I

# Lecture Notes Fall 2003

# ON FINITE DIMENSIONAL SL(2) REPRESENTATIONS
## and some
## Applications to Algebraic Combinatorics

by

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## 1. Basic identities

Throughout these notes $\mathbf{V}$ will be a finite dimensional $K$ vector space. For all our applications there is no loss in specialising $K$ to be the field of rational numbers, but at some point in our development we will need to assume that $K$ is the field of complex numbers. We also assume that we have three linear operators $E$, $F$, $H$ acting on $V$ which satisfy the following three relations

$$(a)\ \ [E,F]=H \quad , \quad (b)\ \ [H,E]=2E \quad , \quad (c)\ \ [H,F]=-2F\ . \tag{1.1}$$

As customary the expression $L=(E+F+H)^*$ denotes the collection of all words in the alphabet $\{E,F,G\}$. Then the symbol $K[L]$ represents the collection of all finite linear combinations with coefficients in $K$ of words of $L$. We intend to consider $K[L]$ as the subalgebra of linear operators on $V$ generated by $E,F$ and $H$. Our purpose here is to decompose the action of $K[L]$ on $\mathbf{V}$ into its irreducible constituents. To this end we need to derive a number of auxiliary identities. These will be stated as separate propositions.

Note first that if $A,B,C$ are operators and we set

$$D_C\,A \;=\; [A,C] \;=\; AC-CA$$

then we can easily verify that

$$D_C\,AB \;=\; (D_CA)\,B \;+\; A\,D_CB \tag{1.2}$$

In other words $D_C$ acts like *differentiation* on products of linear operators. We shall make systematic use of this fact in the sequel. However a word of caution is necessary in regard to this notation. Certain identities satisfied by derivatives in ordinary calculus may not be valid for these differentiations. For instance familiar relation

$$D_CA^m \;=\; m\,A^{m-1}D_CA$$

can be used only when we know before hand that $D_CA$ commutes with $A$. An instance in point is given by the proof of the following

**Proposition 1.1**

*For any integer $m$ we have*

$$\begin{aligned} a)\quad HE^m &= E^m(H+2mI),\ ^{(\dagger)} \\ b)\quad HF^m &= F^m(H-2mI). \end{aligned} \tag{1.3}$$

**Proof**

Note we may write 1.3 a) in the form

$$D_H\, E^m \;=\; -2m\, E^m \;.$$

Note further that for $m=1$ this simply reduces to 1.1 b). Thus we can proceed by induction and assume it true for $m$. This given, using 1.1 (b) and 1.2 we get

$$D_H\, E^{m+1} \;=\; (D_H E)\, E^m \;+\; E\, D_H E^m \;=\; -2E^{m+1} \;-\; 2m\, E^{m+1} \;=\; -2(m+1)\, E^{m+1}$$

as desired. A similar argument gives 1.3 b).

As an immediate corollary we obtain the following remarkable fact.

**Theorem 1.1**

*The operators $E$ and $F$ are necessarily nilpotent.*

**Proof**

Let $E$ satisfy the equation

$$(E-\lambda I)^m \Gamma(E) \;=\; 0\,, \tag{1.4}$$

where $m\geq 1$ and $\Gamma(t)$ is a polynomial that does not vanish at $\lambda$. There is no loss in assuming that $\Gamma(\lambda)=1$. This given we can write

$$\Gamma(t) \;=\; 1 \;+\; (t-\lambda)Q(t)\;.$$

From which we derive that

$$I \;=\; -(E-\lambda I)Q(E) \;+\; \Gamma(E)\;. \tag{1.5}$$

Using 1.3 (a) we deduce that

$$D_H(E-\lambda I)^m \;=\; -2m\, E(E-\lambda I)^{m-1}$$

This may be rewritten as

$$-2m\, E(E-\lambda I)^{m-1} \;=\; (E-\lambda I)^m H \;-\; H(E-\lambda I)^m\;.$$

Multiplying on the right by $\Gamma(E)$ and using 1.4 gives

$$-2m\, E(E-\lambda I)^{m-1}\Gamma(E) \;=\; (E-\lambda I)^m H\, \Gamma(E)\;. \tag{1.6}$$

On the other hand left multiplication of 1.5 by $-2m\, E(E-\lambda I)^{m-1}$ yields

$$-2m\, E(E-\lambda I)^{m-1} \;=\; 2m\, E(E-\lambda I)^m Q(E) \;-\; 2m\, E(E-\lambda I)^{m-1}\Gamma(E)\;.$$

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$^{(\dagger)}$ Here and after the symbol "$I$" represents the identity operator.

Replacing the last term in this identity by the right hand side of 1.6 we get

$$-2m\,E(E-\lambda I)^{m-1} \;=\; 2m\,E(E-\lambda I)^{m}Q(E) \;+\; (E-\lambda I)^{m}H\,\Gamma(E)\;.$$

Since $\Gamma(E)$ commutes with any polynomial in $E$, this relation multiplied on the left by $\Gamma(E)$ may be written in the form

$$-2m\,E(E-\lambda I)^{m-1}\Gamma(E) \;=\; 2m\,E(E-\lambda I)^{m}\Gamma(E)Q(E) \;+\; (E-\lambda I)^{m}\Gamma(E)H\,\Gamma(E)\;.$$

and from 1.4 we derive that

$$E(E-\lambda I)^{m-1}\Gamma(E) \;=\; 0\,.$$

But another use of 1.4 then yields that we must also have

$$\lambda(E-\lambda I)^{m-1}\Gamma(E) \;=\; E(E-\lambda I)^{m-1}\Gamma(E) \;=\; 0\;.$$

If $\lambda\neq 0$ this process can be repeated and come to the final conclusion that 1.4 holds true only if

$$\Gamma(E) \;=\; 0\;.$$

This shows that the minimal equation of $E$ can only be $E^b=0$ for some $b\leq dim\,\mathbf{V}$. A similar argument gives the nilpotency of $F$.

## Proposition 1.2

*For any integer $m$ we have*

$$\begin{aligned} a)\;\; F\,E^m \;&=\; E^mF \;-\; mE^{m-1}(H+(m-1)I)\\ b)\;\; E\,F^m \;&=\; F^mE \;+\; mF^{m-1}(H-(m-1)I)\end{aligned} \qquad 1.7$$

## Proof

We may write 1.7 a) in the form

$$D_FE^m \;=\; mE^{m-1}(H+(m-1)I)\;.$$

Note that $m=1$ gives $D_FE=H$ which is 1.1 (a). Thus we proceed by induction and assume 1.7 a) true for $m$. We then get using 1.2 and the inductive hypothesis

$$\begin{aligned} D_FE^{m+1} \;&=\; (D_FE)\,E^m \;+\; E\,D_FE^m\\ &=\; H\,E^m \;+\; E(mE^{m-1}(H+(m-1)I)\\ \big(\text{by } 1.3\text{ a})\big)\;\;&=\; E^m(H+2mI) \;+\; mE^mH \;+\; m(m-1)E^m\\ &=\; E^m\big(((m+1)H \;+\; (m+m^2)I\big) \;=\; (m+1)E^m(H+mI)\end{aligned}$$

as desired. An analogous argument yields 1.7 b)

This result enables us to identify the eigenvalues of $H$.

## Theorem 1.2

*The eigenvalues of $H$ are all integral and symmetrically distributed around the origin.*

## Proof

Let $\lambda$ be an eigenvalue of $H$ and let $v$ be a corresponding eigenvector. If $\lambda = 0$ there is nothing to show. To be specific assume that $\lambda < 0$. This given, note that 1.3 b) yields

$$H\,F^m v \;=\; F^m(Hv - 2mv) \;=\; -(2m-\lambda)\,F^m v \tag{1.8}$$

Now let $a$ be the smallest integer such that $F^{a+1}v = 0$. Note that, although Theorem 1.1 guarantees there must be such an integer, in this case there is an even simpler reason: 1.8 shows that the vectors $F^m v$ have distinct eigenvalues. Set then

$$u \;=\; F^a v$$

and note that we must have

$$\begin{aligned} &1) \quad u \neq 0 \\ &2) \quad Fu = 0 \quad \text{and} \\ &3) \quad H\,u \;=\; -k\,u \end{aligned} \tag{1.9}$$

where for convenience we have set

$$k = 2a - \lambda \tag{1.10}$$

Indeed, 1) and 2) follow from our choice of $a$ and 3) is given by 1.8 for $m = a$. Equation 1.7 a) then yields that for all $m$

$$\begin{aligned} F\,E^m u &= E^m F u \;-\; mE^{m-1}(H + (m-1)I)u \\ \big(\text{by 2) and 3) of 1.9}\big) \quad &= 0 \;-\; mE^{m-1}(-k+m-1)u \\ &= mE^{m-1}(k+1-m\,)u \end{aligned} \quad . \tag{1.11}$$

Let now $b$ be the smallest integer such that $E^{b+1}u = 0$. Using 1.11 with $m = b+1$ yields

$$0 \;=\; F\,E^{b+1}\,u \;=\; (b+1)(k-b)E^b u \;.$$

Since by our choice of $b$ the vector $E^b u$ cannot vanish we must necessarily have

$$2a - \lambda \;=\; k \;=\; b \;\geq 0 \;. \tag{1.12}$$

This proves that $\lambda$ is an integer as desired. To derive the last assertion note that 1.3 a) and 1.9 3) yield

$$H\,E^m\,u \;=\; (-k+2m)\,E^m u \;. \tag{1.13}$$

Moreover, $E^m\,u \neq 0$ for $0 \leq m \leq k(=b)$. This means that the integers

$$-k\,,\; -k+2\,,\; -k+4\,,\; \ldots\,,\; -4+k\,,\; -2+k\,,\; k$$

are all eigenvalues of $H$. Finally note that 1.13 for $m = a - \lambda$ gives (since $k = 2a - \lambda$)

$$H\,E^{a-\lambda}\,u \;=\; (-k+2a-2\lambda)\,E^{a-\lambda}u \;=\; -\lambda\,E^{a-\lambda}u \;. \tag{1.13}$$

this proves that $-\lambda$ is also an eigenvalue of $A$, provided $E^{a-\lambda} \neq 0$ and this only requires that

$$a - \lambda \leq k\,.$$

But since $k = 2a - \lambda$ this inequality is

$$a - \lambda \leq 2a - \lambda\,.$$

and this is trivially satisfied whatever the value of $\lambda$.

This completes our proof.

The argument we used to prove Theorem 1.2 yields another important result.

**Theorem 1.3**

*Let $v$ be an eigenvector of $H$ with eigenvalue $-k \leq 0$. Suppose further that $Fv = 0$ then the vectors*

$$v\ ,\ E\,v\ ,\ E^2\,v\ ,\ \ldots\ ,\ E^k\,v \tag{1.14}$$

*are a basis of an irreducible $K[L]$-submodule $\mathbf{U}$ of $\mathbf{V}$.*

**Proof**

Our assumptions bring us to the case $a = 0$ of the previous argument. Thus $u = v$ in this case and the relations in 1.11 together with 1.9 2) yield us that $\mathbf{U}$ is invariant under $F$. Likewise the fact that $E^{k+1}u = 0$ shows that $\mathbf{U}$ is invariant under $E$. Since 1.13 shows that each of the elements in 1.14 are eigenvectors of $H$ we see that $\mathbf{U}$ is also invariant under $H$ and the proof is complete.

**Remark 1.1**

It is customary to call the collection in 1.14 an $sl[2]$ "*string*". In this vein we will call here the vectors $v$ and $E^k v$ respectively the "*head*" and the "*tail*" of the string. Our goal is to show that $V$ breaks up into a direct sum of strings. This done, to obtain a basis for $V$ all we need to get is a basis for the subspace $sh(V)$ consisting of all "*string heads*". That is

$$sh(V) \;=\; \bigoplus_{k\geq 0}\big\{v \in V\ :\ Hv = -kv\ \ \&\ \ Fv = 0\big\}\,.$$

## 2. Diagonability of H

Let us recall that an eigenvalue $\lambda$ of an operator $A$ is said to be *tame* if and only if

$$(A-\lambda I)^2\,v \;=\; 0 \qquad \longrightarrow \qquad (A-\lambda I)\,v \;=\; 0\ . \tag{2.1}$$

Note that an operator all of whose eigenvalues are tame satisfies a polynomial equation with simple roots and must necessarily be diagonable. Our goal in this section is to put together the ingredients necessary to establish the diagonability of $H$. We start with the following

**Proposition 2.1**

*If $u$ is an eigenvector of $H$ with eigenvalue $-k \leq 0$ and $F\,u = 0$ then*

$$F^m\,E^m\,u \;=\; \frac{m!k!}{(k-m)!}\,u \qquad (\ \text{for}\quad m = 0,1,2,\ldots,k\ ) \tag{2.2}$$

**Proof**

The identity is trivially true for $m = 0$. We shall thus inductively assume it true for $m$. This given, using 1.7 a) (with $m \to m+1$) we get for $m < k$

$$\begin{aligned} F^{m+1}\,E^{m+1}\,u \;&=\; F^m\,\big(\,FE^{m+1}u\,\big) \;=\; F^m\big(E^{m+1}Fu \;-\; (m+1)(H-mI)E^m\,u\big)\\ &=\; F^m\,(m+1)(k-m)E^m\,u\\ &=\; (m+1)(k-m)F^mE^m\,u\\ \big(\text{by induction}\big) \;&=\; (m+1)(k-m)\,\frac{m!k!}{(k-m)!}\,u\\ &=\; \frac{(m+1)!k!}{(k-m-1)!}\,u \end{aligned}$$

This completes the induction and our proof.

## Proposition 2.2

*For all integers $m$ we have*

$$\begin{aligned} a)\quad (H-2I)^m\, E &= EH^m \\ b)\quad (H+2I)^m\, F &= FH^m \end{aligned} \qquad 2.3$$

## Proof

The case $m=1$ of 2.3 a) reduces to

$$(H-2I)E = EH$$

which is another way of writing 1.1 (b). Assuming inductively that 2.3 a) is true for $m$ we have

$$\begin{aligned} (H-2I)^{m+1}\, E &= (H-2I)\big((H-2I)^m\, E\big) \\ &= (H-2I)\big(EH^m\big) \;=\; EH^{m+1}\,. \end{aligned}$$

This completes the induction and proves 2.3 a). The other relation is established in the same manner.

## Theorem 2.1

*The largest eigenvalue of $H$ is tame*

## Proof

Let $k_o$ be the largest eigenvalue of $H$. By symmetry $-k_o$ must be the smallest. Let $w \in \mathbf{V}$ satisfy

$$(H+k_o)^2\, w \;=\; 0\;. \qquad 2.4$$

Our goal is to show that

$$(H+k_o)\, w \;=\; 0\,.$$

We will show that the alternative

$$(H+k_o)\, w \neq 0\,. \qquad 2.5$$

leads to a contradiction. To this end let us set

$$u \;=\; (H+k_o)\, w\,.$$

Note first that 2.4 gives

$$H\, u \;=\; -k_o u$$

and 1.3 a) implies

$$HFu \;=\; -(k_0+2)Fu\,.$$

To avoid contradicting the choice of $k_o$ we must then conclude that

$$F(H+k_o)\, w \;=\; F\, u \;=\; 0\,. \qquad 2.6$$

Now note that

$$\begin{aligned} (H+2I+k_oI)^2F &= (H+2I)^2F \;+\; 2k_o(H+2I)F \;+\; k_o^2F \\ \big(\text{by } 2.3\text{ b)}\big) &= F\big(H^2+2k_oH+k_o^2I\big) \\ &= F(H+k_oI)^2 \end{aligned}$$

Thus 2.4 gives

$$(H+2I+k_oI)^2Fw \;=\; 0$$

but this may be rewritten as

$$H(H+2I+k_oI)Fw \;=\; -(k_o+2)(H+2I+k_oI)Fw$$

and, again, to avoid contradicting the choice of $k_o$ we must have

$$(H+2I+k_oI)Fw \;=\; 0$$

but this is

$$HFw \;=\; -(2+k_o)Fw$$

and for the same reason we must conclude that

$$Fw \;=\; 0 \qquad 2.7$$

Note next that 1.3 a) gives

$$H\,E^{k_o+1} \;=\; E^{k_o+1}(H+2k_oI+2I)$$

and this may be rewritten as

$$(H-2I-k_oI)E^{k_o+1} \;=\; E^{k_o+1}(H+k_oI)\,.$$

Thus, a fortiori we must also have

$$(H-2I-k_oI)^2E^{k_o+1} \;=\; E^{k_o+1}(H+k_oI)^2\,.$$

But then 2.4 gives

$$(H-2I-k_oI)^2E^{k_o+1}w \;=\; 0\,.$$

Now, here again, in order not to contradict the choice of $k_o$ we must accept that

$$(H-2I-k_oI)E^{k_o+1}w \;=\; 0$$

and for the same reason we are forced to accept that

$$E^{k_o+1}w \;=\; 0\,. \qquad 2.8$$

But this brings us to the final stretch of our proof. Indeed 2.8 gives

$$\begin{aligned} 0 \;=\; F^{k_o+1}E^{k_o+1}w &= F^{k_o}\big(\,FE^{k_o+1}w\big) \\ \big(\text{by 1.7 a)}\big) &= F^{k_o}\big(E^{k_o+1}Fw-(k_o+1)E^{k_o}(H+k_oI)\big)w \\ \big(\text{by 2.7}\big) &= -(k_o+1)F^{k_o}E^{k_o}(H+k_oI)w \\ \big(\text{by 2.2}\big) &= -(k_o+1)(k_o!)^2\,(H+k_oI)w \end{aligned}$$

This contradicts 2.5 and completes our proof.

## Remark 2.1

To prove the tameness of all the other eigenvalues of $H$ we need to separate the subspace generated by the action of $K[L]$ on the eigenvectors of eigenvalue $-k_o$ from the rest of $\mathbf{V}$. Our plan is to show that $V$ may be broken up into the direct sum of two $K[L]$-invariant subspaces

$$\mathbf{V} \;=\; \mathbf{V}_1 \;\oplus\; \mathbf{V}_2 \qquad 2.9$$

where

a) $\mathbf{V_1}$ contains all the eigenvectors of eigenvalue $k_o$ ,
b) $H$ is diagonable on $\mathbf{V_1}$ ,
b) the largest eingenvalue of $H$ in $\mathbf{V_2}$ is less than $k_o$ .

This done, the diagonability of $H$ immediately follows by an inductive argument which descends on the size of the largest eigenvalue of $H$.

Our first step in this program is given by the following basic result

## Theorem 2.2

*Let $k_o$ be the highest eigenvalue of $H$ an let*

$$u_1 \;,\; u_2 \;,\; \ldots \;,\; u_N \qquad 2.10$$

*be a basis for the eigenspace of $H$ corresponding to the eigenvalue $-k_o$. Then the collection*

$$\mathcal{B} \;=\; \bigcup_{i=1}^{N} \{u_i, Eu_i, \ldots, E^{k_o}u_i\} \qquad 2.11$$

*forms an independent set and the linear span*

$$\mathbf{V}_1 \;=\; \mathcal{L}[\; E^m\, u_i \;:\; i=1,2,\ldots,N \;\;;\;\; m=0,1,\ldots,k_o \;] \;. \qquad 2.12$$

*is a $K[L]$-invariant subspace of $\mathbf{V}$ which is a direct sum of irreducible $K[L]$-modules.*

## Proof

Note that each $u_i$ is killed by $F$ since otherwise $H$ would also have the eigenvalue $-k_o-2$. This given, from Theorem 1.3 it follows that the collection

$$\{u_i, Eu_i, \ldots, E^{k_o}u_i\}$$

spans an irreducible $K[L]$-module. Thus we need only show that $\mathcal{B}$ is an independent set. To this end suppose that for some constants $c_{i,m}$ we have

$$\sum_{m=0}^{k_o} \sum_{i=1}^{N} c_{m,i}\, E^m\, u_i \;=\; 0\,, \qquad 2.13$$

and note that the summands

$$\sum_{i=1}^{N} c_{0,i}\, u_i \;,\quad \sum_{i=1}^{N} c_{1,i}\, E\, u_i \;,\quad \sum_{i=1}^{N} c_{2,i}\, E^2\, u_i \quad \cdots \;,\; \sum_{i=1}^{N} c_{k_0,i}\, E^{k_o}\, u_i \;,$$

are eigenvectors of $H$ with respective eigenvalues

$$-k_o\,, \quad -k_o+2\,, \quad -k_o+4\,, \quad \cdots\;,\; -k_o+2k_o\,,$$

Thus 2.13 can hold true only if each of these summands separately vanishes. On the other hand if

$$\sum_{i=1}^{N} c_{m,i}\, E^m\, u_i \;=\; 0$$

then we must also have

$$\sum_{i=1}^{N} c_{m,i}\, F^m E^m\, u_i \;=\; 0$$

but then 2.2 gives

$$\sum_{i=1}^{N} c_{m,i}\, u_i \;=\; 0$$

and the independence of the $u_i\,'s$ forces the vanishing of all the coefficients $c_{m,i}$, proving the independence of $\mathcal{B}$ as desired.

Note that since $H$ acts diagonally on the basis $\mathcal{B}$, to complete our program we need only construct a $K[L]$-invariant complement $\mathbf{V_2}$ of $\mathbf{V}_1$. In fact, the construction of such an invariant complement would prove more than the diagonability of $H$, for it will also provide us with an algorithm for breaking up $\mathbf{V}$ into a direct sum of irreducible $K[L]$-modules.

It develops that the basic tools for carrying out the construction of $\mathbf{V_2}$ are supplied by *Casimir* operator

$$C \;=\; EF \;+\; FE \;+\; H^2/2\ ,$$

we shall present them here as three separate propositions.

## Proposition 2.3

*$C$ commutes with $E$, $F$ and $H$.*

## Proof

Note that 1.1 (b),(c) give

$$D_H\, C \;=\; -2EF \;+\; 2EF \;+\; 2FE \;+\; -2FE \;+\; 0 \;=\; 0$$

Thus $H$ and $C$ commute. Using 1.1 (a) we get

$$D_E\, C \;=\; -EH \;+\; -HE \;+\; (2EH+2HE)/2 \;=\; 0\ .$$

Thus $E$ and $H$ commute as well. The result for $F$ is shown in a similar manner.

## Proposition 2.4

*If $Hu=-ku$ and $Fu=0$ then for any $m$ we have*

$$C\, E^m\, u \;=\; (k+k^2/2)\, E^m\, u \qquad 2.14$$

## Proof

Note that, using 1.1 a), we can rewrite $C$ in the form

$$C \;=\; 2EF \;-\; H \;+\; H^2/2\ . \qquad 2.15$$

This gives

$$C\, u \;=\; 0 \;-\; Hu \;+\; H^2u/2 \;=\; (k+k^2/2)\, u\,,$$

and 2.14 follows immediately from the commutativity of $C$ and $E$.

This result implies that every element of $\mathbf{V}_1$ is an eigenvector of $C$ with eigenvalue $k_o+k_o^2/2$. Setting for convenience

$$t_o \;=\; k_o + k_o^2/2 \qquad 2.16$$

we may express this fact by writing

$$\mathbf{V}_1 \;\subseteq\; Nullspace[C - t_oI] \;. \qquad 2.17$$

We can convert this into an equality. In fact we can prove an even stronger result.

**Proposition 2.5**

*Let $k_o$ be the largest eigenvalue of $H$. Let $\mathbf{V}_1$ be defined as in 2.12 and $t_o$ be as in 2.16. Then for any $w \in \mathbf{V}$ we have*

$$(C - t_oI)^p\, w = 0 \quad \longrightarrow \quad w \;\in\; \mathbf{V}_1 \qquad 2.18$$

*in particular we derive that $t_o$ is a tame eigenvalue of $C$.*

**Proof**

Let $a$ be the least integer such that

$$F^{a+1}\, w \;=\; 0 \;. \qquad 2.19$$

Theorem 1.1 guarantees that such an integer exists. Our proof will proceed by induction on $a$. We start with $a = 0$. So let $Fw = 0$. This given, from 2.15 and 2.16 we derive that

$$\begin{aligned}(C - t_o)w \;&=\; (2EF - H + H^2/2 \;-\; k_o - k_o^2/2)\, w \\ &=\; \Big[ -(H + k_oI) \;+\; \frac{H^2 - k_o^2 I}{2} \Big]\, w \\ &=\; (H + k_oI)\Big(-1 + \frac{H - k_oI}{2}\Big)\, w \\ &=\; \frac{1}{2}(H + k_oI)(H - (k_o+2)I)\, w \;.\end{aligned} \qquad 2.20$$

Thus the condition $(C - t_oI)^p\, w = 0$ may be rewritten as

$$\frac{1}{2^p}(H - (k_o+2)I)^p(H + k_oI)^p\, w \;=\; 0 \;. \qquad 2.21$$

Now, the maximality of $k_o$ guarantees that the operator $H - (k_o + 2)I)$ is invertible. So 2.21 is equivalent to

$$(H + k_oI)^p\, w \;=\; 0 \;.$$

However, since $k_o$ is a tame eigenvalue for $H$ we see that we must also have

$$(H + k_oI)\, w \;=\; 0 \;.$$

This places $w \in \mathbf{V}_1$ and completes the first step of the induction. So let 2.18 be true when $F^a\, w = 0$. Let $F^a w \neq 0$ and $F^{a+1} w = 0$. For convenience set

$$u \;=\; F^a\, w \;. \qquad 2.22$$

The commutativity of $C$ and $H$ yields that

$$(C - t_o)^p\, u \;=\; F^a\,(C - t_oI)^p\, w \;=\; 0$$

Since $Fu = 0$ we are reduced to the case just considered and $u$ must also be an eigenvector of $H$ with eigenvalue $-k_o$. In particular it must be a linear combination of the $u_i\,'s$ introduced in 2.10. So let

$$u \;=\; \sum_{i=1}^{N} c_i\, u_i$$

and set

$$\bar{w} \;=\; \frac{(k_o-a)!}{a!k_o!}\sum_{i=1}^{N} c_i\, E^a\, u_i \;\;.$$

Since each $u_i$ is killed by $F$ we can apply 2.2 and deduce that

$$F^a\,\bar{w} \;=\; \sum_{i=1}^{N} c_i\, u_i \;= u = F^a w \;\;.$$

Now the difference $w-\bar{w}$ is killed by $(C-t_oI)^p$ since $w$ is so by hypothesis and $\bar{w}$ is even killed by $C-t_oI$ since it lies in $\mathbf{V}_1$. Moreover as we have seen $F^a(w-\bar{w})=0$. Thus we can use the induction hypothesis and derive that $w-\bar{w}\in\mathbf{V}_1$. Since $w=\bar{w}+w-\bar{w}$ we have $w\in\mathbf{V}_1$ as desired. This completes the induction and our proof.

We can finally put all our pieces together and produce an invariant complement of $\mathbf{V}_1$. To this end note that since $t_o$ has been shown to be a tame eigenvalue of $C$, the minimal equation of $C$ can be written in the form

$$(C-t_oI)\,Q(C) \;=\; 0 \qquad 2.23$$

with $Q(t)$ a polynomial which does not vanish at $t_o$. There is no loss in assuming that $Q(t_o)=1$. This gives

$$1-Q(t) \;=\; (t-t_o)\Gamma(t) \;\;.$$

Thus we may write

$$I \;=\; Q(C) \;+\; (C-t_oI)\,\Gamma(C) \;\;. \qquad 2.24$$

## Theorem 2.2

*We have the direct sum decomposition*

$$\mathbf{V} \;=\; Q(C)\mathbf{V} \;\oplus\; (C-t_oI)\Gamma(C)\mathbf{V} \;\;. \qquad 2.25$$

*Since*

$$\mathbf{V}_1 \;=\; Q(C)\mathbf{V} \qquad 2.26$$

*and both summands are $K[L]$-invariant, we may take, in 2.9.*

$$\mathbf{V}_2 \;=\; (C-t_oI)\,\Gamma(C)\mathbf{V}$$

## Proof

Note that the sum on the right hand side of 2.25 is direct since if we had two vectors $v_1, v_2$ such that

$$Q(C)\,v_1 \;=\; (C-t_oI)\Gamma(C)\,v_2$$

then hitting this by $(C-t_oI)$ and using 2.23 gives

$$(C-t_oI)^2\Gamma(C)v_2 \;=\; 0 \;\;.$$

But since $t_o$ is tame for $C$ we must also have

$$(C-t_oI)\Gamma(C)v_2 \;=\; 0 \;\;.$$

This implies that the summands in 2.25 share only the zero vector as desired. To show the equality in 2.25 we need only observe that 2.24 implies that every $w \in \mathbf{V}$ has the expansion

$$w \;=\; Q(C)w \;+\; (C - t_o I)\,\Gamma(C)\,w\,. \qquad 2.27$$

Note next that since every element of $Q(C)\mathbf{V}$ is killed by $C - t_o I$, Proposition 2.5 gives us that

$$Q(C)\mathbf{V} \;\subseteq\; \mathbf{V}_1 \;\;.$$

However we can reverse this inequality. In fact, since every element $v \in \mathbf{V}_1$ is killed by $C - t_o I$, the expansion in 2.27 yields that it may be written in the form

$$v \;=\; Q(C)\,v \;\;.$$

Thus

$$\mathbf{V}_1 \subseteq Q(C)\mathbf{V} \;\;.$$

This proves 2.26.

Since the $K[L]$-invariance of the space $(C - t_o I)\Gamma(C)\mathbf{V}$ is an immediate consequence of the commutativity of $C$ with $E$,$F$ and $H$ we see that 2.25 is precisely the decomposition of $\mathbf{V}$ we were looking for and our program for establishing the diagonability of $H$ is now complete.

There are two further results concerning the eigenvectors of $H$ that are worth including here. To begin we have

## Proposition 2.6

*Let $Hu = -ku$ with $k \geq 0$ and let*

$$a)\;\; F^a u \neq 0 \qquad \text{and} \qquad b)\;\; F^{a+1}u = 0 \qquad 2.28$$

*Then there exists a unique set of vectors $u_0, u_1, \ldots, u_a$ such that*

$$u_0 + E\,u_1 + \cdots + E^a u_a = u \qquad 2.29$$

*with*

$$a)\;\; Fu_i = 0 \qquad \text{and} \qquad b)\;\; Hu_i = -(k+2i)u \qquad (\text{for } i = 0, 1, \ldots, a) \qquad 2.30$$

## Proof

Say $a = 0$. Then we can take $u_0 = u$ and we are done. So we may proceed by induction. Suppose the result true up to $a - 1 \geq 0$. Let $F^a u \neq 0$ and $F^{a+1}u = 0$. Now set

$$\overline{u} \;=\; c\,E^a F^a u\,. \qquad \big(\text{ with } c = \tfrac{(k+a)!}{(k+2a)!a!}\;\big) \qquad 2.31$$

Note that 1.3 b with $m = a$ gives

$$HF^a u \;=\; F^a(H - 2aI)u \;=\; F^a(-k-2a)u \;=\; -(k+2a)\,F^a u \qquad 2.32$$

This given, applying formula 2.2 with $m = a$ to 2.29 we get

$$F^a\overline{u} \;=\; c\,F^a E^a \left(F^a u\right) \;=\; c\,\tfrac{(k+2a)!a!}{(k+2a-a)!}\,F^a u \;=\; F^a u \qquad 2.33$$

Since the vector

$$v \;=\; u - \overline{u} \qquad 2.34$$

satisfies the hypothesis $F^a v = 0$, by induction we may write

$$v = u_0 + u_1 + \ldots + u_{a-1}$$

with $Fu_i = 0$, for $i = 0, 1, \ldots a-1$. Combining this with 2.34 we get

$$\begin{aligned} u &= u_0 + E\,u_1 + \ldots + E^{a-1}u_{a-1} + \overline{u} \\ &= u_0 + E\,u_1 + \ldots + E^{a-1}u_{a-1} + cE^a\big(F^a u\big) \end{aligned}$$

This may be rewritten as

$$u \;=\; u_0 + E\,u_1 + \ldots + E^{a-1}u_{a-1} + E^a u_a \tag{3.35}$$

with

$$u_a = c\,F^a u$$

Since by assumption $Fu_a = c\,F^{a+1}u = 0$ we see that 3.35 completes the induction and establishes existence. Finally note that under conditions a) and b) of 2.30 the summand $E^i u_i$ in 2.29 is an eigenvector of the Casimir operator $C$ with eigenvalue $(k+2i) + (k+2i)/2$. Thus the sum in 2.29 can never vanish. This proves uniqueness and completes our proof.

Proposition 2.6 has the following important consequence

## Theorem 2.3

*Let $Hu = -ku$ with $k \geq 0$, suppose that $Fu = 0$ then we cannot have*

$$u = E\,v \tag{2.36}$$

## Proof

Suppose such a $v$ exists giving 2.36. Then let $a$ be such that

$$F^a v \neq 0 \qquad \text{and} \qquad F^{a+1}v = 0\,.$$

Then from the previous proposition we derive that

$$v = v_0 + Ev_1 + \cdots + E^a v_a \qquad (\text{with } F\,v_i = 0\ ) \tag{2.37}$$

with

$$a)\quad F\,v_i = 0 \qquad \text{and} \qquad b)\quad H\ v_i \;=\; -(k+2i)\,v_i\,. \tag{2.38}$$

In particular 2.2 gives

$$F^i\,E^i u_i \;=\; \frac{i!(k+2i)!}{(k+i)!}\,u_i \tag{2.39}$$

Now combining 2.36 and 2.37 we get

$$u = Ev = Ev_0 + E^2 v_1 + \cdots + E^{a+1}v_a$$

in particular, $Fu = 0$ gives

$$0 = FEv_0 + FE^2 v_1 + \cdots + FE^{a+1}v_a \tag{2.40}$$

Since 2.39 and 2.38 a) imply that

$$F^j E^i v_i \;=\; 0 \qquad (\text{for all } j > i\ )\,,$$

applying $F^a$ to 2.40 and using formula 2.39 with $m = a+1$ we derive that

$$0 = F^{a+1}E^{a+1}v_a \;=\; \frac{(a+1)!(k+2a+2)!}{(k+a+1)!}\,v_a$$

This gives $v_a = 0$. We can repeat this argument, successively multiplying 2.35 by $F^{a-1}, F^{a-2}, \ldots$, and obtain that all the $v_i$ must necessarily vanish thereby reaching a contraddiction. This completes our proof.

For our applications of $sl[2]$ theory it will be convenient to study the action of $E$ and $F$ on the eigenspaces of $H$. To this end let $k_o$ as before denote the highest eigenvalue of $H$ and set

$$a) \quad V_k \;=\; \big\{ v\in V \,:\, Hv = k\,v \big\} \qquad (\text{for } -k_o\leq k\leq k_o)$$
$$b) \quad U_k \;=\; \big\{ v\in V \,:\, Hv = -k\,v \;\;\&\;\; F\,v=0 \big\} \qquad (\text{for } 0\leq k\leq k_o)$$

This given we have the following fundamental fact.

## Theorem 2.4

*For any $1\leq k\leq k_o$, the operator $E^k$, yields a non singular map of $V_{-k}$ onto $V_k$ and $F^k$ yields a non singular map of $V_k$ onto $V_{-k}$. In particular it follows that*

$$\dim V_{-k} \;=\; \dim V_k \qquad (\textit{for all } 1\leq k\leq k_o\;) \tag{2.41}$$

## Proof

We will only show that for all $v\in V_{-k}$

$$E^k v \;=\; 0 \;\;\Longrightarrow\;\; v=0 \tag{2.42}$$

since the corresponding result for $F$ is proved in an entirely analogous manner, by inverting the roles of $E$ and $F$.

Now Proposition 2.6 yields that if $v\in V_{-k}$ then for some $a\geq 0$ we can write $v$ in the form

$$v = v_0 + E\,v_1 + \cdots + E^a v_a \tag{2.43}$$

with

$$a)\;\; Fv_i = 0 \qquad \text{and} \qquad b)\;\; Hv_i = -(k+2i)\,v_i \qquad (\text{for } i=0,1,\ldots,a) \tag{2.44}$$

Note that if $a=0$ then a) and b) for i=0 give $Fv=0$ and $Hv=-kv$ thus from Proposition 2.1 we derive that

$$F^kE^kv = k!^2\,v$$

and this proves 2.42 in this case. So we may proceed by induction on $a$. To this end note first that using a) and b) of 2.44 together with Proposition 2.1 we derive that

$$F^mE^m\,v_i \;=\; \frac{m!(k+2i)!}{(k+2i-m)!}\,v_i \qquad (\text{for all } 0\leq m\leq k+2i) \tag{2.45}$$

in particular we have

$$F^iE^i\,v_i \;=\; \frac{i!(k+2i)!}{(k+i)!}\,v_i\,. \tag{2.46}$$

Now if $E^kv=0$ from the expansion in 2.43 we derive that

$$0 \;=\; E^kv_0 + \cdots E^{k+i}\,v_i + \cdots + E^{k+a}v_a$$

and a fortiori we must also have

$$0 \;=\; F^{k+a}E^kv_0 + \cdots F^{k+a}E^{k+i}\,v_i + \cdots + F^{k+a}E^{k+a}v_a\,. \tag{2.47}$$

Now 2.45for $m=k+i$ gives

$$F^{k+i}E^{k+i}\,v_i \;=\; \frac{(k+i)!(k+2i)!}{i!}\,v_i \tag{2.48}$$

and 2.44 a) gives

$$F^{k+a}E^{k+i}\,v_i \;=\; 0 \qquad (\text{for all } 0\leq i<a)\,.$$

Thus using 2.48 with $i=a$ from 2.47 we derive that

$$0 \;=\; F^{k+a}E^{k+a}v_a \;=\; \frac{a!(k+2a)!}{(k+a)!}\, v_a$$

This forces $v_a=0$ and the inductive hypothesis gives $u=0$ completing the proof of 2.42.

This brings us to a result which has been used in various contexts to prove unimodality results.

**Theorem 2.5**

*For every $2\le k\le k_o$*

*(i) the mapping from $V_{-k}$ into $V_{-k+2}$ induced by $E$ is injective.*

*(ii) the mapping from $V_{-k+2}$ into $V_{-k}$ induced by $F$ is onto and thus*

$$\dim U_k \;=\; \dim V_{-k} - \dim V_{-k-2} \qquad 2.49$$

*Analogously we also have*

*(i') the mapping from $V_k$ into $V_{k-2}$ induced by $F$ is injective.*

*(ii') the mapping from $V_{k+2}$ into $V_k$ induced by $F$ is onto.*

*In particular 2.49 and 2.41 yield us the inequalities*

$$\begin{aligned}&\dim V_{-k}\le \dim V_{-k+2}\le\cdots\le \dim V_{-2}\le \dim V_0\\ &\qquad\dim V_0\ge \dim V_2\ge\cdots\ge \dim V_{k-2}\ge \dim V_k\end{aligned} \qquad 2.50$$

*for all even $k\le k_o$, and*

$$\begin{aligned}&\dim V_{-k}\le \dim V_{-k+2}\le\cdots\le \dim V_{-3}\le \dim V_{-1}\\ &\qquad\dim V_1\ge \dim V_3\ge\cdots\ge \dim V_{k-2}\ge \dim V_k\end{aligned} \qquad 2.51$$

*for all odd $k\le k_o$*

**Proof** (Jason)

For $u\in V_{-k}$ we cannot have $Eu=0$ since that would give $E^ku=0$ and contraddict Theorem 2.4. Now for $u\in V_{-k-2}$ Proposition 2.6 gives us the expansion

$$u \;=\; u_0+E\,u_1+\cdots+E^a\,u_a \qquad 2.52$$

with the $u_i$ satisfying a) and b) of 2.30 with $k$ replaced by $k+2$. . In particular from 1.7 a) we derive that

$$FE^{i+1}u_i \;=\; (i+1)(k+2+i)\,E^iu_i \qquad 2.53$$

Thus if we set

$$v \;=\; \sum_{i=0}^{a}\frac{E^iu_i}{(i+1)(k+2+i)}$$

then 2.52 and 2.53 give

$$u \;=\; F\,v \qquad 2.54$$

Since from 1.3 a) and 2.30 b) we easily derive that $v\in V_{-k}$, the equality in 2.54 proves that $F$ maps $V_{-k}$ onto $V_{-k-2}$. In particular this gives the equality

$$\dim_{\text{Range F}}\; V_{-k} \;=\; \dim\; V_{-k-2}$$

and 2.49 immediately follows from the simple identity

$$\dim V_{-k} \;=\; \dim_{\text{Range F}}\; V_{-k} \;+\; \dim_{\text{Ker F}}\; V_{-k}\,.$$

Our next two results bring to light the action of $E, F$ and $H$ on the vector space spanned by the elements of an $sl[2]$ string. To begin we have

## Theorem 2.6

*If $F\,u = 0$ and $H\,u = -m\,u$ and we set*

$$e_j \;=\; (m-j)!\,E^j u \tag{2.55}$$

*then the action of $E, F, H$ on the basis*

$$\big\langle e_0, e_1, \ldots, e_m \big\rangle$$

*is given by the matrices*

$$\begin{aligned} a)\quad \tilde{E} \;&=\; \big\|(m-j)\chi(i=j+1)\big\|_{i,j=0}^{m} \\ b)\quad \tilde{F} \;&=\; \big\|j\,\chi(i=j-1)\big\|_{i,j=0}^{m} \\ c)\quad \tilde{H} \;&=\; \big\|(2j-m)\chi(i=j)\big\|_{i,j=0}^{m} \end{aligned} \tag{2.56}$$

## Proof

Recall that we make the convention that the action of an operator $T$ on a basis is to be expressed by right multiplication by a matrix. Thus to prove 2.56 a) we need to show that the matrix $A = \|a_{ij}\|_{i,i=0}^{m}$ determined by the condition

$$E\big\langle e_0, e_1, \ldots, e_m \big\rangle \;=\; \big\langle e_0, e_1, \ldots, e_m \big\rangle\, A \tag{2.57}$$

Is given by the right hand side of 2.56 a).

Note that the definition in 2.55 gives

$$E\,e_j \;=\; (m-j)!\,E^i u \;=\; (m-j)\,e_{j+1} \tag{2.58}$$

Now, by equating columns, we see that 2.57 is equivalent to the relations

$$E\,e_j \;=\; \sum_{i=0}^{m} e_i\, a_{ij} \tag{2.59}$$

Equating the right hand sides of 2.58 and 2.59 we see that we must have

$$a_{ij} \;=\; \begin{cases} 0 & \text{if} \quad i \neq j+1 \\ \\ m-j & \text{if} \quad i = j+1 \end{cases}$$

and this is simply another way of writing 2.56 a).

The same mechanism yields 2.56 b) and c). Except that for 2.56 b) we first use 1.7 a) for $m = j$ to get

$$FE^j u \;=\; -(j)E^{j-1}(-m+j-1)\,u \;=\; j(m-j+1)\,E^{j-1}u$$

or, equivalently (by 2.55)

$$F\,e_j \;=\; (m-j)!FE^j u \;=\; j(m+1-j)!E^{j-1}u \;=\; j\,e_{j-1}\,.$$

The remaining steps are is as in the previous case. For 2.56 c) we start with 1.3 a), for $m = j$ and get

$$HE^j\,u \;=\; (-m+2j)E^j u$$

or equivalently

$$He_j \;=\; (2j-m)e_j$$

and the remaining steps are is as in the previous cases.

To proceed we need a result which extends the identity in 2.2

## Proposition 2.7

*If $Hu = -m\,u$ and $Fu = 0$ then*

$$F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s < j \\[2ex] \dfrac{s!(m-s+j)!}{(s-j)!(m-s)!}\,E^{s-j}u & \text{if } s \geq j \end{cases} \tag{2.60}$$

## Proof

Formula 1.7 a) gives

$$F\,E^s u \;=\; s(m-s+1)E^{s-1}u\,. \tag{2.61}$$

since $Fu = 0$ the case $j = 1$ of 2.60 holds true. So we may proceed by induction on $j$ and assume 2.60 true for $j$. This given 2.60 immediately gives

$$F^{j+1}\,E^s u \;=\; 0 \qquad\qquad (\text{if } s \leq j)$$

while for $s \geq j+1$ we get

$$\begin{aligned} F^{j+1}\,E^s u \;&=\; \frac{s!(m-s+j)!}{(s-j)!(m-s)!}\,FE^{s-j}u \\ (\text{by 2.61 for } s \to s-j) \;&=\; \frac{s!(m-s+j)!}{(s-j)!(m-s)!}\,(s-j)(m-s+j+1)E^{s-j-1}u \\ &=\; \frac{s!(m-s+j+1)!}{(s-j-1)!(m-s)!}\,E^{s-j-1}u \end{aligned}$$

This completes the induction and the proof.

## Theorem 2.7

*If $F\,u = 0$ and $H\,u = -m\,u$ then the operator*

$$\frac{(m-j)!i!}{j!m!^2(m-i)!}\;F^{m-i}E^mF^j \tag{2.62}$$

*acts on the basis*

$$\big\langle u, Eu, \ldots, E^m u\big\rangle\,.$$

*by the matrix*

$$U^{(ij)} \;=\; \Big\|\chi(r=i)\chi(s=j)\Big\|_{r,s=0}^{m}$$

*and consequently the operator*

$$\sum_{i,j=0}^{m} a_{ij}\;\frac{(m-j)!i!}{j!m!^2(m-i)!}\;E^{m-i}E^mF^j \tag{2.63}$$

*acts by the matrix $A = \|a_{ij}\|_{i,j=0}^{m}$. In the same vein, if we set*

$$e_j \;=\; (m-j)!\,E^j u \qquad (\text{for } j = 0, \ldots, m)$$

*then the operators*

$$\frac{i!}{j!m!^2}\;F^{m-i}E^mF^j \quad \text{and} \qquad\quad \sum_{i,j=1}^{m+1} a_{ij}\;\frac{i!}{j!m!^2}\;F^{m-i}E^mF^j$$

*act on the basis $\langle e_0, e_1, \ldots, e_m\rangle$ likewise by $U^{(ij)}$ and $A$ respectively.*

## Proof

From 2.60 it follows that

$$E^m\,F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s<j\,, \\[2mm] \dfrac{s!(m-s+j)!}{(s-j)!(m-s)!}\,E^m E^{s-j}u & \text{if } s\geq j\,. \end{cases}$$

But

$$E^m E^{s-j}u \;=\; 0 \qquad \forall\; s>j\,.$$

Thus

$$E^m\,F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s\neq j\,, \\[2mm] \dfrac{j!m!}{(m-j)!}\,E^m u & \text{if } s=j\,. \end{cases}$$

In particular

$$F^{m-i}E^m\,F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s\neq j \\[2mm] \dfrac{j!m!}{(m-j)!}\,F^{m-i}E^m u & \text{if } s=j \end{cases} \tag{2.64}$$

and 2.61 for $j\to m-i$ and $s\to m$ gives

$$F^{m-i}E^m u \;=\; \frac{m!(m-i)!}{i!}\,E^i u$$

so that 2.64 becomes

$$F^{m-i}E^m\,F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s\neq j \\[2mm] \dfrac{j!\,m!^2(m-i)!}{(m-j)!\,i!}\;E^i u & \text{if } s=j \end{cases}$$

In other words

$$\frac{(m-j)!\,i!}{j!\,m!^2(m-i)!}\;\;F^{m-i}E^m\,F^j\,E^s u \;=\; \begin{cases} 0 & \text{if } s\neq j \\[2mm] E^i u & \text{if } s=j \end{cases}$$

This proves that the operator in 2.62 acts by $U^{(ij)}$ and consequently the operator in 2.63 acts by $A$. The remaining assertions are immediate consequemces of the definition of the basis $\big\langle e_0,e_1,\ldots,e_m\big\rangle$

N. Wallach pointed out to us the following remarkable corollary of Theorem 2.7.

## Theorem 2.8

*Let $\mathcal{M}_m$ and $\mathcal{M}_n$ respectively denote the spaces of $m\times m$ and $n\times n$ matrices over* $\mathbf{Q}$ *and let $\phi:\mathcal{M}_m\to\mathcal{M}_n$ be a map satisfying, for all $A,B\in\mathcal{M}_m$ and $c\in\mathbf{Q}$:*

$$\begin{aligned} a)\qquad \phi(A+B) \;&=\; \phi(A)+\phi(B) \\ b)\qquad \phi(AB) \;&=\; \phi(A)\phi(B) \\ c)\qquad \phi(c\,AB) \;&=\; c\,\phi(A) \end{aligned} \tag{2.65}$$

*Then $n=Nm$ and $\phi(A)$ (up to a similarity) is none other than a block diagonal matrix consisting of $N$ copies of $A$. More precisely we have a non singular $mN\times mN$ matrix $D$ yielding the decomposition*

$$\phi(A) \;=\; D\Big(\bigoplus_{i=1}^{N} A\Big)\,D^{-1} \tag{2.66}$$

## Proof

Let $\tilde{E}, \tilde{F}, \tilde{H}$ be the matrices defined in 2.56. Let $V_n$ be the $n$-dimensional vector space over $\mathbf{Q}$ and let $E, F$ and $H$ be the linear operators on $V_n$ defined by setting for each $v \in V_n$

$$E\,v \;=\; \phi(\tilde{E})\,v\,, \qquad F\,v \;=\; \phi(\tilde{F})\,v\,, \qquad H\,v \;=\; \phi(\tilde{H})\,v\,,$$

Property 2.65 b) asssures that $E, F$ and $H$ satisfy the the basic relations in 1.1. This immediately yields that $V_n$ decomposes into a direct sum of $sl(2)$-strings

$$V_n \;=\; \bigoplus_{i=1}^{N} \mathcal{L}[u_i, Eu_i, \ldots, E^{k_i}u_i\,]\,. \tag{2.67}$$

with

$$a) \quad H\,u_i = -k_i u_i\;, \qquad b) \quad F\,u_i = 0 \tag{2.68}$$

However, note that the definitions in 2.56 assure that the Casimir matrix

$$\tilde{C} \;=\; 2\tilde{E}\tilde{F} - \tilde{H} + \tilde{H}^2/2$$

reduces to the multiple $(m + m^2/2)\tilde{I}$ of the $m \times m$ identity matrix $\tilde{I}$. But then, the identities in 2.65 yield that the Casimir operator

$$C \;=\; 2EF - H + H^2/2$$

must also be of the form

$$C \;=\; (m + m^2/2)I$$

with $I$ the identity operator on $V_n$. On the other hand, by Proposition 2.4, the relations in 2.68 yield the equality

$$Cu_i \;=\; (k_i + k_i^2/2)u_i$$

and this forces the equality

$$k_i + k_i^2/2 \;=\; m + m^2/2\,. \tag{2.69}$$

But for $k_i \geq 0$ this can only hold true for

$$k_i \;=\; m$$

Since 2.69 must hold for all $i$ the decomposition in 2.67 must reduce to

$$V_n \;=\; \bigoplus_{i=1}^{N} \mathcal{L}[u_i, Eu_i, \ldots, E^{m}u_i\,]\,. \tag{2.70}$$

with

$$a) \quad H\,u_i = -m\,u_i\;, \qquad b) \quad F\,u_i = 0 \tag{2.71}$$

This proves $n = Nm$. We may thus introduce the basis

$$\mathcal{B} \;=\; \bigcup_{i=1}^{n} \{e_0^{(i)}, e_1^{(i)}, \ldots, e_m^{(i)}\} \tag{2.72}$$

with

$$e_s^{(i)} \;=\; (m-i)!\,E^s\,u^{(i)}$$

and conclude from Theorem 2.8 that the operator

$$T_A \;=\; \sum_{i,j=0}^{m} a_{ij}\,\frac{i!}{j!m!^2}\,F^{m-i}E^mF^j$$

acts on each basis block

$$\big\langle e_0^{(i)}, e_1^{(i)}, \ldots, e_m^{(i)} \big\rangle$$

by the matrix $A = \|a_{ij}\|_{i,j=0}^{m}$. However our definition of $E, F, H$ together with the relation in 2.65 gives us that, for any $v \in V_n$ we have

$$T_A\, v \;=\; \phi\Big( \sum_{i,j=0}^{m} a_{ij}\,\frac{i!}{j!m!^2}\,\tilde F^{m-i}\tilde E^m \tilde F^j\Big) v$$

But again from Theorem 2.7 we easily derive that

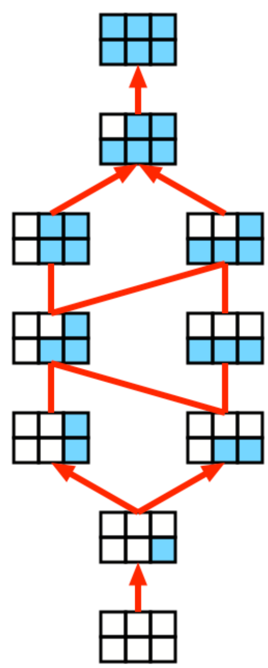
$$\sum_{i,j=0}^{m} a_{ij}\,\frac{i!}{j!m!^2}\,\tilde F^{m-i}\tilde E^m \tilde F^j \;=\; A$$

This gives that $\phi(A)$ acts by $A$ on each basis block proving the decomposition in 2.66, with $D$ the matrix relating the standard basis of $V_n$ with the basis $\mathcal{B}$ in 2.72. This completes our argument.

## 3.1 Spernerity of $\mathbf{L[m,n]}$

We recall that a finite "*poset* " (partially ordered set) is a pair $P = (\Omega, \leq)$ consisting of a finite set $\Omega$ together with a relation "$<$" satsfying the three conditions

(i) $x \leq x \quad \forall x \in \Omega$
(ii) $x \leq y \;\;\&\;\; y \leq x \;\;\Longrightarrow\;\; x = y$
(iii) $x \leq y \;\;\&\;\; y \leq z \;\;\Longrightarrow\;\; x \leq z$



If $x \leq y$ and $x \neq y$ we write "$x < y$". If $x < y$ and there is no element $z \in \Omega$ satisfying $x < z < y$ we write "$x \to y$" and say that "$x$ *is a predecessor of* $y$" or that "$y$ *is a successor of* $y$". We may also say that "$y$ *covers* $x$ ". When one of $x < y$ or $y < x$ holds true we say that $x$ and $y$ are "*comparable* " otherwise $x$ and $y$ are called "*incomparable* ". The "*Hasse diagram of* $P$" is $\Omega$ drawn with an arrow joining each element to its successor, as in the adjacent figure. An element $x \in \Omega$ is said to be "*minimal* " if $x$ has no predecessor in $P$ and "*maximal* " if it has no successor. A "*chain* " of $P$ is a subset $C \subseteq \Omega$, totally ordered by $<$. That is we have

$$C = \{x_o, x_1, x_2, \cdots, x_k\} \qquad \text{with} \qquad \{x_0 < x_1 < x_2 < \cdots < x_k\}$$

In this case we say that "$C$ *joins* $x_0$ *to* $x_k$ and $k$ is called the "*length* " of $C$, the latter is usually denoted "$l(C)$". By contrast we say that a subset $A \subseteq \Omega$ is an "*antichain* " if all pairs of elements of $A$ are incomparable.

A chain $C$ is called "*unrefinable*" or "*saturated* " if and only if

$$C = \{x_o, x_1, x_2, \cdots, x_k\} \qquad \text{with} \qquad \{x_o \to x_1 \to x_2 \to \cdots \to x_k\}$$

A "*maximal chain* " is an saturated chain which is not a proper subset of another chain. A poset whose maximal chains have all the same length is called "*ranked* " . If $P$ is ranked element it is convenient to add to $P$ a "*zero* " element denoted "$\hat{0}$" to be a predecessor of all minimal elements of $P$. Likewise if $P$ has more than one maximal element it is convenient to add to $P$ a "*unit* " element denoted "$\hat{1}$" to be a successor of all maximal elements of $P$. This done it follows that all unrefinable chains joining $\hat{0}$ to a given element $x$ have the same length. This common length is called the "*rank of* $x$ " and is denoted "$r(x)$". The rank of $\hat{1}$ is called the "*rank of* $P$" and is denoted "$r(P)$". The polynomial

$$F_P(q) \;=\; \sum_{x\in\Omega} q^{r(x)} \tag{3.1}$$

is called the "rank generating function of $P$ ". The collection of elements of $P$ of a given rank is called "*a rank row of* $P$". It will be convenient to set

$$\mathcal{R}_k(P) \;=\; \{x \in \Omega \,:\, r(x) = k\,\} \tag{3.2}$$

and refer to it as the "$k^{th}$ *rank row of* $P$ ". The cardinality of $\mathcal{R}_k(P)$ will be denoted $r_k(P)$. In particular, if $r(P) = k_o$ the definition in 3.1 may be written in the form

$$F_P(q) \;=\; \sum_{k=0}^{k_o} r_k(P) q^k \tag{3.3}$$

We say that $P$ is "*rank symmetric* " if and only if

$$r_{k_0-k}(P) \;=\; r_k(P) \qquad \text{(for all } 0 \le k \le k_o) \tag{3.4}$$

In view of 3.3 we see that this condition is equivalent to the identity

$$F_P(q) \;=\; q^{r(P)} F_P(1/q) \tag{3.5}$$

The poset $P$ is said to be "*rank unimodal* " if the sequence $r_k(P)$ weakly increases from 1 to its maximum and then weakly decreases back to 1. More precisely, for some $0 < k_1 < k_o$ we have

$$1 \le r_1(P) \le r_2(P) \le \cdots r_{k_1-1}(P) \le r_{k_1}(P) \ge r_{k_1+1}(P) \ge \cdots r_{k_o-1}(P) \ge 1 \tag{3.6}$$

Note that when $P = (2^{[n]}, \subseteq)$ is the lattice of subsets of the $n$-set $[n] = \{1, 2, \ldots, n\}$ and the rank of a subset $S \subseteq [n]$ is given by its size $|S|$, we have

$$F_P(q) \;=\; (1+q)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} q^k\,. \tag{3.7}$$

In this case the relations in 3.6 are easily shown to be true with $k_1 = [\frac{n}{2}]$. Moreover the symmetry of binomial coefficients yields the relations in 3.4 for $k_o = n$ as well. That is the lattice of subsets $(2^{[n]}, \subseteq)$ is rank symmetric and rank unimodal. It is clear from the definition of rank that

$$x < y \;\rightarrow\; r(x) < r(y)$$

thus in a ranked poset all rank rows are antichains. In a pioneering 1927 paper E. Sperner [] showed that for $P = (2^{[n]}, \subseteq)$

*The size of any antichain of $P$ does not exceed the size of the largest rank row*

This given, it has become customary to call a poset $P$ with this property a "*Sperner* " poset.

In a remarkable 1980 paper R. Stanley [] proved that the lattice $L[m,n]$ of Ferrers diagrams contained in the $m\times n$ rectangle, ordered by inclusion, is a Sperner poset. Stanley obtained this result using Algebraic Geometry. Later it was shown by R. Proctor [] that this result may be obtained as a beautiful application of $SL[2]$ theory. Our goal in this section is to derive the spernerity of $L[m,n]$ by fro our development of $SL[2]$ theory.

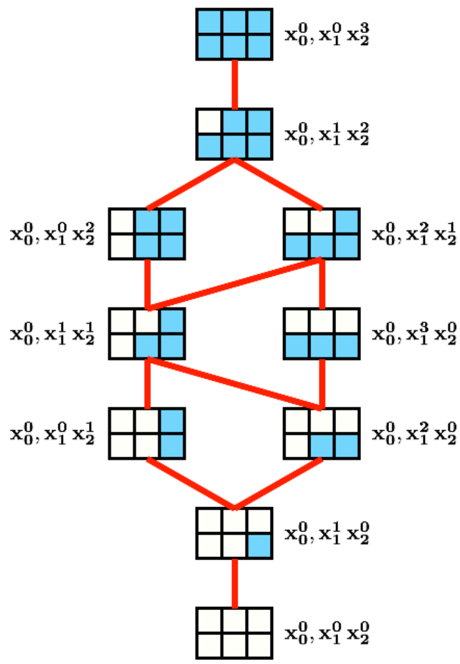
But before we can proceed we need to make some preliminary observations and derive a few auxliary facts.

In the figure above we have depicted the Hasse diagram of the lattice $L[3,3]$. It is easily seen from this example that $L[m,n]$ is a ranked poset, with the rank of a diagram $\pi \in L[m,n]$ given by the function,

$$r(\pi) = \#cells\ of\ \pi\,. \tag{3.8}$$

It is easily seen that $L[3,3]$ is also rank symmetric and rank unimodal. Our first task is to show that this is true in full generality for all $L[m,n]$.

It will be convenient to slightly alter the way we depict $L[m,n]$ as we have done for $L[2,3]$ in the attached display. Here we have depicted $L[2,3]$ with its elements represented by monomials. This will play a crucial role in our development. To be consistent with monomial representation, we have depicted the Ferrers diagrams with right justified rows.




This given we begin by showing that the rank generating function of $L[m,n]$ is given by the $q$-binomial polynomial. More precisely we have

## Proposition 3.1

$$F_{L[m,n]}(q) = \begin{bmatrix} m+n \\ n \end{bmatrix} = \frac{\prod_{i=1}^{m+n}(1-q^i)}{\prod_{i=1}^{m}(1-q^i)\prod_{i=1}^{n}(1-q^i)} \tag{3.9}$$

## Proof

For $\pi \in L[m,n]$ we set (as we did for $L[2,3]$):

$$m[\pi] \;=\; x_o^{p_o} x_1^{p_1}\cdots x_m^{p_m} \tag{3.10}$$

where $p_i$ gives the number of columns of $\pi$ that are of length $i$. Now the generating function of all these monomials is given by the formal series

$$\sum_{p_o\geq 0}\sum_{p_1\geq 1}\cdots\sum_{p_m\geq 0} t^{p_o+p_1+\cdots+p_m}\, x_o^{p_o}x_1^{p_1}\cdots x_m^{p_m} \;=\; \frac{1}{(1-tx_o)(1-tx_1)\cdots(1-tx_m)}\,. \tag{3.11}$$

Since the number of columns of $\pi$ is given by the sum $p_0+p_1+\cdots+p_m$ the coefficient of $t^n$ in 3.11 yields the list of all the monomials that correspond to elements of $L[m,n]$. Now we see from the definition in 3.8 that we have $r(\pi) \;=\; \sum_{i=1}^m i\,p_i$ and 3.9 gives

$$m[\pi]\Big|_{x_i\to q^i} \;=\; x_o^{p_o}x_1^{p_1}\cdots x_m^{p_m}\Big|_{x_i\to q^i} \;=\; q^{r(\pi)}$$

Making the replacements $x_i \to q^i$ on both sides of 3.11 immediately yields the beautiful identity

$$\sum_{n\geq 0} t^n F_{L[m,n]}(q) \;=\; \frac{1}{(1-t)(1-tq)\cdots(1-tq^m)}\,. \tag{3.12}$$

The assertion in 3.9 is then an immediate consequence of the so called $q$-binomial identity. We shall include a proof of it here for sake of completeness. To this end recall that the $q$-analogue of differentiation is defined by setting for any formal series $f(t)$

$$D_q f(t) \;=\; \frac{f(t)-f(qt)}{t(1-q)}\,. \qquad 3.13$$

Now an easy calculation gives that

$$D_q t^k \;=\; [k]_q t^{k-1} \qquad 3.14$$

and

$$D_q \frac{1}{(1-t)(1-tq)\cdots(1-tq^{k-1})} \;=\; \frac{[k]_q}{(1-t)(1-tq)\cdots(1-tq^{k})}\,, \qquad 3.15$$

where, as customary we have set

$$[k]_q \;=\; \frac{1-q^k}{1-q} \;=\; 1+q+q^2+\cdots+q^{k-1}\,. \qquad 3.16$$

Using 3.15 recursively for $k=1,2,\ldots,m-1$ yields that

$$\frac{1}{[m]_q!}D_q^m \frac{1}{1-t} = \frac{1}{(1-t)(1-tq)\cdots(1-tq^m)}. \qquad 3.17$$

On the other hand from 3.14 we derive that we must also have

$$D_q^m \frac{1}{1-t} = \sum_{k\geq m} [k]_q[k-1]_q\cdots[k+1]_q\, t^{k-m}.$$

By changing the summation index to $n=m-k$ we can also rewrite this as

$$\begin{aligned} \frac{1}{[m]_q!}D_q^m \frac{1}{1-t} \;&=\; \sum_{n\geq 0} \frac{[m+n]_q[m+n-1]_q\cdots[n+1]_q}{[m]_q!}\;\; t^n \\ &=\; \sum_{n\geq 0} \begin{bmatrix} n+m \\ m \end{bmatrix}_q t^n \end{aligned} \qquad 3.18$$

Equating the righthand sides of 3.17 an 4.18 we finally obtain the so-called $q$-binomial identity

$$\frac{1}{(1-t)(1-tq)\cdots(1-tq^m)} \;=\; \sum_{n\geq 0} \begin{bmatrix} n+m \\ m \end{bmatrix}_q t^n\,. \qquad 3.19$$

Using this in 3.12 and equating coefficients of $t^n$ gives 3.9 and completes our proof.

Note that from 3.9 we get

$$F_{L[2,3](q)} = \begin{bmatrix} 2+3 \\ 2 \end{bmatrix} = \frac{(1-q^4)(1-q^5)}{(1-q)(1-q^2)} \;=\; 1+q+2q^2+2q^3+2q^4+q^5+q^6$$

which is in perfect agreement with what we infer from our drawing of $L[2,3]$.

To apply $sl[2]$ theory to the study of the $L[m,n]$ posets we need to view the elements of $L[m,n]$ as bases of a vector space. More precisely we shall work with the vector space $V_{m,n}$ which is the linear span over $\mathbf{Q}$ of the set of monomials

$$\mathcal{B}_{m,n} \;=\; \left\{ m(\pi) = x_o^{p_0}x_1^{p_1}\cdots x_m^{p_m} \;:\; \pi\in L[m,n] \,\right\}$$

To construct an $sl[2]$ action on this space we need a few preliminary observations.

We shall begin with the following result which is of independent interest.

## Proposition 3.2

*Given an $m \times m$ matrix $A = \|a_{ij}\|_{i,j=0}^{m}$ let $D_A$ denote the differential operator*

$$D_A \;=\; \sum_{i,j=0}^{m} x_i a_{ij} \partial_{x_j}. \tag{3.20}$$

*It is easy to see that $D_A$ acts as ordinary differentiation on polynomials in $x_0, x_1, \ldots, x_m$. That is for $P, Q \in \mathbf{Q}[x_0, x_1, \ldots, x_m]$ we have*

$$D_A(PQ) = (D_A P)Q + P D_A Q.$$

*But more importantly the map $A \to D_A$ preserves the bracket operation. That is*

$$\big[D_A, D_B\big] = D_{[A,B]}. \tag{3.21}$$

**Proof**

If we let $B = \|b_{ij}\|_{i,j=0}^{m}$, then for any polynomial $P$ we have

$$\begin{aligned} D_A D_B P &= \sum_{i_1,j_1=0}^{m} x_{i_1} a_{i_1,j_1} \partial_{x_{j_1}} \sum_{i_2,j_2=0}^{m} x_{i_2} b_{i-2,j_2} \partial_{x_{j_2}} P \\ &= \sum_{i_1,j,i_2=0}^{m} x_{i_1} a_{i_1,j} b_{j,j_2} \partial_{x_j} x_j \partial_{x_{j_2}} P + \sum_{i_1,j_1 \neq i_2,j_2} x_{i_1} x_{i_2} a_{i_1,j_1} b_{i_2,j_2} \partial_{x_{j_1}} \partial_{x_{j_2}} P \\ &= D_{AB} P + \sum_{i_1,j_1,\neq i_2,j_2} x_{i_1} x_{i_2} a_{i_1,j_1} b_{i_2,j_2} \partial_{x_{j_1}} \partial_{x_{j_2}} P \end{aligned}$$

Repeating this calculation with the roles of $A$ and$B$ interchanged and subtracting we see that the second terms on the right hand sides cancel and the first terms yield the right hand side of 3.21 as desired.

This proposition implies that if $\mathcal{G}$ is a Lie algebra of $(m+1) \times (m+1)$ matrices then the operators $D_A$ with $A \in \mathcal{G}$ yield a representation of $\mathcal{G}$ on the space of polynomials in the variables $x_0, x_1, \cdots, x_m$. In particular we can obtain a representation of $sl[2]$ on the space of polynomials in $x, y$ by means of the operators that correspond to the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that in this case the definition in 3.20 with $m = 1$ $x_0 = x$ and $x_1 = y$ gives

$$D_E = x\partial_y\ , \qquad D_F P(x,y) = y\partial_x\ , \qquad D_H = (x\partial_x - y\partial_y)\,.$$

Let us now restrict this action to homogeneous polynomials of degree m in $x, y$ and express it in terms of the monomial basis

$$\big\{ x_j = x^j y^{m-j} : j = 0, 1, \ldots, m \big\}\,. \tag{3.22}$$

If we agree to set

$$x_{-1} = x_{m+1} = 0$$

a simple calculation yields that for $i = 0, 1, \ldots, m$ we have

$$D_E x_j = (m-j)x_{j+1}\ , \qquad D_F x_j = j\, x_{j-1}\ , \qquad D_H x_j = (2j-m)x_i \qquad 3.23$$

We thus see that this construction brings us back to the matrices $\tilde{E}, \tilde{F}, \tilde{H}$ given tn 2.56. We thus obtain another proof that these matrices satisfy the relations

$$[\tilde{E}, \tilde{F}] = \tilde{H}\ , \quad [\tilde{H}, \tilde{E}] = 2\tilde{E}\ , \quad [\tilde{H}, \tilde{F}] = -2\tilde{F}\ ,$$

It thus follows, from Proposition 3.2 that the differential operators

$$a)\quad D_{\tilde{E}} = \sum_{j=0}^{m-1}(m-j)x_{j+1}\partial_{x_j}\ , \qquad b)\quad D_{\tilde{F}} = \sum_{j=1}^{m} j\, x_{j-1}\partial_{x_j}\ , \qquad c)\quad D_{\tilde{E}} = \sum_{j=0}^{m}(2j-m)x_j\partial_{x_j}\ , \qquad 3.24$$

yield a representation of $sl[2]$ on the polynomials in $x_0, x_1, \ldots, x_m$.

Now it develops that we have the following remarkable identities

## Proposition 3.3

*The action of the operators $D_{\tilde{E}}, D_{\tilde{F}}, D_{\tilde{H}}$ on the monomial basis $\{m(\pi)\ :\ \pi \in L[m.n]\}$ may be expressed in the form*

$$\begin{aligned} a)\qquad D_{\tilde{E}}m(\pi) &= \sum_{\pi^+} m(\pi^+)\,\chi(\pi\to\pi^+)w(\pi,\pi^+) \\ b)\qquad D_{\tilde{F}}m(\pi) &= \sum_{\pi^-} m(\pi^-)\,\chi(\pi^-\to\pi)w(\pi,\pi^-) \\ c)\qquad D_{\tilde{H}}m(\pi) &= \big(2r(\pi)-mn\big)\,m(\pi) \end{aligned} \qquad 3.25$$

*where $w(\pi,\pi^+)$ and $w(\pi,\pi^-)$ denote suitable integer weights.*

**Proof**

Note that for $\pi \in L[m,n]$ we may write

$$\begin{aligned} D_{\tilde{E}}m(\pi) &= \sum_{j=0}^{m-1}(m-j)x_{j+1}\partial_{x_j}\, x_0^{p_0}x_1^{p_1}\cdots x_m^{p_m} \\ &= \sum_{j=0}^{m-1}(m-j)p_j\ \tfrac{x_{j+1}}{x_j}\ m(\pi) \end{aligned}$$

But when $p_j > 0$, the monomial $\frac{x_{j+1}}{x_j}\, m(\pi)$ is none other than the monomial of the partition obtained by removing from $\pi$ a column of length $j$ and adding a column of length $j+1$. The resulting partition $\pi^+$ is easily seen to be one of the successors of $\pi$ in the $L[m,n]$ poset. This proves 3.25 a) with

$$w(\pi,\pi^+)\ =\ (m-j)p_j\ .$$

Similarly from 3.24 b) we derive that

$$\begin{aligned} D_{\tilde{F}}m(\pi) &= \sum_{j=1}^{m} j\, x_{j-1}\partial_{x_j}\, x_0^{p_0}x_1^{p_1}\cdots x_m^{p_m} \\ &= \sum_{j=1}^{m} j\, p_j\ \tfrac{x_{j-1}}{x_j}\ m(\pi) \end{aligned}$$

Here again, when $p_j > 0$, the monomial $\frac{x_{j-1}}{x_j}\, m(\pi)$ is none other than the monomial of the partition obtained by removing from $\pi$ a column of length $j$ and adding a column of length $j-1$. The resulting partition $\pi^-$ is easily seen to be one of the predecesors of $\pi$ in the $L[m,n]$ poset. This proves 3.25 b).

In the same vein 3.24 c) gives

$$\begin{aligned} D_{\tilde{H}} m(\pi) &= \sum_{j=0}^{m} (2j-m)\, x_j \partial_{x_j}\, x_0^{p_0} x_1^{p_1} \cdots x_m^{p_m} \\ &= \sum_{j=0}^{m} (2j-m)\, p_j \; m(\pi). \end{aligned} \qquad 3.26$$

But when $\pi \in L[m,n]$ we have the relations

$$\sum_{j=0}^{m} j p_j = area(\pi)\,, \qquad \sum_{j=0}^{m} p_j = n\,.$$

Since, as we have noted that $area(\pi) = r(\pi)$ we see that 3.25 is simply another way of writing 3.25 c). This completes our proof.

At this point it is good to shorten the symbol introduced in 3.2 and simply use $\mathcal{R}_s$ to denote the $s^{th}$ rank row of $L[m,n]$. In other words $\mathcal{R}_s$ consists of all the partitions $\pi \in L[m,n]$ of area $s$, We will use $n_s$ to denote the cardinality of $\mathcal{R}_s$. It is also convenient to denote by $\pi^c$ the partition corresponding to the complement of $\pi$ in the $m \times n$ rectangle.

We are finally in a position to apply $sl[2]$ theory to the study of $L[m,n]$. We begin with

## Theorem 3.1

*The $L[m,n]$ poset is rank symmetric and rank unimodal. In particular the coefficients of the $q$-binomial polynomials form a unimodal sequence.*

## Proof

We see from 3.25 c) that all the monomials $m(\pi)$ are eigenvectors of $D_{\tilde{H}}$. Since these monomials are a basis of $V_{m,n}$, it follows from 3.25 that the eigenspace $V_{-k}$ of $D_{\tilde{H}}$ corresponding the eigenvalue $-k \leq 0$ is given by the linear span

$$V_{-k} \;=\; \mathcal{L}[m(\pi)\;:\;\pi \in \mathcal{R}_{(mn-k)/2}\;]$$

Thus we immediately derive from Theorem 2.5 that

$$n_{s-1} \leq n_s \qquad (\text{for } s \leq mn/2)$$

and

$$n_{s+1} \leq n_s \qquad (\text{for } s \geq mn/2)\,.$$

This proves rank unimodality. Rank symmetry immediately follows from the simple fact that the map $\pi \to \pi^c$ is an involution of $L[m,n]$ which maps a rank row onto a rank row of complementary rank. Nevertheless it is interesting to note that Theorem 2.4 shows that also rank symmetry follows from $sl[2]$ theory.

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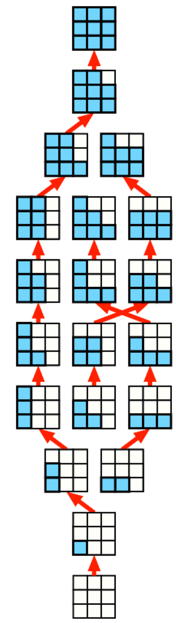
**Theorem 3.1**

$L[m,n]$ *is a Sperner poset*

**Proof**



Note first that if a poset $P$ can be decomposed into the disjoint union of $d$ chains then its largest antichain can have no more that $d$ elements. This is clear since the intersection of an antichain with any chain can contain at most one element. Thus one of the standard ways of proving Spernerity is to produce a chain cover whose cardinality is equal to that of the largest rank row. It develops that $sl[2]$ theory provides us precisely with the tools we need to construct such a chain cover for $L[m,n]$. To proceed we need an auxiliary definition.

For a given $0 \leq s < k_o = \lceil \frac{mn}{2} \rceil$, a map

$$\phi_s \ : \ \mathcal{R}_s \longrightarrow \mathcal{R}_{s+1}$$

will be called "' *admissible* " if and only if

(i) $\phi_s$ is injective,

(ii) $\forall \quad \pi \in \mathcal{R}_s$ we have $\pi \rightarrow \phi_s(\pi)$. 3.27

Now, note that if we are in possession of a sequence of admissible maps $\phi_0 \ , \ \phi_1 \, , \ldots , \ \phi_s \, , \ldots , \ \phi_{k_o}$

then the construction of the desired chain cover is immediate. We simply erase from the Hasse diagram of $L[m,n]$ all containement arrows except those corresponding to the sets of pairs

$$\mathcal{A} \ = \bigcup_{0\leq s\leq k_o} \big\{ \big(\pi, \phi_s(\pi)\big)\ ;\ \pi \in \mathcal{R}_s \, \big\} \quad \text{and} \quad \mathcal{B} \ = \bigcup_{0\leq s<k_o} \big\{ \big(\pi^c, \phi_s(\pi)^c\big)\ ;\ \pi \in \mathcal{R}_s \, \big\} \qquad 3.28$$

It is easily seen that the remaining arrows depict our chain cover. In the adjacent diagram we illustrate such a construction for $L[3,3]$. There the darker arrows come from the pairs in $\mathcal{B}$ of 3.28. It should be clear that the number of chains produced by this construction equals te number of elements in $\mathcal{R}_{k_o}$.

This given, our proof is completed by combining Theorem 2.5 with Proposition 3.3. In fact, it follows from Theorem 2.5 that, for each $s \leq k_o$, the $n_s \times n_{s+1}$ matrix giving the action of $D_{\tilde{E}}$ on the monomials corresponding to the elements of rank $s$, must necessarily have rank equal to $n_s$. This implies that this matrix must contain an $n_s \times n_s$ non vanishing subdeterminant. In turn this subdeterminant will have at least one non vanishing term. But the nature of the matrix of $D_{\tilde{E}}$ as given by 3.25 a) yields that any non vanishing determinantal term yields a map of $\mathcal{R}_s$ into $\mathcal{R}_{s+1}$ which satisfies properties (i) and (ii) of 3.27. Choosing one of these injections for each rank $s \leq k_o$ and calling it $\phi_s$ yields the sequence of admissible maps which yields the chain cover that proves Spernerity.

**Remark 3.1**

A problem that has remained open since the early $80's$ is to prove the existence of a minimal chain cover for $L[m,n]$ consisting entirley of "saturated symmetric chains ". These are saturated chains which join a element of rank $s$ to an elemnt of rank $mn-s$. The decomposition of $V_{m,n}$ as a direct sum of strings resulting from the $sl[2]$ action may be viewed as an " *algebraic level* " form of saturated symmetric chain cover. Unfortunately, , the above argument falls short of providing such a cover at the "*combinatorial level* ", and it is easily seen from the previous figure what could cause lack of symmetry. Nevertheless the single replacement of the arrow $(2) \rightarrow (3)$ by the arrow $(2) \rightarrow (2,1)$ yields a symmetric chain cover for $L[3,3]$.