The action of a finite Group G on a G invariant Vector space

Let G be a finite group and $\mathbf{V}$ be a finite dimensional vector space over a field K. We say that G "*acts*" on $\mathbf{V}$ if for each $\alpha \in G$ we have a linear transformation T_α of $\mathbf{V}$ onto itself such that

$$T_\alpha T_\beta \;=\; T_{\alpha\beta} \qquad \text{(for all } \alpha,\beta \in G) \tag{1}$$

It follows from this that, given a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_d\}$ of $\mathbf{V}$ we can construct a d-dimensional representation $\alpha \to A(\alpha)$ by simply settiing $A(\alpha) = \|a_{i,j}(\alpha)\|_{i,j=1}^{d}$ when

$$T_\alpha b_j \;=\; \sum_{i=1}^{d} b_i a_{i,j}(\alpha) \tag{2}$$

Or in matrix form

$$T_\alpha \langle b_1, b_2, \ldots, b_d \rangle \;=\; \langle b_1, b_2, \ldots, b_d \rangle A(\alpha) \tag{3}$$

It is easy to see that the character χ^A of this representation is independent of the chosen basis. If we are in possession of a complete set $\{A^\lambda\}_{\lambda=1}^{k}$ of irreducible representations of G and their corresponding system of characters $\{\chi^\lambda\}_{\lambda=1}^{k}$, a fundamental problem is to calculate the coefficients m_λ in the expansion

$$\chi^A \;=\; \sum_{\lambda=1}^{k} m_\lambda \chi^\lambda \tag{4}$$

In these notes we give an algorithmic solution of this problem which turns out to be quite efficient in many practical applications.

To state the result we need some definitions. Note that from our hypotheses we can define the action of an element $f \in \mathcal{A}(G)$ on $\mathbf{V}$ by simply setting

$$T_f b \;=\; \sum_{\alpha \in G} f(\alpha) T_\alpha b \qquad \text{(for all } b \in \mathbf{V}) \tag{5}$$

moreover property (1) assures that

$$T_f T_g \;=\; T_{fg} \qquad \text{(for all } f, g \in \mathcal{A}(G)). \tag{6}$$

This given, for any $f \in \mathcal{A}(G)$ we can set

$$T_f \mathbf{V} = \{T_f b \,:\, v \in \mathcal{B}\} \tag{7}$$

Clearly $T_f\mathbf{V}$ is a subspace of $\mathbf{V}$, in particular is easily seen that for our basis $\mathcal{B}$, given above we have

$$T_f \mathbf{V} \;=\; \mathcal{L}[T_f b_1, T_f b_2, \ldots, T_f b_d] \tag{8}$$

In other words, the collection $T_f\mathcal{B} = \{T_f b_1, T_f b_2, \ldots, T_f b_d\}$ spans $T_f\mathbf{V}$.

At this point it will be convenient to drop the "$T_f b$" notation for the action of a group algebra element f on an element $b \in \mathbf{V}$ and simply write "fb". Using this notation we can state

Theorem 1

If n_λ is the dimension of the representation A^λ then

$$m_\lambda = \tfrac{1}{n_\lambda} \dim \chi^\lambda \mathbf{V} \tag{9}$$

The proof of this result will be obtained by combining three auxiliary Propositions and a Theorem.

Let us recall that we can construct in $\mathcal{A}(G)$ a basis $\cup_{\lambda=1}^{k}\{e^{\lambda}_{i,j}\}_{i,j=1}^{n_\lambda}$ we the following multiplicative properties

$$e^{\mu}_{r,s}e^{\lambda}_{i,j} \;=\; \begin{cases} 0 & \text{if } \lambda\neq\mu \\ 0 & \text{if } \lambda=\mu \text{ but } s\neq i \\ e^{\lambda}_{r,j} & \text{if } \lambda=\mu \text{ and } s=i \end{cases} \tag{10}$$

We can also show that for $h_\lambda = |G|/n_\lambda$ we have

$$e^{\lambda} = \chi_\lambda/h_\lambda = e^{\lambda}_{1,1} + e^{\lambda}_{2,2} + \cdots e^{\lambda}_{n_\lambda,n_\lambda} \qquad\qquad (\text{for all } \lambda\vdash n) \tag{11}$$

Moreover, the action of G on the linear span of the sub-basis

$$\{e^{\lambda}_{1,i}, e^{\lambda}_{2,i}, \ldots, e^{\lambda}_{n_\lambda,i}\}$$

is a representation with character χ^{λ}. More precisely we have

$$\alpha\langle e^{\lambda}_{1,i}, e^{\lambda}_{2,i}, \ldots, e^{\lambda}_{n_\lambda,i}\rangle \;=\; \langle e^{\lambda}_{1,i}, e^{\lambda}_{2,i}, \ldots, e^{\lambda}_{n_\lambda,i}\rangle A(\alpha) \qquad\qquad (\text{for all } \alpha\in G) \tag{12}$$

Keeping in mind all these facts, note first that we have the following basic result

Proposition 1

We always have the following direct sum decomposition

$$e^{\lambda}\mathbf{V} \;=\; \bigoplus_{i=1}^{n_\lambda} e^{\lambda}_{i,i}\mathbf{V}, \tag{13}$$

in particular it follows that

$$\dim e^{\lambda}\mathbf{V} \;=\; \sum_{i=1}^{n_\lambda} \dim e^{\lambda}_{i,i}\mathbf{V}. \tag{14}$$

Proof

Note first that if $b\in e^{\lambda}\mathbf{V}$ then , since $e^{\lambda}b=b$, from (11) it follows that

$$b = \sum_{i=1}^{n_\lambda} e^{\lambda}_{i,i}b. \tag{15}$$

To show (13) we need only show that this decomposition is unique. To this end suppose that $b\in e^{\lambda}\mathbf{V}$ and that for some elements $b_i\in e^{\lambda}_{i,i}\mathbf{V}$ (for $1\le i\le n_\lambda$) we have

$$b \;=\; \sum_{i=1}^{n_\lambda} b_i \tag{16}$$

then since from (11) it follows that $e^{\lambda}_{j,j}\chi^{\lambda} = h_\lambda e^{\lambda}_{j,j}$ and for $i\neq j$ $e^{\lambda}_{j,j}b_i = 0$, by multiplying both sides of (16) by $e^{\lambda}_{j,j}$ we get

$$e^{\lambda}_{j,j}b \;=\; e^{\lambda}_{j,j}b_j = b_j$$

completing our proof.

We still need two more facts to derive Theorem 1.

Proposition 2

For any non vanishing element $b \in e^{\lambda}_{i,i}\mathbf{V}$ *the collection*

$$\{e^{\lambda}_{1,i}b, e^{\lambda}_{2,i}b, \ldots, e^{\lambda}_{n_\lambda,i}b\} \;=\; \{e^{\lambda}_{1,i}, e^{\lambda}_{2,i}, \ldots, e^{\lambda}_{n_\lambda,i}\}b \tag{17}$$

is a basis of a subspace of $e^{\lambda}\mathbf{V}$ *on which* G *acts by the representation* A^{λ}.

Proof

It is sufficient to show that the elements of the collection in (15) do not vanish and that they are independent. This done, the last assertion is simply an immediate consequence of (12). To prove independence it is sufficient to show that any vanishing non trivial linear combination of the elements forces b itself to vanish. Suppose then that for some constants c_r we have

$$\sum_{r=1}^{n_\lambda} c_r e^{\lambda}_{r,i} b \;=\; 0 \tag{18}$$

say $c_s \neq 0$ for some $1 \leq s \leq n_\lambda$. Then multitplying both sides of this identity by $e^{\lambda}_{i,s}$ and using (10) (and the associativity property in (1)) we obtain

$$0 \;=\; c_s e^{\lambda}_{i,s} e^{\lambda}_{s,i} b \;=\; c_s e^{\lambda}_{i,i} b \;=\; c_s b$$

and the non vanishing of c_s gives $b = 0$ as desired.

We are now in a position to prove a more refined version of Theorem 1.

Theorem 2

Let $\{b_1, b_2, \ldots, b_l\}$ *be a basis for* $e^{\lambda}_{1,1}\mathbf{V}$*, then by the previous proposition each collection* $\{e^{\lambda}_{1,1}b_s, e^{\lambda}_{2,1}b_s, \ldots, e^{\lambda}_{n_\lambda,1}b_s\}$ *is a basis of a subspace of* $e^{\lambda}\mathbf{V}$ *on which* G *acts by the representation* A^{λ}*. These* l *subspaces are disjoint and independent. More precisely the collection*

$$\mathcal{B} = \bigcup_{s=1}^{l} \{e^{\lambda}_{1,1}b_s, e^{\lambda}_{2,1}b_s, \ldots, e^{\lambda}_{n_\lambda,1}b_s\} \tag{19}$$

is a basis for $e^{\lambda}\mathbf{V}$ *and all the subspaces* $e^{\lambda}_{i,i}\mathbf{V}$ *have same dimension* l *with bases*

$$\mathcal{B}_i = \{e^{\lambda}_{i,1}b_1, e^{\lambda}_{i,1}b_2, \ldots, e^{\lambda}_{i,1}b_l\} \tag{19}$$

In particular it follows that the multiplicity m_λ *we are seeking for is none other than the common dimension of the subspaces* $e^{\lambda}_{i,i}\mathbf{V}$.

Proof

The independence of $\mathcal{B}$ is quickly disposed off. In fact, suppose if possible that for some constants $c_{i,s}$, we have

$$\sum_{i=1}^{n_\lambda}\sum_{s=1}^{l} c_{i,s} e^{\lambda}_{i,1} b_s \;=\; 0, \tag{20}$$

we then only need to hit both sides of this relation by $e^{\lambda}_{1,j}$ and reduce it to

$$0 = \sum_{s=1}^{l} c_{j,s} e^{\lambda}_{1,j} e^{\lambda}_{j,1} b_s = \sum_{s=1}^{l} c_{j,s} e^{\lambda}_{1,1} b_s = \sum_{s=1}^{l} c_{j,s} b_s,$$

which plainly contradicts the independence of $\{b_1, b_2, \ldots, b_l\}$ unless all the $c_{j,s}$ simultaneously vanish.

But now the independence of $\mathcal{B}_i$ and the fact that $\mathcal{B}_i \subseteq e^{\lambda}_{i,i}\mathbf{V}$ gives the relation

$$\dim e^{\lambda}_{1,1}\mathbf{V} \leq \dim e^{\lambda}_{i,i}\mathbf{V}. \tag{21}$$

But there is nothing special about $e^{\lambda}_{1,1}\mathbf{V}$, since we could have operated on $e^{\lambda}_{i,i}\mathbf{V}$ just the same way we did on $e^{\lambda}_{1,1}\mathbf{V}$ and we would have obtained the reverse of (21). This forces the stated dimension equalities and (13) yields that

$$\dim e^{\lambda}\mathbf{V} \;\; = \;\; l \times n_{\lambda}, \tag{22}$$

proving not only Theorem 1 but also that $\mathcal{B}$ is a basis of $e^{\lambda}\mathbf{V}$.