DISCRETE MATHEMATICS

In the last fifteen years we have witnessed an explosion of results and information in a field which is underscriptively referred to as *Discrete Mathematics*. Its beauty, achievements and usefulness in today's world of information processing is unquestioned. Courses in Discrete Mathematics, specialized to the needs of CS students, are now a requirement for all CS majors throughout the country.

However, the subject in its full generality has a unique capability of transmitting deep mathematical concepts in a visual and constructive manner. It promotes inductive thinking and encourages creative experimentation. Moreover, it has a mathematically *formative* character which rivals that of calculus as a discipline that trains into rigorous rational thinking. Yet it can do that in an enjoyable and encouraging manner. In summmary, unlike calculus it does not require from the student a monumental amount of acquiescence to delayed gratification. This given we believe that the time has come to offer basic courses in Discrete Mathematics at an early undergraduate level to the general student population. These courses should also be of benefit to the students who choosing not to take Calculus may end up never to acquire or enjoy any University level Mathematics.

We shall give a brief description of the goals and contents of a Discrete Mathematics course as offered by a Mathematics department to provide enlightment and service to the general undergraduate community. We shall begin with a description of the subject, and the aspects that distinguish it from the traditional service courses such as Calculus, Statistics, Numerical Analysis, etc. We shall also include at the end some representative samples of the material and the manner in which it is presented.

Crudely speaking Discrete Mathematics only deals with "discrete" (meaning finite) objects and their relationships. These relationships are encoded in the visual appearance of the objects under study and ultimately translate themselves into mathematical facts and identities. From this point of view Discrete Mathematics provides an imagery containing and extending in an an extremely rich manner that which was provided, in classical times, by Synthetic Geometry. The advent of Symbolic Manipulation Software, combined the present developments in Discrete Mathematics provides new avenues in the teaching of mathematics. Experience shows that even beginners, such as undergraduates exposed to a first introductory course, can quickly encounter the joy of discovery by experimentation This is further magnified if the experimentation is carried out on the computer with software such as MATHEMATICA or MAPLE presently available in our undergraduate labs. This possibility may not be as readily implemented in traditional service courses. In summary our new courses will present a unique opportunity to place UCSD in a leading position in the introduction of innovative undergraduate teaching.

The topics covered in Ma15 a,b may be divided into 4 general Areas

I *Formal Languages and Combinatorial Structures.*

Binomial coefficients, Permutation enumeration, trees, heaps, lattice paths, Lagrange inversion and formal power series.

II *Ordinary and Exponential Generating Functions*
Partition enumeration, recurrence relations, combinatorial species and its mathematical applications. Enumeration and construction of some discrete molecular structures.

III *Graph Theory*
Spanning trees, chromatic polynomials, combinatorial linear algebra. Network flows. The max-flow min-cut theorem and network flow problems and algorithms.

IV *Codes and Information*
Hoffman codes, comma free codes, error correcting codes, sorting and searching.

The emphasis in this course is on concrete examples from combinatorics and computer science rather than on general theory. Prerequisites: High School Algebra only.

TREES AND THE COEFFICIENTS IN A POWER SERIES EXPANSION IN A FRESHMAN DISCRETE MATHEMATICS COURSE

The genesis of the present day developments in Discrete Mathematics can be ultimately traced to the needs and problems of information processing and storage. Thus the easiest examples arise in this context. Presently, our means of communication with the electronic world has a strictly *1-dimensional form.* This is the case whether it be *words* typed on a keyboard, *sounds* through a microphone or sequences of $0, 1$-s produced by a scanner as it "reads" a page of printed material. So the first problem we encounter is the conversion of higher dimensional input into one dimensional form. Have a look at the picture below.

We may store a lengthy verbal description of it in a computer as follows:

It is the geneological tree of a cell belonging to a family of cells which either die or split into two cells one of type "A" and one of type "B". The great grand father has two children which we call a and b. Cell a splits into aa and ab, aa dies while ab splits into aba and abb which both die. Cell b splits into ba and bb and both die. If we draw A-cells to the left and B cells to the right, succesive offsprings in successive rows then finally connect each parent cell to its offsprings we get the tree like structure we are looking at.

This may be adequate if the information is only stored for human retrieval and consumption. However, matters change considerably, if we need the computer to *process* such information or even just *store* it in large quantities. The needs of *versatility* and *efficiency* do necessarily play an essential role. A manner which has become standard in recent literature is to represent these objects (which are refererd to nowdays as *Planar Complete Binary Trees* or *PCB trees* for brief) by *words* in two letters. S and D. These words are obtained as follows. We first label all cells that split by an S and those that die by a D as in the figure below.

Then read the labels by traveling along the edges of the tree *always going down a left edge before going down a right edge.* This manner of reading yields the word

$$SSDSDDSDD$$

This not only compresses the long winded description above into a single word, but as we shall see it enables us to discover some mathematics... using trees... as a guide! To this end, note that if we denote our tree by T and the trees that hang from the offsprings of the great-grandfather by T_a and T_b and denote their corresponding words by $w(T)$, $w(T_a)$ and $w(T_b)$ we see that

$$w(T) \ = \ S\, w(T_a)\, w(T_b)\ . \tag{1}$$

Now let us do something really wild. Let us denote by $\mathcal{F}$ the *sum* of all the words that correspond in this manner to complete planar binary trees. That is

$$\begin{aligned} \mathcal{F} \ &= \ w(\circ) \ + \ w(\qquad) \ + \ w(\qquad) \ + \ w(\qquad) \ + \\ &\qquad + \ w(\qquad) \ + \ w(\qquad) \ + \ w(\qquad) \ + \ w(\qquad) \ + \ w(\qquad) \ + \ \cdots \\ &= \ D \ + \ SDD \ + \ SSDDD \ + \ SDSDD \ + \\ &\qquad + \ SSSDDDD \ + \ SSDSDD \ + \ SSDDSDD \ + \ SDSSDDD \ + \ SDSDSDD \ + \ \cdots \end{aligned}$$

A moment's reflection should suggest that this *sum* must satisfy the equation

$$\mathcal{F} \ = \ D \ + \ S\,\mathcal{F}\,\mathcal{F} \tag{2}$$

This is because either the initial cell *dies* yielding a tree with a single vertex which is then labelled D (this gives the first term on the right hand side of (2) or it splits as in the example above. The second term in (2) is then obtained by summing over all possible choices of T_a and T_b in equation (1). Let us now do another wild thing. In these words replace all D's by a 1 and all S's by an x. We shall then obtain a series (an infinite formal sum) of the form

$$f(x) \;=\; 1 \;+\; x \;+\; 2\,x^2 \;+\; 5\,x^3 \cdots \;+\; c_k\,x^k \;+\; \cdots \tag{3}$$

where c_k is equal to the number of trees with k *nodes* carrying an S label. Note further that from (2) we derive that $f(x)$ must also satisfy the equation

$$f(x) \;=\; 1 \;+\; x\,(f(x))^2 \;. \tag{4}$$

Recalling the (high school algebra) formula for the solution of the quadratic equation we immediately derive from this that

$$f(x) \;=\; \frac{1 \;-\; \sqrt{1 \;+\; 4x}}{2\,x} \tag{5}$$

Now, one of the very first results derived in a good Discrete Math. course is that

$$c_k \;=\; \frac{1}{2k+1}\binom{2k+1}{k} \;=\; \frac{(k+2)(k+3)(k+4)\cdots(k+k)}{2\,3\,4\cdots k} \;. \tag{6}$$

Combining all this together we get what is now referred to as a *combinatorial* proof of the *power series expansion*

$$\frac{1 \;-\; \sqrt{1 \;+\; 4x}}{2\,x} \;=\; \sum_{k\geq 0}\frac{1}{2k+1}\binom{2k+1}{k}\;x^k \;. \tag{7}$$

The point of all these considerations is to illustrate one of the aspects of the present day developments of the field of Discrete Mathematics:

The conversion or encoding of discrete structures into words allows us to translate properties of these structure into equations involving "languages" (formal sums) of the corresponding words. Finally, by specializations of the letters used in the words, these equations then yield a variety of surprising formulas and identities. This process, has been shown to produce new identities as well as unifying large families of identities which heretofore demanded separate verification

This methodology is not limited to such simple applications. Nor is it limited to results in the so called *Theory of Special Functions.* The combinatorial approach to a mathematical problem consists in giving a discrete setting to the problem. It then turns out that this provides a bridge for carrying information and identities from one area of mathematics to another. Rather deep results and applications have thus emerged in such disparate areas as Algebra, Representation Theory, Probability Theory and Statistics, Numerical analysis, the Analysis of Data structures, etc. etc. and the list is growing daily.

Let us pursue our example a bit further. Another structure we meet in Discrete Mathematics is the *Lattice path* these are paths which travel along straight edges joining points with integer coordinates. Now let us take the letters of our tree word $SSDSDDSDD$ an replace each in succession by *lattice path edges*. More precisely we replace each S by a NORTH-EAST step of (like the one that joins $(0,0)$ with $(1,1)$) and each D by a SOUTH-EAST step (like the one that joins $(0,0)$ with $(1,-1)$). This produces the lattice path below.

If we now carry this out again with other Planar Complete Binary trees we quickly discover that the path obtained always remains above the $x - axis$ until the last edge which necessarly dips to a point of y-coordinate -1! Lattice paths which start at the origin and end on the x-axis always remaining weakly above it, are usually referred to as DYCK *paths*. Our construction of a path from a word of a PCB tree can be easily shown to yield a bijection between trees with $2n+1$ nodes and DYCK paths with $2n$ edges showing that these two families of objects are equinumerous. However this bijection has a further surprising byproduct. It takes the mystery out of formula (6). In fact, the binomial coefficient $\binom{2k+1}{k}$ simply counts the number of words with k S's and $k+1$ D's. This given, to prove (6) we need only observe that each of the latter words has $2k+1$ distinct circular rearrangenents only one of which is the word of a PCB tree.

It may be good at this point to focus a bit on the various features exhibited by our example. Note that the factorization of tree words given in (1) stems right out of the geometry of PCB trees. This shows that the *language* of PCB trees satisfies the equation in (2), which can be easily translated into an algorithm for constructing all PCB tree words. In particular (2) yields (4) which ultimately can be translated into (5). On the other hand (6) does come into evidence only after we study the *lattice paths* corresponding to PCB tree words. These paths reveal the nature of the restrictions that make a word with n S's and $n+1$ D's into a tree word. So it is only out of the interplay between Trees, Words and Lattice Paths that the full picture emerges. Moreover, we can see that the basic role that these various discrete objects play in our discovery is to provide us with a visualization of their mathematical relationships. This phenomenon is not unique to this example. For instance the whole mathematical theory of continued fractions and their relationship to classical orthogonal polynomials and associated moment problems can be derived from a few simple equations associated with the language of Planar Incomplete Binary trees. The latter are trees where a node is also allowed to have a single a-child or a single b-child.

These developments permit to present and teach with the greatest of ease at a very elementary undergraduate level material that is only encountered in upper division or even graduate courses. To cite another example: The Cayley-Hamilton theorem to the effect that every matrix satisfies its own characteristic polynomial. Surprisingly, a good percentage of our graduate students have never seen a proof of it and some do not even know the statement. Yet the theorem has a very elementary purely combinatorial visualization that makes its validity entirely obvious. In fact, the combinatorial argument shows that it is valid even when the elements of the given matrix are just letters of an alphabet!

We might point out that the theory of series comes in later Calculus courses (most commonly the 4^{th}). It is a pain to teach and the students hate it. The reason is simple: the Cauchy point of view (that *all series diverge unless they are proved otherwise*) kills all attempts at experimentation and forces the instruction to concentrate on tools for testing and proving *convergence*. The latter are fundamentally abstruse and totally useless. For, in practice, all series we encounter and use do in fact converge! However, the secret (already known to Euler and Gauss) for turning the subject into a useful and enjoyable learning experience , is not to demand that the series terms be numbers, rather they should be *objects*, say *words* in an alphabet. Convergence then simply means that any particular word appears only a finite number of times in the sum. All operations on series that make the coefficient of any object computable in a finite number of steps are permitted. This viewpoint opens up an extremely rich variety of formal sums gravid of combinatorial and mathematical content. Teaching the theory of series in this manner becomes pure pleasure, especially when the students, liberated from the stifling Cauchy principle, discover non trivial mathematical identities by experimentation. The actual Taylor expansions of classical special functions, as exponentials, logarithms, the trigonometric functions, Bessel functions, and even more general hypergeometric functions, become trivial specializations of languages of words of appropriate families of lattice paths. Their coefficients readily computable by counting the corresponding objects.

To help advisors and students in the evaluation of our course material we are planning to prepare and distribute sample video tapes covering some of the classroom instruction.