe that it follows from the recursion

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^{2n}\, e^{-x^2/2}\, dx \;=\; \frac{2n-1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^{2n-2}\, e^{-x^2/2}\, dx \;,$$

(which we can immediately obtain by a single integration by parts) and the initial condition $\mu_o = 1$ which is given by the classical integral

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \;=\; \sqrt{\pi}\;.$$

This completes our section.

8. The Legendre polynomials

The Legendre polynomials $\{L_n(x)\}$ may defined by means of the formula

$$L_n(x) \;=\; \frac{1}{2^n n!}\, D^n(x^2-1)^n \;. \qquad 8.1$$

These polynomials are easily shown to be orthogonal with respect to the measure $d\mu = dx/2$ concentrated on the interval $[-1,1]$. Indeed, we do have that

Proposition 8.1

$$\frac{1}{2}\int_{-1}^{1} P_n(x)\, P_m(x)\, dx \;=\; \begin{cases} 1)\ 0 & \text{if } n\neq m \text{ and} \\ 2)\ \frac{1}{2n+1} & \text{if } n=m \end{cases} \;. \qquad 8.2$$

Proof

For convenience let $\Gamma_n = D^n(x^2-1)^n$. Then an integration by parts gives

$$\int_{-1}^{1} \Gamma_n(x)\,\Gamma_m(x)\, dx \;=\; \int_{-1}^{1} D^{n-1}(x^2-1)^n\, D^{m+1}(x^2-1)^m\, dx$$

repeating this process will necessarily yield zero after $m+1$ integration by parts if $m<n$. This establishes the first equality in 8.2. On the other hand if $m=n$ then n succesive integrations by parts yield

$$\int_{-1}^{1} \Gamma_n(x)\,\Gamma_n(x)\, dx \;=\; (-1)^n\,(2n)! \int_{-1}^{1} (x^2-1)^n\, dx \;. \qquad 8.3$$

But integrating by parts again we get

$$\int_{-1}^{1} (1-x^2)^n\, dx \;=\; 2n \int_{-1}^{1} (1-x^2)^{n-1}\, x^2\, dx \;.$$

Now this can be rewritten in the form

$$(2n+1)\int_{-1}^{1} (1-x^2)^n\, dx \;=\; 2n \int_{-1}^{1} (1-x^2)^{n-1}\, dx \;.$$

Iterating and combining with 8.3 yields

$$\int_{-1}^{1} \Gamma_n^2(x)\, dx \;=\; 2\,\frac{(2n)!!}{(2n+1)!!}\,(2n)! \;=\; 2\,\frac{((2n)!!)^2}{2n+1}$$

Since $\Gamma_n = 2^n\, n!\, P_n$ we see that this is precisely the second identity in 8.2.

Now the definition in 8.1 gives that the leading term in $L_n(x)$ is $\frac{(2n-1)!!}{n!}\, x^n$. Thus to apply our machinery we need to take

$$Q_n(x) \;=\; \frac{n!}{(2n-1)!!}\, L_n(x) \;. \qquad 8.4$$

We can then rewrite 8.2 2) in terms of the Q_n to read

$$\int_{-1}^{1} Q_n^2(x)\,dx \;=\; \frac{(n!)^2}{(2n-1)!!\,(2n+1)!!} \;. \tag{8.5}$$

We are now in a position to determine the coefficients c_n and λ_n occurring in the three-term recursion. Indeed, we need not work very hard for c_n since P_n and therefore also Q_n is an odd function of x for odd n and an even function for n even. Thus, since μ is symmetrically distributed, the integral formula 3.10 gives that $c_n = 0$ for all n. As for λ_n, we can apply formula 4.33 a) and obtain from 8.5 that

$$\lambda_n \;=\; \frac{\int_{-1}^{1} Q_n^2(x)\,dx}{\int_{-1}^{1} Q_{n-1}^2(x)\,dx} \;=\; \frac{n^2}{(2n-1)(2n+1)} \;. \tag{8.6}$$

This gives the recursion

$$Q_{n+1} \;=\; x\,Q_n \;-\; \frac{n^2}{(2n-1)(2n+1)}\,Q_{n-1} \;.$$

Using 8.4 we deduce that the Legendre polynomials themselves must satisfy the recursion

$$(n+1)\,L_n(x) \;=\; (2n+1)x\,L_n(x) \;-\; n\,L_{n-1}(x) \;. \tag{8.7}$$

Note further that in this case the sequence of moments is given by

$$\mu_n \;=\; \frac{1}{2}\int_{-1}^{1} x^n\,dx \;=\; \begin{cases} 0 & \text{if } n \text{ is odd, and} \\[2ex] \frac{1}{(n+1)} & \text{if } n \text{ is even.} \end{cases}$$

Thus this time theorem 3.1 may be used to derive a rather neat continued fraction expansion. Namely

Theorem 8.1

$$\cfrac{x}{1-\cfrac{1^2\cdot x^2/1\cdot 3}{1-\cfrac{2^2\cdot x^2/3\cdot 5}{1-\cfrac{3^2\cdot x^2/5\cdot 7}{1-\cdots}}}} \;=\; \sum_{k\geq 0}\frac{x^{2k+1}}{2k+1} \;=\; log\sqrt{\frac{1+x}{1-x}} \tag{8.8}$$

9. The Laguerre polynomials.

These polynomials depend on a parameter α which may be assumed to be an independent variable. They can be defined by setting

$$L_n^{(\alpha)}(x) \;=\; (-1)^n\,e^x\,x^{-\alpha}\,D^n\left(x^{n+\alpha}\,e^{-x}\right) \;. \tag{9.1}$$

Note that an integration by parts, gives

$$\int_0^{+\infty} x^m\, L_n^{(\alpha)}(x) x^\alpha\, e^{-x}\, dx \;=\; (-1)^{n-1}\, m \int_0^{+\infty} x^{m-1}\, D^{n-1}\big(x^{n+\alpha}\, e^{-x}\big)\, dx \;. \tag{9.2}$$

We see again that if $m < n$ then $m+1$ successive integration by parts yield that this integral vanishes. If $m \geq n$ we can carry out n successive integrations by parts and get

$$\int_0^{+\infty} x^m\, L_n^{(\alpha)}(x) x^\alpha\, e^{-x}\, dx \;=\; \frac{m!}{(m-n)!} \int_0^{+\infty} x^{m+\alpha}\, e^{-x}\big)\, dx \;.$$

Finally, m more integrations by parts give

$$\int_0^{+\infty} x^m\, L_n^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx \;=\; \frac{m!}{(m-n)!}\, (m+\alpha)(m+\alpha-1)\cdots(\alpha+1) \int_0^{+\infty} x^\alpha\, e^{-x}\, dx \tag{9.3}$$

Letting μ be the measure concentrated on $[0,\infty)$ defined by setting

$$d\mu \;=\; \frac{x^\alpha\, e^{-x}}{\Gamma(\alpha+1)} \quad \text{with} \quad \Gamma(\alpha+1) \;=\; \int_0^{+\infty} x^\alpha\, e^{-x}\, dx \;,$$

we can translate our findings here into the identities

$$\int_0^{+\infty} x^m\, L_n^{(\alpha)}(x)\, d\mu \;=\; \begin{cases} 0 & \text{if } m < n \text{ and} \\ \frac{m!}{(m-n)!}\, (m+\alpha)(m+\alpha-1)\cdots(\alpha+1) & \text{for } m \geq n \end{cases} \tag{9.4}$$

In particular, we see that the polynomials $L_n^{(\alpha)}(x)$ are orthogonal with respect to μ. Moreover, since the leading term in $L_n^{(\alpha)}(x)$ is again x^n we can apply our theory without rescaling. Now formulas 4.35 and 9.4 for $n = m$ immediately give that

$$\lambda_n \;=\; \frac{n!\, (n+\alpha)(n+\alpha-1)\cdots(\alpha+1)}{(n-1)!\, (n+\alpha-1)(n+\alpha-1)\cdots(\alpha+1)} \;=\; n(n+\alpha) \;. \tag{9.5}$$

To compute c_n here we shall use 4.33 which we rewrite in the form

$$c_n \;=\; \frac{\chi_n}{d_{n-1}}\frac{d_{n-1}}{d_n} \;-\; \frac{\chi_{n-1}}{d_{n-2}}\frac{d_{n-2}}{d_{n-1}} \;. \tag{9.6}$$

Now 4.38 gives that

$$\frac{\chi_n}{d_{n-1}} \;=\; \frac{(-1)^n}{\Gamma(\alpha+1)} \int_0^{\infty} x^{n+1}\, D^n\, x^{\alpha+n} e^{-x}\, dx \;,$$

and using 9.4 for $m = n+1$ we get

$$\frac{\chi_n}{d_{n-1}} \;=\; (n+1)!\, (\alpha+n+1)(\alpha+n)\cdots(\alpha+1) \;. \tag{9.7}$$

On the other hand 4.36 and 9.4 for $m = n$ give

$$\frac{d_n}{d_{n-1}} \;=\; n!\, (\alpha+n)(\alpha+n-1)\cdots(\alpha+1) \;. \tag{9.8}$$

Using 9.7 and 9.8 for n and $n-1$ we can finally conclude that

$$\begin{aligned} c_n = &\ \frac{(n+1)!\ (\alpha+n+1)(\alpha+n)\cdots(\alpha+1)}{n!\ (\alpha+n)(\alpha+n-1)\cdots(\alpha+1)} - \frac{n!\ (\alpha+n)(\alpha+n-1)\cdots(\alpha+1)}{(n-1)!\ (\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)} = \\ = &\ 2n+\alpha+1\ . \end{aligned} \qquad 9.10$$

The moments μ_n are also easily computed in this case. In fact, we have

$$\mu_n = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} x^{n+\alpha} e^{-x}\, dx = (\alpha+n)(\alpha+n-1)\cdots(\alpha+1)\ . \qquad 9.11$$

Formulas 9.5, 9.6 and 9.11 yield us the continued fraction expansion

Theorem 9.1

$$\cfrac{1}{1-(\alpha+1)x-\cfrac{1\cdot(\alpha+1)x^2}{1-(\alpha+3)x-\cfrac{2\cdot(\alpha+2)x^2}{1-(\alpha+5)x-\cfrac{3\cdot(\alpha+3)x^2}{1-(\alpha+7)x-\cdots}}}} = \qquad 9.12$$

$$= \sum_{n\geq 0} (\alpha+n)(\alpha+n-1)\cdots(\alpha+1)\, x^n\ .$$

Note that letting $\alpha=0$ in 9.12 yields the remarkable identity

$$\cfrac{1}{1-1\cdot x-\cfrac{1^2\cdot x^2}{1-3\cdot x-\cfrac{2^2\cdot x^2}{1-5\cdot x-\cfrac{3^3\cdot x^2}{1-7\cdot x-\cdots}}}} = \sum_{n\geq 0} n!\, x^n\ . \qquad 9.13$$

In closing we should note that some authors, whose name may be best left out here, have had some difficulty swallowing this identity. Their problem was that they could not see how to fit it into their Cauchy-limited world. Apparently Gauss had no such difficulty for he knew very well how to make sense out of it.

From our point of view 9.13 in none other than a purely combinatorial identity. It simply expresses the existence of a bijection between permutations and a certain class of labelled Motzkin paths. In fact, using the interpretation of the moments given in 4.5, we can see that 9.12 can be established by showing that

$$\sum_{\pi\in\Pi_{oo}(n)} w(\pi)\ |_{a_n=n+1;b_n=n+\alpha;c_n=2n+1+\alpha} = (\alpha+n)(\alpha+n-1)\cdots(\alpha+1) \qquad 9.14$$

To get this we need a bijection between permutations in S_n and labelled Motzkin paths in $\Pi_{oo}(n)$ where the label $L(e)$ of an edge e which starts at height k should be

$$L(e) = \begin{cases} \in\{1,2,\ldots,k\} & \text{if } e \text{ is a NORTHEAST edge,} \\ \in\{1,2,\ldots,k,\alpha\} & \text{if } e \text{ is a SOUTHEAST edge,} \\ \in\{1,2,\ldots,2k+1,\alpha\} & \text{if } e \text{ is an EAST edge,} \end{cases} \qquad 9.15$$

Now a very beautiful bijection doing precisely that was given by Françon and Viennot in [5]. It can also be found in Flajolet's paper [4].

We should point out that, just as was the case for the Flajolet bijection given in section 7, The Françon-Viennot bijection yields a variety of continued fraction expansions. Using it we may obtain generating functions of combinatorial sequences enumerating permutations according to various statistics. Notably, it gives a combinatorial explanation to the Stieltjes computation of the Laplace transform of $tan\ x$. However, in order to keep these notes to a reasonable size we shall have to refer the reader to the original papers for this additional material.

REFERENCES

[1] G. Andrews, The Theory of Partitions, Encyc. of Math. and its Applications, 2 Addison-Wesley, Reading (1976).

[2] G. Andrews and R. J. Baxter, A motivated Proof of The Roegers-Ramanujan Identities, Amer. Math. Monthly 96 #5 (1989) 401-409.

[3] P. Cartier and D. Foata, Problèmes Combinatoires de commutation et réarrangements, Springer Lecture Notes in Math. 85 (1969).

[4] P. Flajolet, Combinatorial aspects of Continued Fractions, Discrete Math. 32 (1980) 125-161.

[5] J. Françon and G. Viennot, Permutations selon les pics, creux, double montées, double descentes, Discrete Math. 28 (1979) 21-35.

[6] A. M. Garsia and S. C. Milne, A Rogers-Ramanujan Bijection, J. Comb. Theory Ser A 31 (1981) 289-329.

[7] I. Schur, Ein Betrag zur additiven Zahlentheorie und zur Theorie der Kettenbruche, S-B Preuss. Akad. Wiss. Phys. - Math. Kl, (1917) 302-321.

[8] G. Viennot, Une Théorie Combinatoire des Polynomes Orthogonaux Generaux, Lecture notes UQAM Montréal (1983)

[9] G. Viennot, Heaps of Pieces, Lecture Notes Univ. of Bordeaux # I-8614, 351 cours de la Libération 33405 Talence, France. See also Proceedings of the Montréal Colloquium In Enumerative Combinatorics, UQAM Montréal (1985).