CHAPTER 9

COMBINATORIAL SPECIES

1. Definitions

The objects we are dealing with here may be visualized as *play cards* displaying an image containing a number of *cells* (usually represented by circles) labeled by the integers $1, 2, \ldots, n$. These cards can be used to represent various combinatorial objects such as *permutations, cycles, trees, graphs, arborescences, simplices,etc....* A few examples will get accross what we have in mind. Recall that a *derangement* is a permutation without fixed points, in other words, a derangement is a permutation all of whose cycles have lengths greater than one. Thus a card representing the derangent $\alpha = (1, 9, 13, 11, 8, 5)(2, 12, 7, 4)(3, 6, 10)$ may de depicted as

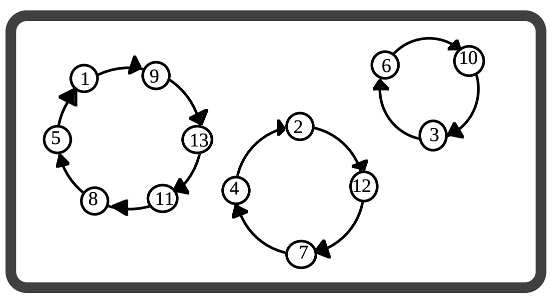
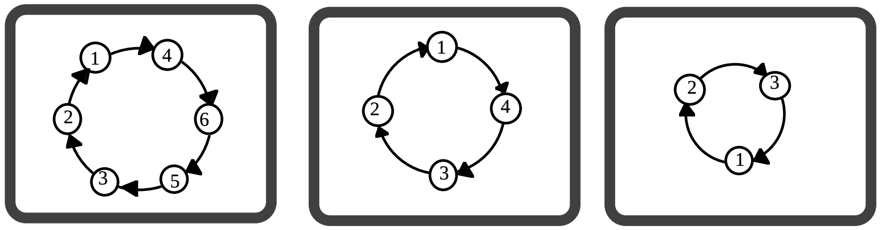


Fig 1.1

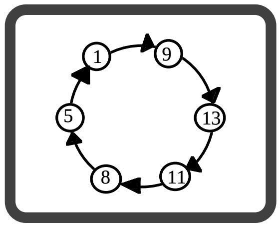
This given, a collection of combinatorial objects may be represented by a *deck of cards*. This imagery may also be used to view a complex object, such as the derangement α displayed above, as obtained by assembling it out of simpler objects: namely its cycles. More precisely, we may start with the 3 cards C_1, C_2, C_3 given below



then choose a set partition Π of $\{1, 2, \ldots, 13\}$, in this case we want

$$\Pi = (\ \{1, 5, 8, 9, 11, 13\}\ ,\ \{2, 4, 7, 12\}\ ,\ \{3, 6, 10\}\ )$$

and finally *substitute* the 3 parts of Π respectively into C_1, C_2, C_3, where in this context, the substitution of a *set* $S = \{i_1 < i_2 < \ldots < i_k\}$ into a *card* C with k cells, simply means replacing the label r by the integer i_r. Thus, if we call $C(S)$ the resulting card, then substituting $\{1, 5, 8, 9, 11, 13\}$ into the cycle card C_1 given above results in

$C_1(\{1, 5, 8, 9, 11, 13\}) =$ 

In this vein we see that the card illustrating the derangement α given above is simply obtained by glueing together the three cards

$$C_1(\{1, 5, 8, 9, 11, 13\}) \quad , \qquad C_2(\{2, 4, 7, 12\}) \quad , \qquad C_3(\{3, 6, 10\})$$

Such constructions were used for a number of years in the 70's to illustrate "*Exponential Structures*". To be brief, these structures were built in a manner that reflected formal substitution into the exponential series. This idea was considerably enriched by André Joyal in 1980 by extending the constructions to reflect formal substitution into arbitrary formal series, thereby creating what is now referred to as the "*The theory of Combinatorial Species*".

In the present writing, a *Combinatorial Species* is simply a "*Deck of cards*" $\mathcal{A}$ representing a finite or infinite family of combinatorial objects. Since, as we have required, each of these cards contains a number of cells, we shall denote by "$\mathcal{A}\,\big|_n$" or simply "$\mathcal{A}_n$" the subcollection consisting of the cards of $\mathcal{A}$ which contain precisely n cells. However, a basic assumption is made that each of the subcollections $\mathcal{A}_n$ contains only a finite number of objects. This given, to each species $\mathcal{A}$ we can associate a formal power series $A(x)$ whose coefficients yield the cardinalities of the subcollections $\mathcal{A}_n$. More precisely we set

$$GF(\mathcal{A}) \;=\; A(x) \;=\; \sum_{n\geq 0} \big(\#\,\mathcal{A}_n\big)\;\frac{x^n}{n!}$$

and refer to it as the *Exponential Generating Function of the species* $\mathcal{A}$. For instance, if $\mathcal{P}$ is the species consisting of all permutations, then $\mathcal{P}_n$ consists of all the permutations of $\{1, 2, \ldots, n\}$ and we shall then set

$$GF(\mathcal{P}) = P(x) = \sum_{n\geq 0} \; x^n \;=\; \frac{1}{1-x} \;.$$

On the other hand, if the species $\mathcal{C}$ consists of all cycles, then $\#\mathcal{C}_n = (n-1)!$ an we will set

$$GF(\mathcal{C}) = C(x) = \sum_{n\geq 1} \;\frac{x^n}{n} \;=\; log\Big(\frac{1}{1-x}\Big) \;.$$

The fundamental idea of Joyal consisted in extending to Combinatorial Species a number of the basic operations we may perform on general formal power series. To be specific, he succeeds in defining "*Addition*", "*Multiplication* ", "*Formal Substitution*" and "*Differentiation* " for Species in such a manner that each of these operations results in the analogous operation being carried out on the corresponding exponential generating functions. The important by-product of this construction is that the enumeration of more complex combinatorial structures may be thus reduced to the enumeration of simpler ones.

In some instances it will be good to assume that the species $\mathcal{A}$ we work with has either one card with no cells or none at all, in that case we shall express this condition by writing $\#\mathcal{A}_0 = 1$ or $\mathcal{A}_0 = \emptyset$ as the case may be. This given, for two species $\mathcal{A}$ and $\mathcal{B}$ the Joyal's constructions are as follows:

1) *Addition*: The species $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is simply the set theoretical disjoint union of $\mathcal{A}$ and $\mathcal{B}$.

2) *Multiplication*: In this case $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is defined by setting $\mathcal{C} = \sum_{n\geq 0} \mathcal{C}_n$ with

$$C_n \;=\; \sum_{h+k=n} \mathcal{A}_h \times \mathcal{B}_k \qquad 1.1$$

where $\mathcal{A}_h \times \mathcal{B}_k$ is the collection of cards c obtained by taking a card $a \in \mathcal{A}_h$, a card $b \in \mathcal{B}$ and two subsets S, T decomposing the set $\{1, 2, \ldots, n\}$ with $|S| = h$ and $|T| = k$ then letting c be the card obtained by glueing together $a(S)$ and $b(T)$. In other words we set

$$\mathcal{A}_h \times \mathcal{B}_k \;=\; \sum_{a\in\mathcal{A}_h} \sum_{b\in\mathcal{B}_h} \sum_{\substack{S+T=\{1,2,\ldots,n\} \\ |S|=h\ ,\ |T|=k}} \boxed{a(S)\ b(T)}$$

Notice that this definition entails that

$$\frac{\#(\mathcal{A}_h \times \mathcal{B}_k)}{n!} \;=\; \frac{1}{n!}\binom{n}{h}\; \#(\mathcal{A}_h)\; \#(\mathcal{B}_k) \;=\; \frac{\#(\mathcal{A}_h)}{h!}\; \frac{\#(\mathcal{B}_k)}{k!}\ . \qquad 1.2$$

3) *Formal Substitution*: If $\mathcal{B}_0 = \emptyset$ we can construct each card of the species $\mathcal{A}(\mathcal{B})$ by the following 4-step procedure:

(i) Pick $k \geq 1$ and $a \in \mathcal{A}_k$.

(ii) Pick a partition $\pi = (S_1, S_2, \ldots S_k)$ of $\{1, 2, \ldots, n\}$.

(iii) Pick $b_i \in \mathcal{B}_{|S_i|}$ for $i = 1, 2, \ldots, k$.

(iv) Place $b_i(S_i)$ in the i^{th} cell of a.

3) *Differentiation*: The species $\mathcal{C} = \mathcal{A}'$ is simply obtained by erasing the highest labelled cell from each card of $\mathcal{A}$. Note that this gives

$$\#\mathcal{A}'_n \;=\; \#\mathcal{A}_{n+1} \qquad 1.3$$

These definitions assure the following basic identities:

Theorem 1.1

$$\begin{array}{lrcl}
1) & GF(\mathcal{A}+\mathcal{B}) & = & GF(\mathcal{A}) \;+\; GF(\mathcal{B}) \\
2) & GF(\mathcal{A}\times\mathcal{B}) & = & GF(\mathcal{A}) \;\times\; GF(\mathcal{B}) \\
3) & GF(\mathcal{A}(\mathcal{B})) & = & GF(\mathcal{A})\big(\,GF(\mathcal{B})\,\big) \\
4) & GF(\mathcal{A}') & = & GF(\mathcal{A})'
\end{array} \qquad 1.4$$

Proof

The identity in 1.4 1) is immediate. For 1.4 2) we note that from 1.2 we deduce that

$$\frac{1}{n!}\,\#\big(\mathcal{A}\times\mathcal{B}\,\big|_n\big) \;=\; \sum_{h+k=n} .\frac{\#\mathcal{A}_h}{h!}\;\frac{\#\mathcal{B}_k}{k!}$$

In other words we have

$$GF(\mathcal{A}\times\mathcal{B})\,\Big|_{x^n} \;=\; GF(\mathcal{A})\times GF(\mathcal{B})\,\Big|_{x^n}$$

as desired.

For 1.4 3) we recall that the number of partitions of the set $\{1, 2, \ldots, n\}$ with p_i parts of size i is given by the formula

$$\frac{n!}{(1!)^{p_1}(2!)^{p_2}\cdots(n!)^{p_n}\; p_1!p_2!\cdots p_n!}$$

Thus from our 4-step construction of $\mathcal{A}(\mathcal{B})\big|_n$ we derive that

$$\#\big(\, \mathcal{A}(\mathcal{B})\big|_n\,\big) \;=\; \sum_{k=1}^{n} \alpha_k \sum_{\substack{p_1+p_2+\cdots+p_n=k \\ p_1+2p_2+\cdots+np_n=n}} \frac{n!\;\beta_1^{p_1}\beta_2^{p_2}\cdots\beta_n^{p_n}}{(1!)^{p_1}(2!)^{p_2}\cdots(n!)^{p_n}\;p_1!p_2!\cdots p_n!} \qquad 1.5$$

where for convenience we have set

$$\#\mathcal{A}_k \;=\; \alpha_k \qquad \text{and} \qquad \#CB_i=\beta_i\ . \qquad 1.6$$

Now 1.5 may also be rewritten as

$$\frac{1}{n!}\,\#\big(\, \mathcal{A}(\mathcal{B})\big|_n\,\big) \;=\; \sum_{k=1}^{n} \frac{\alpha_k}{k!} \sum_{p_1+p_2+\cdots+p_n=k} \frac{k!}{p_1!p_2!\cdots p_n!}\; \Big(\frac{x\beta_1}{1!}\Big)^{p_1}\Big(\frac{x^2\beta_2}{2!}\Big)^{p_2}\cdots\Big(\frac{x^n\beta_n}{n!}\Big)^{p_n}\;\Big|_{x^n}$$

and the multinomial theorem gives that

$$\frac{1}{n!}\,\#\big(\, \mathcal{A}(\mathcal{B})\big|_n\,\big) \;=\; \sum_{k=1}^{n} \frac{\alpha_k}{k!}\bigg(\Big(\frac{x\beta_1}{1!}\Big) \;+\; \Big(\frac{x^2\beta_2}{2!}\Big) \;+\;\cdots\;+\; \Big(\frac{x^n\beta_n}{n!}\Big)\bigg)^k\;\Big|_{x^n}\ .$$

Since adding terms containing x to a power greater than n do not affect the operation of taking the coefficient of x^n we may also write

$$\frac{1}{n!}\,\#\big(\, \mathcal{A}(\mathcal{B})\big|_n\,\big) \;=\; \sum_{k=1}^{\infty} \frac{\alpha_k}{k!}\bigg(\frac{x\beta_1}{1!} \;+\; \frac{x^2\beta_2}{2!} \;+\;\cdots\bigg)^k\;\Big|_{x^n}\ . \qquad 1.7$$

Now 1.6 gives

$$GF(\mathcal{A})=\sum_{k=0}^{\infty}\ \alpha_k\ \frac{x^k}{k!} \qquad \text{and} \qquad GF(\mathcal{B})=\sum_{k=1}^{\infty}\ \beta_n\ \frac{x^n}{n!} \quad (\text{since}\quad \mathcal{B}_0=\emptyset\ )\ .$$

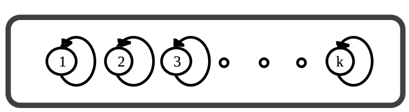
Thus 1.7 is simply

$$GF(\mathcal{A}(\mathcal{B}))\Big|_{x^n} \;=\; \sum_{n=0}^{\infty}\frac{\alpha_k}{k!}\ \Big(GF(\mathcal{B})\Big)^k\ \Big|_{x^n} \;=\; GF(\mathcal{A})\big(GF(\mathcal{B})\big)\Big|_{x^n}\ ,$$

and this gives 1.4 3).

2. Examples

There are some rather striking immediate applications of the theory yielding, with minimum effort, a number of interesting identities. To begin we need to introduce some notation. Here and after let $\mathcal{P}$ denote the species of all permutations, $\mathcal{D}$ the species of all derangements and $\mathcal{S}$ be the special species where $\mathcal{S}_k$ consists of only one card with a row of k cells labelled $1, 2, \ldots, k$ from left to right. It will be convenienient, in this section to visualize this card as depicted below



$Fig\ 2.1$

with $\mathcal{S}_0$ containing only the empty card. This given we have

Theorem 2.1

The species of permutations is the product of the derangement species by the special species, that is we have

$$\mathcal{P} \;=\; \mathcal{D} \times \mathcal{S} \ . \qquad 2.1$$

In particular we deduce that

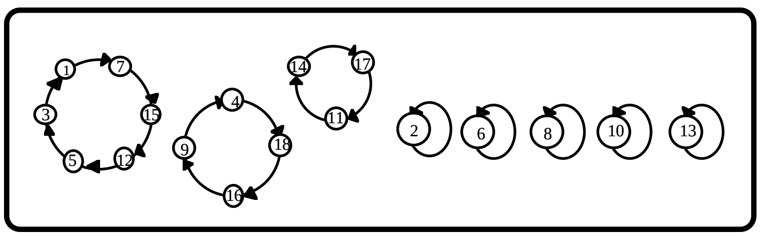
$$GF(\mathcal{D}) \;=\; \frac{e^{-x}}{1-x} \ . \qquad 2.2$$

Thus

$$\#\mathcal{D}_n \;=\; n!\Big(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\Big) \qquad 2.3$$

Proof

We need only one picture to prove 2.1. By definition, to construct a card c in $\mathcal{D} \times \mathcal{S}\,\big|_{18}$ we may pick a card d in $\mathcal{D}_{13}$ and a card s in S_5 then 2 subsets A, B of cardinalities 13 and 5 decomposing $\{1,2,\ldots 18\}$ and take c to be the card obtained by glueing $d(A)$ and $s(B)$. Note that if d is chosen to be the card given in Fig 1.1, s to be the card in Fig 2.1 for $k = 5$ and chose $A = \{1,3,4,5,7,9,11,12,14,15,16,17,18\}$ and $B = \{2,6,8,10,13\}$ this construction results in the card given below



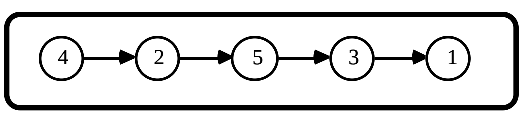
which is seen to depict the permutation $\sigma = (1,7,15,12,53)(4,18,16,9)(11,14,17)(2)(6)(8)(10)(13)$. Since all the permutations can be uniquely constructed in this manner, 2.1 must necessarily follow. Now Theorem 1.1 yields that

$$\frac{1}{1-x} \;=\; GF(\mathcal{P}) \;=\; GF(\mathcal{D}) \times GF(\mathcal{S}) \;=\; GF(\mathcal{D}) \times \sum_{n\geq 0} \frac{x^n}{n!} \;=\; GF(\mathcal{D}) \times e^x$$

and this clearly is another way of writing 2.2. Now 2.3 follows upon equating coefficients of x^n on both sides of 2.2. This completes our proof.

Remark 2.1

Note that if $\mathcal{L}$ is the species of all linear orders, a typical card of which may be depicted as given below



Then again we must have

$$GF(\mathcal{L}) \;=\; \frac{1}{1-x}$$

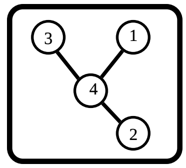
Now the reader should have no difficulty seeing that there is a bijection between the species obtained by removing the highest labelled cell from each card of $\mathcal{L}$ and the cards of the species $\mathcal{L} \times \mathcal{L}$. In other words we must have

$$\mathcal{L}' \;=\; \mathcal{L} \times \mathcal{L}$$

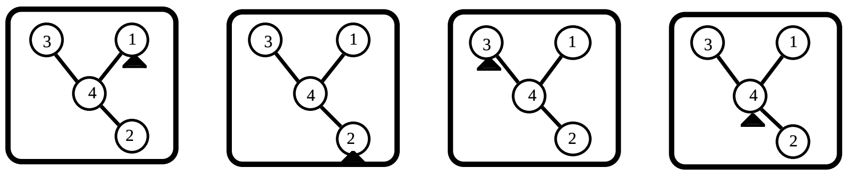
Thus this simple observation results (via 1.4 4)) in a purely combinatorial proof of the identity

$$\frac{d}{dx}\,\frac{1}{1-x} \;=\; \frac{1}{(1-x)^2}$$

Given a card f of a species $\mathcal{F}$, we let $\mathcal{R}f$ denote the collection of cards obtained by placing a pointer on one of the cells of f. For instance if $\mathcal{F}$ is the family of all Cayley trees and f is the card given below



Then $\mathcal{R}f$ is the following collection of cards



We shall refer to these cards as the *rooted* versions of f. This given the *rooted version of a species* $\mathcal{F}$ is simply the species obtained by rooting in all possible manners the cards of $\mathcal{F}$, with the proviso that we discard the cards of $\mathcal{F}_0$. Let us denote the resulting species by $\mathcal{R}\mathcal{F}$. The following property of rooting is immediate:

Proposition 2.1

$$GF(\mathcal{R}\mathcal{F}) \;=\; x\;GF(\mathcal{F}') \qquad 2.5$$

Proof

Since each card $f \in \mathcal{F}_n$ can be rooted in exactly n different ways we must conclude that

$$\#\mathcal{R}\mathcal{F}_n \;=\; n\;\#\mathcal{F}_n$$

This gives that

$$GF(\mathcal{R}\mathcal{F}) \;=\; \sum_{n\geq 1} \frac{\#\mathcal{F}_n}{(n-1)!}\,x^n \;=\; x\sum_{n\geq 1} \frac{\#\mathcal{F}_n}{(n-1)!}\,x^{n-1} \;=\; x\,\frac{d}{dx}\,GF(\mathcal{F})$$

and 2.5 follows from 1.4 4).

An immediate corollary of this result is the following result for rooted Cayley trees.

Theorem 2.2

If $\mathcal{C}$ denotes the collection of Cayley trees then

$$GF(\mathcal{R}\mathcal{C}) \;=\; x\;e^{GF(\mathcal{R}\mathcal{C})} \qquad 2.6$$

Proof

Note that if we remove the highest labelled cell from a Cayley tree the result may be viewed as a forest (which could be empty) of Cayley trees rooted at the cell that was connected to the cell we removed. Now, it is not difficult to see that, the species of forests of Rooted Cayley trees may be simply obtained by composing the special species $\mathcal{S}$ we considered above with the species of Cayley trees. In other words we must have

$$\mathcal{C}' \;=\; \mathcal{S}(\mathcal{RC}) \;.$$

Since $GF(S) = e^x$ we deduce from this that

$$GF(\mathcal{C}') \;=\; e^{GF(\mathcal{RC})} \qquad 2.7$$

Now 2.5 gives that

$$GF(\mathcal{RC}) \;=\; x\;\, GF(\mathcal{C}') \;, \qquad 2.8$$

and 2.6 follows by combining 2.7 and 2.8