ON FINITE DIMENSIONAL SL(2) REPRESENTATIONS

Lecture Notes Winter 1993

(With some help from N. Wallach)

1. Basic identities

Throughout these notes **V** will be a finite dimensional K vector space with three linear operators E, F, H satisfying the relations

$$(a)\ \ [E,F]=H \quad , \quad (b)\ \ [H,E]=2E \quad , \quad (c)\ \ [H,F]=-2F\ . \tag{1.1}$$

As customary the expression $L=(E+F+H)^*$ will denote the collection of all words in the alphabet $\{E,F,G\}$. Then the symbol $K[L]$ will represent the collection of all finite linear combinations with coefficients in K of words of L. We should note that at some point we shall need to assume that K is the field of complex numbers. Our purpose here is to decompose the action of $K[L]$ on **V** into its irreducible constituents. To this end we need to derive a number of auxiliary identities. These will be stated as separate propositions.

Note first that if A,B,C are operators and we set

$$D_C\ A \ = \ [A,C] \ = \ AC-CA$$

then we can easily verify that

$$D_C\ AB \ = \ (D_CA)\ B \ + \ A\ D_CB \tag{1.2}$$

In other words D_C acts like *differentiation* on products of linear operators. We shall make systematic use of this fact in the sequel. An instance in point is given by the proof of the following

Proposition 1.1

For any integer m we have

$$\begin{aligned} a)\quad HE^m \ &= \ E^m(H+2m) \\ b)\quad HF^m \ &= \ F^m(H-2m) \end{aligned} \tag{1.3}$$

Proof

Note we may write 1.3 in the form

$$D_H\ E^m \ = \ -2m\ E^m\ .$$

Note further that for $m=1$ this simply reduces to 1.1 b). Thus we can proceed by induction and assume it true for m. This given, using 1.1 (b) and 1.2 we get

$$D_H\ E^{m+1} \ = \ (D_HE)\ E^m \ + \ E\ D_HE^m \ = \ -2E^{m+1} \ - \ 2m\ E^{m+1} \ = \ -2(m+1)\ E^{m+1}$$

as desired. A similar argument gives 1.3 b).

As an immediate corollary we obtain the following remarkable fact.

Theorem 1.1

The operators E and F are necessarily nilpotent.

Proof

Let E satisfy the equation

$$(E-\lambda I)^m \Gamma(E) \;=\; 0 \tag{1.4}$$

Where $\Gamma(t)$ is a polynomial that does not vanish at λ. There is no loss in assuming that $\Gamma(\lambda)=1$. This given we can write

$$\Gamma(t) \;=\; 1 \;+\; (t-\lambda)Q(t)\ .$$

From which we derive that

$$I \;=\; -(E-\lambda I)Q(E) \;+\; \Gamma(E)\ . \tag{1.5}$$

Using 1.3 (a) we deduce that

$$D_H(E-\lambda I)^m \;=\; -2m\, E(E-\lambda I)^{m-1}$$

This may be rewritten as

$$-2m\, E(E-\lambda I)^{m-1} \;=\; (E-\lambda I)^m H \;-\; H(E-\lambda I)^m\ .$$

Multiplying on the right by $\Gamma(E)$ and using 1.4 gives

$$-2m\, E(E-\lambda I)^{m-1}\Gamma(E) \;=\; (E-\lambda I)^m H\, \Gamma(E)\ . \tag{1.6}$$

On the other hand left multiplication of 1.5 by $-2m\, E(E-\lambda I)^{m-1}$ yields

$$-2m\, E(E-\lambda I)^{m-1} \;=\; 2m\, E(E-\lambda I)^m Q(E) \;+\; -2m\, E(E-\lambda I)^{m-1}\Gamma(E)\ .$$

Replacing the last term in this identity by the right hand side of 1.6 we get

$$-2m\, E(E-\lambda I)^{m-1} \;=\; 2m\, E(E-\lambda I)^m Q(E) \;+\; (E-\lambda I)^m H\, \Gamma(E)\ .$$

Left multiplication by $\Gamma(E)$ and another use of 1.4 yields that we must also have

$$\lambda(E-\lambda I)^{m-1}\Gamma(E) \;=\; E(E-\lambda I)^{m-1}\Gamma(E) \;=\; 0\ .$$

If $\lambda\neq 0$ this process can be repeated and come to the final conclusion that 1.4 holds true only if

$$\Gamma(E) \;=\; 0\ .$$

This shows that the minimal equation of E can only be $E^b = 0$ for some $b \leq dim\,\mathbf{V}$. The same argument gives the nilpotency of F.

Proposition 1.2

For any integer m we have

$$\begin{aligned} a)\ F\,E^m &= E^m F - mE^{m-1}(H+(m-1)I) \\ b)\ E\,F^m &= F^m E - mF^{m-1}(H+(m-1)I) \end{aligned} \qquad 1.7$$

Proof

We may write 1.7 a) in the form

$$D_F E^m = mE^{m-1}(H+(m-1)I)\ .$$

Note that $m=1$ gives $D_F = H$ which is 1.1 (a). Thus we proceed by induction and assume 1.7 a) true for m. We then get using 1.2 again and 1.3 a)

$$\begin{aligned} D_F E^{m+1} &= (D_F E)\,E^m + E(mE^{m-1}(H+(m-1)I) \\ &= E^m(H+2mI) + mE^m H + m(m-1)E^m \\ &= E^m((m+1)H + (m+m^2)I) = (m+1)E^m(H+m) \end{aligned}$$

as desired. An analogous argument yields 1.7 b)

This result enables us to identify the eigenvalues of H.

Theorem 1.2

The eigenvalues of H are all integral and symmetrically distributed around the origin.

Proof

Let λ be an eigenvalue of H and let v be a corresponding eigenvector. Note that 1.3 b) yields that for any m

$$H\,F^m v = F^m(Hv - 2mv) = (\lambda - 2m)\,F^m v \qquad 1.8$$

Now let a be the smallest integer such that $F^{a+1}v = 0$. Note that, although Theorem 1.1 guarantees there must be such an integer, in this case there is an even simpler reason: 1.8 shows that the vectors $F^m v$ have distinct eigenvalues. Set then

$$u = F^a v$$

and note that we must have

$$\begin{aligned} &1)\quad u \neq 0 \\ &2)\quad Fu = 0 \quad \text{and} \\ &3)\quad H\,u = -k\,u \end{aligned} \qquad 1.9$$

where for convenience we have set

$$k = 2a - \lambda \tag{1.10}$$

Indeed, 1) and 2) follow from our choice of a and 3) is given by 1.8 for $m = a$. Equation 1.7 a) then yields that for all m

$$F\,E^m\,u \;=\; -\;mE^{m-1}(Hu + (m-1)u) \;=\; mE^{m-1}(k+1-m)u\ . \tag{1.11}$$

Let b be the smallest integer such that $E^{b+1}u = 0$. Using 1.9 with $m = b+1$ yields

$$0 \;=\; F\,E^{b+1}\,u \;=\; (b+1)(k-b)E^b u\ .$$

Since by our choice of b the vector $E^b\,u$ cannot vanish we must necessarily have

$$2a - \lambda \;=\; k \;=\; b\ \geq 0\ . \tag{1.12}$$

This proves that λ is an integer as desired. To derive the last assertion note that 1.3 and 1.9 3) yield

$$H\,E^m\,u \;=\; (-k+2m)\,E^m u\ . \tag{1.13}$$

Moreover, $E^m\,u \neq 0$ for $0 \leq m \leq k(=b)$. This means that the integers

$$-k\ ,\ -k+2\ ,\ -k+4\ ,\ \ldots -4+k\ ,\ -2+k\ ,\ k$$

are all eigenvalues of H. Now in case $\lambda \leq 0$ then $2a - k = \lambda \leq 0$ yields $a \leq k$ and we can take $m = k - a$ in 1.13 and obtain an eigenvector of eigenvalue $-k + 2k - 2a = -\lambda$. To obtain the eigenvalue $-\lambda$ when $\lambda > 0$ we need to reverse the roles of E and F in the above construction. Equivalently, we can replace H by $-H$ and obtain the desired symmetry when $\lambda > 0$ from the case just considered. This completes our proof.

Remark 1.1

The above proof yields another important fact. note that all the results we derived about the vector u introduced in our argument stem from properties 1),2),3) of 1.9. In particular we deduce that the vectors

$$u\ ,\ E\,u\ ,\ E^2\,u\ ,\ \ldots\ ,\ E^k\,u \tag{1.14}$$

are a basis of a subspace $\mathbf{U}$ of $\mathbf{V}$ that is invariant under the action of $K[L]$. In fact, the fact that $E^{k+1} = 0$ shows that $\mathbf{U}$ is invariant under E. The relations in 1.10 and 1.9 2) yield that $\mathbf{U}$ is invariant under F and those in 1.13 show that $\mathbf{U}$ is invariant under H. These observations lead us to the following basic fact.

Theorem 1.3

Let u be an eigenvector of H with eigenvalue $-k \leq 0$. Suppose further that $Fu = 0$ then the vectors in 1.14 are a basis of an irreducible $K[L]$-submodule of $\mathbf{V}$.

2. Tameness of the eigenvalues of H

Let us recall that an eigenvalue λ of an operator A is said to be *tame* if and only if

$$(A-\lambda I)^2\, v \;=\; 0 \qquad \longrightarrow \qquad (A-\lambda I)\, v\ . \tag{2.1}$$

Note that an operator all of whose eigenvalues are tame satisfies a polynomial equation with simple roots and must necessarily be diagonable. Our goal in this section is to put together the ingredients necessary to establish the diagonability of H. We start with the following

Proposition 2.1

If u is an eigenvector of H with eigenvalue $-k\leq 0$ and $F\,u=0$ then

$$F^m\, E^m\, u \;=\; \frac{m!k!}{(k-m)!}\, u \qquad (\text{ for } \quad m=0,1,2,\ldots,k\ ) \tag{2.2}$$

Proof

The identity is trivially true for $m=0$. We shall thus inductively assume it true for m. This given, using 1.7 a) (with $m\rightarrow m+1$) we get

$$\begin{aligned} F^{m+1}\, E^{m+1}\, u \;&=\; F^m\, \big(\, FE^{m+1}u\, \big) \\ &=\; F^m\, (m+1)(k-m)E^m\, u \\ &=\; (m+1)(k-m)F^mE^m\, u \;=\; (m+1)(k-m)\, \frac{m!k!}{(k-m)!}\, u \\ &=\; \frac{(m+1)!k!}{(k-m+1)!}\, u \end{aligned}$$

This completes the induction and our proof.

Proposition 2.2

For all integers m we have

$$\begin{aligned} a)\quad EH^m \;&=\; (H-2I)^m\, E \\ b)\quad FH^m \;&=\; (H+2I)^m\, F \end{aligned} \tag{2.3}$$

Proof

The case $m=1$ of 2.3 a) reduces to $EH=(H-2I)E=HE-2E$ which is another way of writing 1.1 (b). The general relation is obtained by successive applications of the first. Indeed we have

$$E\, H^m \;=\; (H-2I)E\, H^{m-1} \;=\; (H-2I)^2\, E\, H^{m-2} \;=\; \cdots$$

The other relation is established in the same manner.

Theorem 2.1

The largest eigenvalue of H is tame

Proof

Let k_o be the largest eigenvalue of H. By symmetry $-k_o$ must be the smallest. Let $w \in \mathbf{V}$ satisfy

$$(H+k_o)^2\, w \;=\; 0 \;. \tag{2.4}$$

We are to derive that

$$u \;=\; (H+k_o)\, w \;=\; 0 \;. \tag{2.5}$$

We shall show that $u \neq 0$ leads to a contraddiction. Note first that repetitive uses of $(H+2I)F = FH$ yields

$$(H+2I+k_oI)^2 Fw \;=\; (H+2I+k_oI)F(H+k_oI)w \;=\; F(H+k_oI)^2 w \;=\; 0 \;.$$

Thus to avoid a contraddiction at this early stage we must accept that

$$a) \quad Fw = 0 \qquad \text{and} \qquad b) \quad Fu = F(H+k_oI)w = 0 \;. \tag{2.6}$$

Similarly, repetitive uses of $(H-2I)E = EH$ yield

$$(H - k_oI - 2I)^2 E^{k_o+1} w \;=\; E^{ko+1}(H+k_oI)^2 w \;=\; 0 \;, \tag{2.7}$$

Thus again to avoid a contraddiction we must accept

$$E^{k_o+1} w \;=\; 0 \;. \tag{2.8}$$

But on the other hand a) and b) of 2.6 (using first 1.7 a) then 2.2 with $m = k_o + 1$) give

$$\begin{aligned} F^{k_o+1}E^{k_o+1}w = F^{k_o}\big(\, FE^{k_o+1}w\big) &= -(k_o+1)F^{k_o}E^{k_o}(H+k_oI)w \\ &= \; -(k_o+1)(k_o!)^2\, (H+k_oI)w \\ &= -(k_o+1)(k_o!)^2\, u \;\; \neq 0 \;\text{, (!!!)} \end{aligned}$$

which plainly contraddicts 2.8. This completes our proof.

Remark 2.1

To prove the tameness of all the other eigenvalues of H we need to separate the $K[L]$-invariant subspace $\mathbf{V}_1$ generated by the eigenvectors of eigenvalue $-k_o$ from the rest of $\mathbf{V}$. More precisely we need to be able to construct a direct sum decomposition

$$\mathbf{V} \;=\; \mathbf{V}_1 \;\oplus\; \mathbf{V}_2 \tag{2.9}$$

with $\mathbf{V}_2$ also $K[L]$-invariant. This done we will be in a position to apply Theorem 2.1 to $\mathbf{V}_2$ and successively obtain the diagonability of H. To this end let

$$u_1 \;,\; u_2 \;,\; \ldots \;,\; u_N \tag{2.10}$$

be a basis for the eigenspace of H corresponding to the eigenvalue $-k_o$. Note that each u_i is killed by F since otherwise H would have the eigenvalue $-k_o - 2$. Thus in view of Remark 1.1 the vectors $u_i, Eu_i, \ldots, E^{k_o} u_i$ are a basis of an invariant subspace. Note further that $\{E^m\, u_i\}_{m,i}$ is an independent set. In fact, the equality

$$\sum_{m=0}^{k_o}\ \sum_{i=1}^{N} c_{m,i}\, E^m\, u_i \;=\; 0$$

forces

$$\sum_{i=1}^{N} c_{m,i}\, E^m\, u_i \;=\; 0 \qquad 2.11$$

separately for each m since the terms in this sum are all eigenvectors of H with eigenvalue $-k_o - m$ and different eigenspaces are independent. On the other hand if we apply F^m to 2.11 and use 2.2 we get

$$\sum_{i=1}^{N} c_{m,i}\, u_i \;=\; 0$$

and the vanishing of the coefficients $c_{m,i}$ is then forced by the independence of the $u_i\,'s$. This implies that the subspace

$$\mathbf{V}_1 \;=\; \mathcal{L}[\, E^m\, u_i \;:\; i = 1, 2, \ldots, N \;;\; m = 0, 1, \ldots, k_o \;] \;. \qquad 2.12$$

is a direct sum of irreducible $K[L]$-submodules. Moreover, H acts diagonally on the basis $\{E^m\, u_i\}_{m,i}$.

This given, we see that the construction of a $K[L]$-invariant complement of $\mathbf{V}_1$ (*) would prove more than the diagonability of H. In fact, it would provide us with a proceedure for breaking up $\mathbf{V}$ into a direct sum of irreducible $K[L]$-modules.

It develops that the tools for carrying out this construction are provided by *Casimir* operator

$$C \;=\; EF \;+\; FE \;+\; H^2/2 \;, \qquad 2.13$$

we shall present them as three separate propositions.

Proposition 2.3

C commutes with E, F and H.

Proof

Note that 1.1 (b) and (c) give

$$D_H\, C \;=\; -2EF \;+\; 2EF \;+\; 2FE \;+\; -2FE \;+\; 0 \;=\; 0$$

Thus H and C commute. Using 1.1 (a) we get

$$D_E\, C \;=\; -EH \;+\; -HE \;+\; (2EH + 2HE)/2 \;=\; 0 \;.$$

(*) That is a $K[L]$-submodule $\mathbf{V}_2$ breaking up $\mathbf{V}$ as indicated in 2.9

Thus E and H commute as well. The result for F is shown in the same manner.

Proposition 2.4

If $Hu = -ku$ and $Fu = 0$ then for any m we have

$$C\,E^m\,u \;=\; (k + k^2/2)\,E^m\,u \qquad 2.14$$

Proof

Note that we can rewrite C in the form

$$C \;=\; 2EF \;+\; -H \;+\; H^2/2\ . \qquad 2.15$$

This gives

$$C\,u \;=\; -Hu \;+\; H^2u/2 \;=\; (k+k^2/2)\,u\ ,$$

and 2.14 follows immediately from the commutativity of C and E.

This result implies that every element of $\mathbf{V}_1$ is an eigevector of C with eigenvalue $k_o + k_o^2/2$. Setting for convenience

$$t_o \;=\; k_o + k_o^2/2 \qquad 2.16$$

we may express this fact by writing

$$\mathbf{V}_1 \;\subseteq\; Nullspace[C - t_oI]\ . \qquad 2.17$$

We can convert this into an equality. In fact we can prove an even stronger result.

Proposition 2.5

Let k_o be the largest eigenvalue of H. Let $\mathbf{V}_1$ be defined as in 2.1 and t_o be as in 2.16. Then for any $w \in \mathbf{V}$ we have

$$(C - t_oI)^p\,w = 0 \quad\longrightarrow\quad w \;\in\; \mathbf{V}_1 \qquad 2.18$$

in particular we derive that t_o is a tame eigenvalue of C.

Proof

Let a be the least integer such that

$$F^{a+1}\,w \;=\; 0\ . \qquad 2.19$$

Theorem 1.1 guarantees that such an integer exists. Our proof will proceed by induction on a. We start with $a = 1$. So let $Fw = 0$. This given, the expression in 2.15 yields

$$\begin{aligned}(C - t_o)w \;&=\; \big[\,-(H + k_oI) \;+\; \frac{H^2 - k_o^2I}{2}\,\big]\,w\\ &=\; (H + k_oI)\,\big(-1 + \frac{H - k_oI}{2}\big)\,w \qquad 2.20\\ &=\; \frac{1}{2}(H + k_oI)(H - (k_o + 2)I)\,w\ .\end{aligned}$$

Thus the condition $(C - t_o I)^p \, w = 0$ may be rewritten as

$$\frac{1}{2^p}(H - (k_o + 2)I)^p (H + k_o I)^p \, w \;=\; 0 \;. \qquad 2.21$$

Now, the maximality of k_o guarantees that the operator $H + (k_o + 2)I)$ is invertible. So 2.21 is equivalent to

$$(H + k_o I)^p \, w \;=\; 0 \;.$$

However, since k_o is a tame eigenvalue for H we see that we must also have

$$(H + k_o I) \, w \;=\; 0 \;.$$

This places $w \in \mathbf{V}_1$ and completes the first step of the induction. So let 2.18 be true when $F^a \, w = 0$. Let $F^a w \neq 0$ and $F^{a+1} w = 0$. For convenience set

$$u \;=\; F^a \, w \;. \qquad 2.22$$

The commutativity of C and H yields that

$$(C - t_o)^p u \;=\; F^a \, (C - t_o I)^p w \;=\; 0$$

Since $Fu = 0$ we are reduced to the case just considered and u must also be an eigenvector of H with eigenvalue $-k_o$. In particular it must be a linear combination of the $u_i\,'s$ introduced in 2.10. So let

$$u \;=\; \sum_{i=1}^{N} c_i \, u_i$$

and set

$$\bar{w} \;=\; \frac{(k_o - a)!}{a! k_o!} \sum_{i=1}^{N} c_i \, E^a \, u_i \;.$$

Since each u_i is killed by F we can apply 2.2 and deduce that

$$F^a \, \bar{w} \;=\; \sum_{i=1}^{N} c_i \, u_i \, u = F^a w \;.$$

Now the difference $w - \bar{w}$ is killed by $(C - t_o I)^p$ since w is so by hypothesis and $\bar{w}$ is even killed by $C - t_o I$ since it lies in $\mathbf{V}_1$. Moreover as we have seen $F^a(w - \bar{w}) = 0$. Thus we can use the induction hypothesis and derive that $w - \bar{w} \in \mathbf{V}_1$. Since $w = \bar{w} + w - \bar{w}$ we have $w \in \mathbf{V}_1$ as desired. This copmpletes the induction and our proof.

We can finally put all our pieces together and produce an invariant complement for $\mathbf{V}_1$. To this end note that since t_o has been shown to be a tame eigenvalue of C, the minimal equation of C can be written in the form

$$(C - t_o I) \, Q(C) \;=\; 0 \qquad 2.23$$

with $Q(t)$ a polynomial which does not vanish at t_o. There is no loss in assuming that $Q(t_o) = 1$. In other words we may write

$$1 = Q(t) \ + \ (t - t_o)\Gamma(t) \ .$$

Thus

$$I \ = \ Q(C) \ + \ (C - t_o I)\,\Gamma(C) \ . \tag{2.24}$$

This gives us the decomposition

$$\mathbf{V} \ = \ Q(C)\mathbf{V} \ \oplus \ (C - t_o I)\Gamma(C)\mathbf{V} \ . \tag{2.25}$$

Note that the sum is direct here since if we had two vectors v_1, v_2 such that

$$Q(C)\,v_1 \ = \ (C - t_o I)\Gamma(C)\,v_2$$

then hitting this by $(C - t_o I)$ and using 2.23 gives

$$(C - t_o I)^2\Gamma(C)v_2 \ = \ 0 \ .$$

But since t_o is tame for C we must also have

$$(C - t_o I)\Gamma(C)v_2 \ = \ 0 \ .$$

This implies that the summands in 2.24 share only the zero vector as desired. Moreover, since every element of $Q(C)\mathbf{V}$ is killed by $C - t_o I$, Proposition 2.5 gives us that

$$Q(C)\mathbf{V} \ \subseteq \ \mathbf{V}_1 \ .$$

However equality must hold here. In fact, since every element $v \in \mathbf{V}_1$ is killed by $C - t_o I$, 2.24 yields that it may be written in the form

$$v \ = \ Q(C)\,v \ .$$

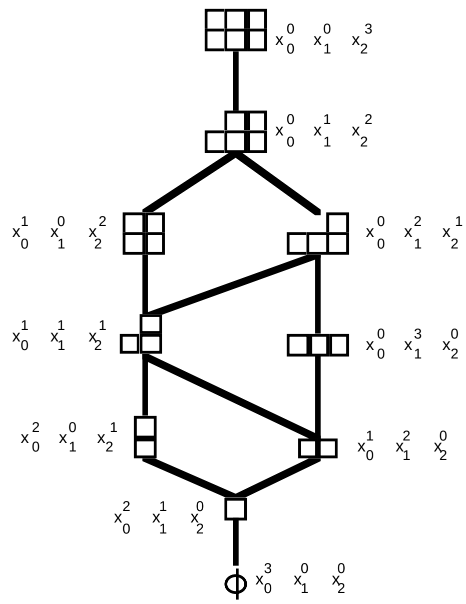
Thus

$$Q(C)\mathbf{V} \ = \ \mathbf{V}_1 \ .$$

Since the $K[L]$-invariance of the space $(C - t_o I)\Gamma(C)\mathbf{V}$ is an immediate consequence of the commutativity of C with E,F and H we see that 2.25 is precisely the decomposition of $\mathbf{V}$ we were looking for. That is we may take $\mathbf{V}_2 = (C - t_o I)\Gamma(C)\mathbf{V}$ in 2.9. This completes our development.

3.1 Spernerity of $L[m,n]$

Following standard notation, $L[m,n]$ denotes here the collection of Ferrers diagrams contained in an $m \times n$ rectangle. It is easy to see that the ordinary containment partial order makes $L[m,n]$ into a distributive lattice. In the figure below we have depicted $L[2,3]$ with its elements indexed by monomials that will be of use in our developments. We have also used the *australian* way of depicting Ferrers diagrams. That is with right justified rows of legnths increasing from bottom to top.



We notice that $L[m,n]$ is also a ranked poset, and if we set, for a diagram $\pi \in L[m,n]$,

$$rank[\pi] \;=\; \#squares\ of\ \pi \ .$$

then the rank generating function

$$F_{L[m,n]}(q) \;=\; \sum_{\pi \in L[m,n]} q^{rank[\pi]} \ .$$

is given by the familiar $q-binomial$ coefficient. More precisely we have the following identity

Proposition 3.1

$$F_{L[m,n]}(q) \;=\; \begin{bmatrix} m+n \\ n \end{bmatrix}_q \;=\; \frac{\prod_{i=1}^{m+n}(1-q^i)}{\prod_{i=1}^{m}(1-q^i)\prod_{i=1}^{n}(1-q^i)} \qquad 3.1$$

Proof

Note that we can represent an element π of $L[m,n]$ by a monomial

$$m[\pi] \;=\; x_o^{p_o} x_1^{p_1} \cdots x_m^{p_m}$$

where p_i gives the number of columns of π that are of length i. We did this for $L[2,3]$ in the figure above. Now the generating function of all these monomials is given by the formal series

$$\sum_{p_o\geq 0}\sum_{p_1\geq 0}\cdots\sum_{p_m\geq 0} t^{po+p_1+..+p_m}\; x_o^{p_o} x_1^{p_1}\cdots x_m^{p_m} \;=\; \frac{1}{(1-tx_o)(1-tx_1)\cdots(1-tx_m)} \;. \quad 3.2$$

In fact, the coefficient of t^n here yields the list of all the monomials that correspond to the elements of $L[m,n]$. Since replacing x_i by q^i in $m[\pi]$ yields $q^{rank[\pi]}$ we can easily deduce (setting $x_i = q^i$ in 3.2) that

$$\sum_{n\geq 0} t^n\, F_{L[m,n]}(q) \;=\; \frac{1}{(1-t)(1-tq)\cdots(1-tq^m)} \;. \qquad 3.3$$

The assertion in 3.1 is then an immediate consequence of the so called $q-binomial$ identity. We shall include a proof here for sake of completeness. Recall that the $q-analogue$ of differentiation is defined by setting for any formal series $f(t)$

$$\partial_q f(t) \;=\; \frac{f(t)-f(qt)}{t(1-q)} \;. \qquad 3.4$$

Now an easy calculation gives that

$$\partial_q\, t^k \;=\; [k]_q\, t^{k-1} \qquad 3.5$$

and

$$\partial_q\, \frac{1}{(1-t)(1-tq)\cdots(1-tq^{k-1})} \;=\; \frac{[k]_q}{(1-t)(1-tq)\cdots(1-tq^k)} \;,$$

where, as customary we have set

$$[k]_q \;=\; \frac{1-q^k}{1-q} \;=\; 1+q+q^2+\cdots+q^{k-1} \;. \qquad 3.6$$

Using 3.6 recursively for $k=1,2,\ m-1$ yields that

$$\frac{1}{[m]_q!}\,\partial_q^m \frac{1}{1-t} \;=\; \frac{1}{(1-t)(1-tq)\cdots(1-tq^m)} \;. \qquad 3.7$$

On the other hand from 3.5 we derive that we must also have

$$\partial_q^m \frac{1}{1-t} \;=\; \sum_{k\geq m} [k]_q[k-1]_q\cdots[k+1]_q\, t^{k-m} \;.$$

By changing the summation index to $n = m-k$ we can also rewrite this as

$$\frac{1}{[m]_q!}\,\partial_q^m \frac{1}{1-t} \;=\; \sum_{n\geq 0} \frac{[m+n]_q[m+n-1]_q\cdots[n+1]_q}{[m]_q!}\, t^n \qquad 3.8$$

and we can easily see that the identity in 3.1 is obtained by equating the righthand sides 3.3, 3.7 and 3.8.

Note that from 3.1 we get

$$F_{L[2,3]}(q) \;=\; \begin{bmatrix} 2+3 \\ 2 \end{bmatrix}_q \;=\; \frac{(1-q^4)(1-q^5)}{(1-q)(1-q^2)} \;=\; 1+q+2q^2+2q^3+2q^4+q^5+q^6$$

which in perfect agreement with what we mey infer from our drawing of $L[2,3]$.

We recall that a subset S of a poset $P=(\Omega,\leq)$ is said to be an *antichain* if no two elements of S are comparable in P. It is clear that in a ranked poset all the rank rows are antichains. This given if P is ranked and the coefficients of the rank generating function of P form a unimodal sequence, we may ask if the size of the largest antichain of P could be that of the largest rank row. Posets with this property are customarily referred to as a *Sperner posets*. It develops that the spernerity of $L[m,n]$ may be established by a beautiful argument which uses the elements of the representation theory of $sl[2]$ that we have just presented. Before we can see how this comes about we need to make some preliminary observations and introduce some notation.

Remark 3.1

Note that if P can be decomposed into the disjoint union of d chains then its largest antichain can have no more that d elements. This is clear since the intersection of an antichain with any chain can contain at most one element. Thus one of the standard ways of proving spernerity is to produce a disjoint chain cover whose cardinality is equal to that of the largest rank row. We shall use this idea here. More precisely we shall show that each rank row below the midrank can be injectively mapped *along containment edges* into the next higher row. We can easily see that these maps can be used to construct the desired chain cover. In fact, if we start with an element of a lower row which is not an image of an element of the previous row and follow the maps we will eventually end up in the middle rank row. Doing this with each element that has no preimage constructs the lower portion of the chains we need. Since complentation of a diagram yields an edge preserving map of $L[m,n]$ into itself, the remaining portions of the chains can be constructed by *turning the poset upside down* and essentially using the same maps.

For instance, in the figure above, we can use the vertical edges together with the edge $x_0^2x_1^1x_2^0 \;\longrightarrow\; x_0^2x_0^1x_2^1$ to construct the desired maps. This given the resulting chain cover consists of the left most path from $x_o^3x_1^0x_2^0$ to $x_o^0x_1^0x_2^3$ and the path

$$x_o^1x_1^2x_2^0 \;\longrightarrow\; x_o^0x_3^3x_2^0 \;\longrightarrow\; x_o^0x_1^2x_2^1 \;.$$

We should note that in this particular construction the two chains we obtain are *symmetric*. i.e. they both go from one rank to the complementary rank. This is just an accident. In general the chains produced by the construction we have sketched above will not have this property. It should be noted that the decomposition of $L[m,n]$ into a minimal chain cover consisting of symmetric chains is a problem that has frustrated a generation of combinatorists.

To see how the representation theory of $sl[2]$ delivers us the desired maps we need some further facts. Given an $n\times n$ matrix $A=\|a_{ij}\|$ we shall denote by D_A the differential operator

$$D_A \;=\; \sum_{i,j=1}^{n} x_i a_{ij}\,\partial_{x_j} \;. \qquad 3.9$$

It is easy to check that D_A acts as ordinary differentiation. More precisely, if P and Q are polynomials in $x_1,x_2,\ldots,x_n$ then

$$D_A\,(PQ) \;=\; (D_A\,P)\,Q \;+\; P\,D_A\,Q \;.$$

More importantly we have the following basic fact.

Proposition 3.2

The map $A\to D_A$ preserves the bracket operation. That is

$$[D_A,D_B] \;=\; D_{[A,B]} \;. \qquad 3.10$$

Proof

If we let $B=\|b_{ij}\|$, then for any polynomial P we have

$$\begin{aligned} D_AD_BP \;&=\; \sum_{i_1,j_1=1}^{m} x_{i_1}a_{i_1,j_1}\partial_{x_{j_1}} \sum_{i_2,j_2=1}^{m} x_{i_2}b_{i_2,j_2}\partial_{x_{j_2}}\;P \\ &=\; \sum_{i_1,j,i_2}^{m} x_{i_1}a_{i_1,j}b_{j,j_2}\partial_{x_j}x_j\partial_{x_{j_2}}\;P \;+\; \sum_{i_1,j_1\neq i_2,j_2} x_{i_1}x_{i_2}a_{i_1,j_1}b_{i_2,j_2}\partial_{x_{j_1}}\partial_{x_{j_2}}\;P \\ &=\; D_{AB}\;P \;+\; \sum_{i_1,j_1,i_2,j_2} x_{i_1}x_{i_2}a_{i_1,j_1}b_{i_2,j_2}\partial_{x_{j_1}}\partial_{x_{j_2}}\;P \end{aligned}$$

Repeating this calculation with the roles of A and B interchanged and subtracting we see that the second terms on the right hand sides cancel and the first terms yield the right hand side of 3.10 as desired.

This proposition implies that if $\mathbf{g}$ is a Lie algebra of $m\times m$ matrices then the operators D_A with $A\in\mathbf{g}$ yield a representation of $\mathbf{g}$ on the space of polynomials in the variables $x_1,x_2,\ldots,x_m$. In particular we can obtain a representation of $sl[2]$ on the space of polynomials in x,y by means of the operators that correspond to the matrices

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad,\quad F=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad,\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\;.$$

We see that in this case the definition in 3.9 gives

$$D_E\;P(x,y)=x\partial_y\;P(x,y)\quad,\quad D_F\;P(x,y)=y\partial_x\;P(x,y)\quad,\quad D_H\;P(x,y)=(x\partial_y-y\partial_x)\;P(x,y)\;.$$

Let us now restrict this action to homogeneous polynomials of degree m in x,y and express it in terms of the monomial basis

$$\{\;x_i=x^iy^{m-i}\;:\;i=0\ldots m\}\;. \qquad 3.11$$

If we agree to set $x_{-1}=x_{m+1}=0$ the result of this simple calculation may be expressed in the form

$$D_E\, x_i=(m-i)x_{i+1} \quad , \quad D_F\, x_i=i\, x_{i-1} \quad , \quad D_H\, x_i=(2m-i)x_i \quad (\, i=0,\ldots,m\,) \qquad 3.12$$

The next step is to forget that $x_i=x^iy^{m-i}$ and construct the matrices $\bar{E},\bar{F},\bar{H}$ that express the actions of D_E,D_F,D_H on the basis $\{x_o,x_1,\ldots,x_m\}$ as given by 3.12. To get accross what we have in mind here we carry this out for $m=3$. Note that in this case the first of 3.12 may be rewritten in matrix form as

$$D_E\{x_o,x_1,x_2,x_3\} \;=\; \{3x_1,2x_2,x_3,0\} \;=\; \{x_o,x_1,x_2,x_3\}\begin{pmatrix}0&0&0&0\\3&0&0&0\\0&2&0&0\\0&0&1&0\end{pmatrix} .$$

In the same vein we write

$$D_F\{x_o,x_1,x_2,x_3\} \;=\; \{0,x_o,2x_1,3x_2\} \;=\; \{x_o,x_1,x_2,x_3\}\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix} ,$$

and

$$D_H\{x_o,x_1,x_2,x_3\} \;=\; \{-3x_o,-x_1,x_2,3x_3\} \;=\; \{x_o,x_1,x_2,x_3\}\begin{pmatrix}-3&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&1&3\end{pmatrix} .$$

Thus for $m=3$ we must take

$$\bar{E}=\begin{pmatrix}0&0&0&0\\3&0&0&0\\0&2&0&0\\0&0&1&0\end{pmatrix} \quad , \quad \bar{F}=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix} \quad , \quad \bar{H}=\begin{pmatrix}-3&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&1&3\end{pmatrix} .$$

From proposition 3.2 we immediately derive that we must have (for any choice of m)

$$[\bar{H},\bar{E}]=2\bar{E} \quad , \quad [\bar{H},\bar{F}]=-2\bar{F} \quad , \quad [\bar{E},\bar{F}]=\bar{H} \;. \qquad 3.12$$

We can thus repeat the process and obtain a further representation of $sl[2]$ on polynomials in the *variables* $\{x_o,x_1,\ldots,x_m\}$ by means of the differential operators $D_{\bar{E}},D_{\bar{F}},D_{\bar{H}}$. More precisely, using 3.9 we get, for any polynomial $P=P(x_o,x_1,\ldots,x_m)$

$$\begin{aligned} D_{\bar{E}}P &= \sum_{i=1}^{m}(m+1-i)x_i\partial_{x_{i-1}}\, P \\ D_{\bar{F}}P &= \sum_{i=0}^{m-1}(i+1)x_i\partial_{x_{i+1}}\, P \\ D_{\bar{H}}P &= \sum_{i=0}^{m}(2i-m)x_i\partial_{x_i}\, P \end{aligned} \qquad 3.13$$

We are now ready to apply our $sl[2]$ machinery to the study of the $L[m,n]$ poset. The idea is very simple and it is based on our representation of the elements of $L[m,n]$ by monomials. In fact, note that if

$$m[\pi] \;=\; x_o^{p_o} x_1^{p_1} \cdots x_m^{p_m}$$

then

$$n = p_o + p_1 + \cdots + p_m \quad \text{and} \quad rank[\pi] = p_1 + 2p_2 + 3p_3 + \cdots + mp_m \ . \qquad 3.14$$

Thus $L[m,n]$ itself may be viewed as the natural monomial basis of the space of homogeneous polynomials of degree n in $x_o, x_1, \ldots, x_m$. By a slight abuse of notation we shall then denote this space by $\mathcal{L}[L[m,n]]$ and refer to it as *the linear span of* $L[m,n]$. Note further that we can write the action of $D_{\bar{H}}$ on $m(\pi)$ in the form

$$D_{\bar{H}} m(\pi) \;=\; (\sum_{i=0}^{m} (2i-m)p_i)\, m(\pi) \;=\; (2\, rank(\pi) - nm) m(\pi) \ . \qquad 3.15$$

Which shows that the rank rows of $L[m,n]$ may in turn be viewed as natural bases for the various eigenspaces of the operator $D_{\bar{H}}$. But we are not finished. The actions of $D_{\bar{E}}$ and $D_{\bar{F}}$ have even more remarkable combinatorial interpretations. In fact, from 3.13 we get

$$D_{\bar{E}} m(\pi) \;=\; \sum_{i=1}^{m} p_{i-1}(m+1-i) x_i m(\pi)/x_{i-1} \ . \ . \qquad 3.16$$

Now we can rewrite this in a combinatorially more revealing form. Note that, when $p_{i-1} > 0$ the monomial $x_i m(\pi)/x_{i-1}$ simply corresponds to the diagram obtained by adding one square on the top of the right most column of length $i-1$ of π. In other words the latter diagram is one of the immediate successors of π in $L[m,n]$. Let us denote such a diagram by π^+ and write $\pi \rightarrow \pi^+$ to express that the pair $(\pi\ ,\ \pi^+)$ is an upwards pointing edge of the Hasse diagram of $L[m,n]$. Moreover, if $m(\pi^+) = x_i m(\pi)/x_{i-1}$ set $w(\pi, \pi^+) = p_{i-1}(m+1-i)$ and refer to this as the *weight* of the pair $(\pi\ ,\ \pi^+)$. This given we may rewrite 3.16 in the form

$$D_{\bar{E}} m(\pi) \;=\; \sum_{\pi^+} \chi(\pi \rightarrow \pi^+)\, w(\pi, \pi^+)\, m(\pi^+) \ . \qquad 3.17$$

We can express the action of $D_{\bar{F}}$ in a similar manner by letting π^- denote a diagram whose monomial is $x_i m(\pi)/x_{i+1}$ and set $w(\pi -, \pi) = p_{i+1}(i+1)$. The second of 3.13 then gives

$$D_{\bar{F}} m(\pi) \;=\; \sum_{\pi^-} \chi(\pi^- \rightarrow \pi)\, w(\pi^-, \pi)\, m(\pi^-) \ . \qquad 3.18$$

These two expressions enable us to extract combinatorial information from linear algebraic properties of the operators $D_{\bar{E}}, D_{\bar{F}}, D_{\bar{H}}$. Our foray into the representation theory of $sl[2]$ theory tells us that $\mathcal{L}[L[m,n]]$ decomposes into a direct sum of strings. Recall that a string which starts with an eigenvector of $D_{\bar{H}}$ of eigenvalue $-k < 0$ will hit all the eigenspaces corresponding to the eigenvalues

$$-k\ ,\ -k+2\ ,\ \ldots\ ,\ k-2\ ,\ k\ \ .$$

Note that, in view of 3.15, the eigenspace of $D_{\bar{H}}$ spanned by the row of rank r of $L[m,n]$ corresponds to the eigenvalue $-h = 2r - nm$. Thus if $r < nm/2$ and a string hits the linear span of this row, then it will necessarily hit the span of the complementary rank row. That is the one of rank $r' = nm - r$ (the latter corresponds to the eigenvalue $h = nm - 2r$). We derive from this that the operator $D_{\bar{E}}^{h}$ yields a non singular map between the span of the row of rank $r = (nm-h)/2$ and that of rank r'. In fact, $D_{\bar{E}}^{h}$ simply maps an element of eigenvalue $-h$ of a string (which starts at eigenvalue $-k \leq -h$) into its element of eigenvalue h. The injectivity of $D_{\bar{E}}^{h}$ is thus clear and the surjectivity follows immediately from the fact that rows of complementary ranks have the same number of elements.

Going back to our monomial basis, we see that if the rows of rank r, $r+1$ and r' have respectively d_r, d_{r+1} and $d_{r'} = d_r$ elements, then the action of $D_{\bar{E}}$ and $D_{\bar{E}}^{h-1}$ respectively restricted to the spans these rows may be expressed (in terms of the corresponding monomial bases) respectively as a $d_r \times d_{r+1}$ and a $d_{r+1} \times d_r$ matrices A and B. Since the product of these two matrices is a non singular $d_r \times d_r$ matrix, the Cauchy-Binet theorem guarantees the existence of a $d_r \times d_r$ submatrix A' of A with non zero determinant. Clearly, each non-vanishing determinantal term of A' provides us with an injective map ϕ of the row of rank r of $L[m,n]$ into the row of rank $r+1$. However, in view of the nature of the action of $D_{\bar{E}}$, as expressed by 3.17, we can immediately conclude that such a map will necessarily consist of pairs $\pi \to \phi(\pi)$ giving edges of the Hasse diagram of $L[m,n]$.

This proves the existence of the desired bijections and the proof of spernerity of $L[m,n]$ can thus be completed by the argument outlined in Remark 3.1