If A is an ordered subset of $X_n = \{x_1, \ldots, x_n\}$ we let $\Delta(A)$ denote the Vandermonde determinant in the elements of A. To be precise (since order counts), if $A = \{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\}$ we set

$$\Delta(A) \;=\; det\|x_{a_i}^{j-1}\|_{i,j=1}^{m} \ , \qquad I.1$$

with the convention that $\Delta(A) = 1$ if A is an empty subset. Let $\mu = \{0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n\}$ be a partition of n and T be an injective tableau T of shape μ with entries $1, 2, \ldots, n$. Recalling that we use the French notation of depicting tableaux, let $C_n, C_{n-1}, \ldots, C_1$ denote the columns of T ordered from left to right. Note, that with our conventions concerning partitions, if $\mu' = \{0 \le \mu'_1 \le \mu'_2 \le \cdots \le \mu'_n\}$ is the conjugate of μ, then C_i has length μ'_i. Of course, if the diagram of μ has only k columns then $C_{n-k}, \ldots, C_2, C_1$, are necessarily empty. Now let A_i be the ordered subset of X_n obtained by selecting the variables x_j with indices in C_i in the order occurring in T (from bottom to top). This given, we set

$$\Delta_T(x) \;=\; \Delta(C_n)\Delta(C_{n-1})\ldots\Delta(C_1) \ , \qquad I.2$$

and refer to it as the *Garnir element* corresponding to T, we shall also say that $\Delta_T(x)$ is *standard* , *of shape* μ,...etc, if the same holds true for T itself.

For instance if

$$T \;=\; \begin{array}{ccc} 7 & & \\ 2 & 3 & 6 \\ 1 & 4 & 5 \end{array}$$

Then

$$\Delta_T(x) \;=\; det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_7 & x_7^2 \end{pmatrix} \times\, det\begin{pmatrix} 1 & x_4 \\ 1 & x_3 \end{pmatrix} \times\, det\begin{pmatrix} 1 & x_5 \\ 1 & x_6 \end{pmatrix} \ .$$

These polynomials were introduced by Garnir in [9], in his reconstruction of Young's natural representation. It will be convenient here and after to denote by $\lambda(T)$ the shape of a tableau T and by $ST(\mu)$ the collection of all standard tableux of shape μ. It is not difficult to show that the space

$$\Gamma_\mu \;=\; \mathcal{L}\{\Delta_T \ : \ T \in ST(\mu)\ \} \ , \qquad I.3$$

linear span of the standard Garnir elements of shape μ, is an irreducible S_n-module with character given by χ^μ in the Young indexing. To be precise, the S_n-action we refer to here is the usual action by permutation of variables. That is for a polynomial $P(x) = P(x_1, x_2, \ldots, x_n)$ and a permutation $\sigma \in S_n = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ we set

$$\sigma P(x) \;=\; P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) \ . \qquad I.5$$

We should note first that the relation of the Garnir module Γ_μ to Young's natural is quite immediate. In fact, we only need to point out that each Garnir element $\Delta_T(x)$ may also be written in the form

$$\Delta_T(x) \;=\; N(T)\, m_T(x) \;=\; \frac{1}{\mu!}\, N(T)P(T)m_T(x) \ . \qquad 4.1$$

Where, as customary $P(T)$ and $N(T)$ denote the so called *positive* and *negative* , *row* and *column groups* of T respectively and

$$m_T(x) \;=\; \prod_{i=1}^{n} x_i^{h_i(T)-1} \qquad 4.2$$

with $h_i(T)$ denoting the height of the label "i" in T. The expression $m_T(x)$ will occur quite often in this writing, we shall refer to it as the *monomial of* T. Perhaps all of this can be best understood through an example. For instance if

$$T \;=\; \begin{array}{ccc} 7 & & \\ 2 & 3 & 6 \\ 1 & 4 & 5 \end{array}$$

Then as we have seen

$$\Delta_T(x) \;=\; det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_7 & x_7^2 \end{pmatrix} \times det\begin{pmatrix} 1 & x_4 & \\ 1 & x_3 & \end{pmatrix} \times det\begin{pmatrix} 1 & x_5 \\ 1 & x_6 \end{pmatrix} \;.$$

Now, using Young's notation, we can rewrite these determinants in the form

$$det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_7 & x_7^2 \end{pmatrix} = [1,2,7]'\, x_2x_7^2 \;, \quad det\begin{pmatrix} 1 & x_4 \\ 1 & x_3 \end{pmatrix} = [4,3]'x_4 \;, \quad det\begin{pmatrix} 1 & x_5 \\ 1 & x_6 \end{pmatrix} = [5,6]'x_6 \;,$$

and this is an instance of the first equation in 4.1. Clearly the last equality in 4.1 holds true since the monomial of T is left invariant by any permutation of the row group of T.

Let now $T_1, T_2, \ldots, T_{n_\mu}$ denote the standard tableaux of shape μ (in the usual lexicographic order say) and σ_{ij} be the permutation that sends T_j into T_i, we recall that the Young *natural* representation corresponding to the partition μ may be defined as the integral matrix $A(\sigma) = \|a_{ij}(\sigma)\|$ which occurs in the relations (see Garsia-Wachs [13] eq. 2.13)

$$\sigma N(T_j)P(T_j) \;=\; \sum_{i=1}^{n_\mu} a_{ij}(\sigma)\, N(T_i)P(T_i)\, \sigma_{ij} \;. \qquad 4.3$$

Applying this group algebra element to the monomial of T_j we finally get

$$\sigma \Delta_{T_j}(x) \;=\; \sum_{i=1}^{n_\mu} \Delta_{T_i}(x)\, a_{ij}(\sigma) \;. \qquad 4.4$$

It will be convenient here to denote by $\mathcal{T}(\mu)$ the collection of injective tableaux of shape μ. We shall for a moment alter our the definition of $\mathbf{\Gamma}_\mu$ given in the introduction and set

$$\mathbf{\Gamma}_\mu \;=\; \mathcal{L}\{\Delta_T(x) \;:\; T \in \mathcal{T}(\mu)\ \} \;.$$

We can immediately see from 4.4 that we have not changed the space.

Theorem 4.1*(Garnir)*

The polynomials $\{\Delta_{T_i}(x)\}_{i=1}^{n_\mu}$ *form a basis for* $\mathbf{\Gamma}_\mu$. *Moreover, this space is an irreducible* S_n*-module with character* χ^μ*, in the Young indexing.*

Proof.

Using 4.4 we derive that the image of a linear combination

$$P(x) \;\;=\;\; \sum_{j=1}^{n_\mu} c_j \; \Delta_{T_j}(x)$$

by the action of an element f of the group algebra $\mathbf{Q}(S_n)$ is given by acting on the coefficient sequence $(c_1, c_2, \ldots, c_n)$ by the matrix

$$A(f) \;\;=\;\; \sum_{\sigma\in S_n} f(\sigma)A(\sigma) \;\; .$$

Using, the irreducibility of the representation $A(\sigma)$ we deduce that, as long as the coefficients c_i are not all zero, we can choose f in such a manner that the polynomial $f\;P(x)$ reduces to one of the $\Delta_{T_i}(x)$. Since none of the Garnir elements is equal to zero, it follows that $P(x)$ itself cannot be zero. This shows the independence of the standard Garnir elements. Thus they must form a basis. Finally, we see (again from 4.4) that in terms of this basis the action of a permutation σ is precisely given by Young's natural. Thus the last assertion of the theorem holds true in the strongest possible sense.