The Egge-Loehr-Warrington Result

(a manipulatorial proof)

Preliminaries

Let us recall that an integral vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is called a "*composition of* n" and write "$\alpha \models n$", if and only if $\alpha_i > 0$ for $1 \leq i \leq k$ and $\sum_{i=1}^n \alpha_i = n$. The number k is called the "*length of* α" and is denoted "$l(\alpha)$". In these notes it will convenient to call an integral vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ "*a weak composition of* n" and write "$\gamma \| \models n$" if $\gamma_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^n \gamma_i = n$.

For any weak composition $\gamma \| \models n$ set

$$s_\gamma[X] \;=\; \frac{\det \| x_i^{\gamma_j + n - j} \|_{i,j=1}^n}{\det \| x_i^{n-j} \|_{i,j=1}^n} \qquad \text{P.1}$$

In these notes, it will be convenient to keep n fixed so as to avoid appending it to many of our symbols. In particular, to save space we will use "X" to denote the alphabet $x_1, x_2, \ldots, x_n$ and use the symbol $\Delta[X]$ to denote the Vandermonde determinant

$$\Delta[X] \;=\; \det \| x_i^{n-j} \|_{i,j=1}^n$$

In a recent paper Egge-Loehr-Warrington proved the following basic result

Theorem P.1

If a symmetric function $f[X]$ has the expansion

$$f[X] \;=\; \sum_{\alpha \models n} a_\alpha F_\alpha[X] \qquad \text{P.2}$$

in terms of the Gessel fundamental basis of quasi-symmetric functions then its Schur expansion is simply obtained by the replacement

$$F_\alpha[X] \;\rightarrow\; s_{\bar{\alpha}}[X] \qquad \text{P.3}$$

where $\bar{\alpha}$ is the weak composition obtained by adding $n - l(\alpha)$ zeros to α

It develops that this result can be quickly derived from of the following

Basic Fact

For a given n, let $\mathcal{A}_n$ denote the polynomial operator

$$\mathcal{A}_n \;=\; \sum_{\sigma \in S_n} sign(\sigma)\sigma \qquad \text{P.4}$$

then every symmetric function $f(x_1, x_2, \ldots, x_n)$ satisfies the following identity

$$f(x_1, x_2, \ldots, x_n) \;=\; \frac{1}{\Delta[X]}\, \mathcal{A}\, f(x_1, x_2, \ldots, x_n)\, x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n} \qquad \text{P.5}$$

This is an immediate consequence of the determinantal expansion

$$\Delta[X] \;=\; \sum_{\sigma \in S_n} sign(\sigma) x_{\sigma_1}^{n-1} x_{\sigma_2}^{n-2} \cdots x_{\sigma_n}^{n-n}$$

and the fact that for any symmetric function $f[X]$ we have

$$f(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) \;=\; f(x_1, x_2, \ldots, x_n) \qquad\qquad (\text{for all } \sigma \in S_n)$$

This given, to prove Theorem P.1 we need only derive the identity

$$\tfrac{1}{\Delta[X]}\, \mathcal{A}\, F_\alpha[X]\, x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n} \;=\; s_{\bar\alpha}[X] \qquad \text{P.6}$$

To this end, let $S(\alpha) = \{1 \le i_1 < i_2 < \cdots < i_{k-1} \le n-1\}$ the subset of $[1, n-1]$ corresponding to the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, and recall that by definition we have

$$F_\alpha[X] \;=\; \sum_{\substack{1 \le a_1 \le a_2 \le \cdots a_i \le a_{i+1} \le \cdots \le a_n \le n \\ i \in S(a) \rightarrow a_i < a_{i+1}}} x_{a_1} x_{a_2} \cdots x_{a_n} \qquad \text{P.7}$$

Now P.6 is simply due to the fact that the terms in the sum

$$\sum_{\substack{1 \le a_1 \le a_2 \le \cdots a_i \le a_{i+1} \le \cdots \le a_n \le n \\ i \in S(a) \rightarrow a_i < a_{i+1}}} \tfrac{1}{\Delta[X]}\; \mathcal{A}\, x_{a_1} x_{a_2} \cdots x_{a_n} x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n} \qquad \text{P.8}$$

cancel each other in pairs leaving out a variety of vanishing terms together with the single term

$$\tfrac{1}{\Delta[X]}\; \mathcal{A}\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n} \qquad \text{P.9}$$

and P.6 follows, since by definition

$$s_{\bar\alpha}[X] \;=\; \tfrac{1}{\Delta[X]} \mathcal{A}\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} x_1^{n-1} x_2^{n-2} \cdots x_n^{n-n} \qquad \text{P.10}$$

This is essentially the basic discovery of Egge, Loehr and Warrington.

For sake of completeness we will prove it in the next section.

1. The Proof

If $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \models n$ then

$$\begin{aligned} \mathcal{A}\, x_1^{\gamma_1+n-1} x_2^{\gamma_2+n-2} \cdots x_i^{\gamma_i+n-i} x_{i+1}^{\gamma_{i+1}+n-i-1} \cdots x_n^{\gamma_n+n-n} \;&= \\ =\; -\,\mathcal{A}\, x_1^{\gamma_1+n-1} x_2^{\gamma_2+n-2} \cdots x_i^{\gamma_{i+1}-1+n-i} x_{i+1}^{\gamma_i+1+n-i-1} \cdots x_n^{\gamma_n+n-n} & \end{aligned} \qquad 1.1$$

This may be rewritten as

$$s_{\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \cdots, \gamma_n}[X] \;=\; -\, s_{\gamma_1, \ldots, \gamma_{i+1}-1, \gamma_i+1, \cdots, \gamma_n}[X] \qquad 1.2$$

Note that any weakly decreasing sequence $a = (a_1 \le a_2 \le \cdots \le a_n)$ can be converted into a weak composition $\gamma(a)$ via the identity

$$x_{a_1} x_{a_2} \cdots x_{a_n} \;=\; x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n} \qquad 1.3$$

Clearly, the map $a \rightarrow \gamma(a)$ is invertible and we will denote by $a(\gamma)$ the weakly decreasing sequence a that yields 2.13. We are thus led to focus on the pairing of weakly decreasing sequences obtained by writing

$$a(\gamma_1, \ldots \gamma_i, \gamma_{i+1}, \cdots, \gamma_n) \iff a(\gamma_1, \ldots \gamma_{i+1}-1, \gamma_i+1, , \cdots, \gamma_n) \qquad 1.4$$

Now we see that from left to right we need to replace the string of i in a by the number $i+1$ minus one and the number of $i+1$ by the number of i plus one. From right to left we do the reverse.

Going back to our problem note that if for some $\underline{\text{non} - \text{vanishing}}$ summand in P.7 we have

$$x_{a_1}x_{a_2}\cdots x_{a_n} \;\;\neq\;\; x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$$

there will be a largest $0 \leq s < k$ such that

$$x_{a_1}x_{a_2}\cdots x_{a_n} \;\;=\;\; x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^{\gamma_{s+1}}x_{s+2}^{\gamma_{s+2}}\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}$$

with $\gamma_{s+1}+\gamma_{s+2}+\cdots+\gamma_{s+r} = \alpha_{s+1}$ and $\gamma_{s+r} > 0$. Now the maximality of s forces two possible cases:

(1) $\gamma_{s+1} = 0$ and there will be a first $1 < i < r$ such that $\gamma_{s+i} > 0$, or

(2) $\gamma_{s+1} > 0$ and $r > 1$,

In case (1) we pair off the term

$$\tfrac{1}{\Delta[X]}\mathcal{A}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^0\cdots x_{s+i-1}^0x_{s+i}^{\gamma_{s+i}}\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n} \qquad 1.5$$

with the term

$$\tfrac{1}{\Delta[X]}\mathcal{A}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^0\cdots x_{s+i-1}^{\gamma_{s+i}-1}x_{s+i}^1\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$$

note that we must have $\gamma_{s+1} > 1$ for otherwise our term would vanish.

In case (2) we need to consider whether

(a) $\gamma_{s+2} > 0$ or

(b) $\gamma_{s+2} = 0$

in case (a) we pair off the term

$$\tfrac{1}{\Delta[X]}\mathcal{A}\,x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^{\gamma_{s+1}}x_{s+2}^{\gamma_{s+2}}\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$$

with the term

$$\tfrac{1}{\Delta[X]}\mathcal{A}\,x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^{\gamma_{s+2}-1}x_{s+2}^{\gamma_{s+1}+1}\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$$

In case (b) we look for the first $\gamma_{s+i} > 0$ and note that we must have such an $i \leq r$ since $\gamma_{s+r} > 0$. But then our term is

$$\tfrac{1}{\Delta[X]}\mathcal{A}\,x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^{\gamma_{s+1}}x_{s+2}^0\cdots x_{s+i-1}^0x_{s+i}^{\gamma_{s+i}}\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$$

and we pair it off with

$$\tfrac{1}{\Delta[X]}\mathcal{A}\,x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_s^{\alpha_s}x_{s+1}^{\gamma_{s+1}}x_{s+2}^0\cdots x_{s+i-1}^{\gamma_{s+i}-1}x_{s+i}^1\cdots x_{s+r}^{\gamma_{s+r}}\cdots x_n^{\gamma_n}x_1^{n-1}x_2^{n-2}\cdots x_n^{n-n}$$

here again we must have $\gamma_{s+i} > 1$ for otherwise our term would vanish.

It is easily seen that all the switches we have performed preserve the required ascents of the corresponding summands in P.7, moreover it is not difficult to check that our pairings are none other than the 2-cycles of an involution of the terms of P.8 whose fixed points are the vanishing terms together with the term in P.9.

Since each non vanishing term in P.8 is paired with a term of opposite sign, our proof of P.6 is finally complete.

2. Additional identities.

The polynomial operator

$$P[X] \;\rightarrow\; \tfrac{1}{\Delta[X]}\;\; \mathcal{A}\, P[X] \tag{2.1}$$

may of course be used in a similar manner to obtain a result analogous to Theorem P.1 for any other Quasi-symmetric basis . The following result shows that the Quasi-Monomials are definitely the easiest to deal with, while computer data reveal that the Quasi-Schur may be the hardest.

For two vectors α and β let us denote by "$\alpha \sqcup\!\sqcup \beta$" the collection of all vectors obtained by shuffling the components of α with the components of β (as if α and β were two decks of cards). Let us also write "$\gamma \in \alpha \sqcup\!\sqcup \beta$" to express that γ is an element of the collection $\alpha \sqcup\!\sqcup \beta$.

Now, for any composition $\alpha \models n$, let

$$z(\alpha) \;\;=\;\; [0,0,\ldots,0] \quad (n-l(\alpha) \text{ zeros}) \tag{2.2}$$

and define a new family of symmetric polynomials $\{\Pi[\alpha]\}_{\alpha\models n}$ by setting

$$\pi_\alpha[X] \;\;=\;\; \sum_{\gamma\in\alpha\sqcup\!\sqcup z[\alpha]} s_\gamma[X] \tag{2.3}$$

Theorem 2.1

For any symmetric function $f[X]$ we have

$$f[X] \;\;=\;\; \sum_{\alpha\models n} a_\alpha M_\alpha[X] \;\;=\;\; \sum_{\alpha\models n} a_\alpha \pi_\alpha[X]. \tag{2.4}$$

That is, we can pass from Quasi-Monomial expansions to Schur expansions by the replacement $M_\alpha[X]\rightarrow\pi_\alpha[X]$.

Proof

By definition the monomial symmetric function $m_\mu[X]$ is given by the the sum

$$m_\mu[X] \;\;=\;\; \sum_{\gamma\models\!\models n;\lambda(\gamma)=\mu} x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n} \tag{2.5}$$

Thus applying te operator in 2.1 to both sides of this identity we obtain

$$m_\mu[X] \;\;=\;\; \sum_{\gamma\models\!\models n;\lambda(\gamma)=\mu} s_\gamma[X] \tag{2.6}$$

Now , since each weak composition γ yields a (strong) composition by zero removals we may rewrite the identity in 2.6 in the form

$$m_\mu[X] \;\;=\;\; \sum_{\alpha\models n;\lambda(\alpha)=\mu} \pi_\alpha[X] \tag{2.7}$$

But at the same time the expansion of $m_\mu[X]$ in terms of the Quasi-Monomial basis can also be written as

$$m_\mu[X] \;\;=\;\; \sum_{\alpha\models n;\lambda(\alpha)=\mu} M_\alpha[X] \tag{2.8}$$

Thus 2.4 is true for the monomial basis and must therefore be true for any symmetric function.

This result has the following interesting corollary, which is essentially implicit in the path adopted by Egge, Loehr and Warrington to obtain their result.

Theorem 2.2

$$s_{\bar{\alpha}}[X] \;=\; \sum_{\beta\models n} \pi_\alpha[X]\chi(\beta \preceq \alpha) \tag{2.9}$$

where "$\preceq$" denotes the refinement partial order.

Proof

It is well known that we have

$$F_\alpha[X] \;=\; \sum_{\beta\models n} M_\alpha[X]\chi(\beta \preceq \alpha) \tag{2.10}$$

and 2.9 follows by applying the operator in 2.1 to both sides of this identity.

Egge, Loehr and Warrington introduce two interesting rectatngular matrices K and K' that give a compositional version of the Kotska matrix and its inverse. Using the present notations these matrices and and their properties may be obtained by purely manipulatorial steps.

To this end, note that he classical expansion

$$s_\lambda[X] \;=\; \sum_{\alpha\models n} M_\alpha[X]K_{\lambda,\alpha} \tag{2.11}$$

gives

$$K_{\lambda,\alpha} \;=\; \big\langle s_\lambda \,,\, h_\alpha \big\rangle \tag{2.12}$$

with $h_\alpha = h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_{l(\alpha)}}$. Of course this is due to fact that we have

$$K_{\lambda,\alpha} = K_{\lambda,\mu} \qquad\qquad (\text{when} \quad \lambda(\alpha) = \mu). \tag{2.13}$$

In matrix notation we may write 2.11 as

$$\big\langle s[x]\big\rangle \;=\; \big\langle M[X]\big\rangle K^T \tag{2.14}$$

with

$$K \;=\; \|K_{\lambda,\alpha}\|_{\substack{\lambda\vdash n\\ \alpha\models n}} \tag{2.15}$$

with partitions as well as compositions arranged in lex order.

Next, define for any $\lambda \vdash n$ and any $\alpha \models n$

$$K'_{\lambda,\alpha} \;=\; \big\langle s_\lambda[X] \,,\, \Pi_\alpha[X]\big\rangle \tag{2.18}$$

and set

$$K' \;=\; \|K'_{\lambda,\alpha}\|_{\substack{\lambda\vdash n\\ \alpha\models n}} \tag{2.19}$$

Below we display the two matrices K' and K^T for $n = 4$.

$$Kp4 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix} \qquad KT4 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix}$$

By a slight abuse of notation let us denote by $K'_{\lambda,\mu}$ the entries of the inverse of the transpose of the classical Kostka matrix and note that we have

$$K'_{\lambda,\mu} \;=\; \big\langle m_\mu[X]\,,\, s_\lambda[X]\big\rangle \;=\; \sum_{\lambda(\alpha)=\mu} K'_{\lambda,\alpha} \tag{2.20}$$

Using this and 2.13 we derive that

$$\sum_{\alpha\models n} K'_{\lambda,\alpha}K^T_{\alpha,\nu} \;=\; \sum_{\mu\vdash n}\Big(\sum_{\lambda(\alpha)=\mu} K'_{\lambda,\alpha}\Big)K^T_{\nu,\mu} \;=\; \sum_{\mu\vdash n} K'_{\lambda,\mu}K^T_{\nu,\mu} \;=\; \chi(\lambda=\nu)$$

In summary we have

$$K'K^T \;=\; I \tag{2.21}$$

where I is the $p_n \times p_n$ identity matrix.

The combinatorial significance of 2.11 is the starting point of Egge, Loehr and Warrington derivation of in their result.