The Character of the $\mathbf{SL_n}$ Harmonics

Recall that the action of an $n\times n$ matrix $M=\|m_{i,j}\|_{i,j=1}^n$ on a polynomial $P(x)=P(x_1,x_2,\ldots,x_n)$ is defined by setting

$$T_M P(x) \;=\; P(xM)$$

The matrix expressing the action of T_M on the homogeneous polynomials of degree m in term of the monomial basis $\langle x^p\rangle_{|p|=m}$ is denoted here by $S^m(M)$ and its entries may be computed from the identities

$$T_M x^q \;=\; \sum_{|p|=m} x^p S^m_{p,q}(M)$$

That is

$$S^m_{p,q}(M) \;=\; (xM)^q\Big|_{x^p}$$

It follows from the Macmahon Master Theorem that the generating function of the traces of the matrices $S^m(M)$ is given by the formula

$$\sum_{m\geq 0} q^m trace\, S^m(M) \;=\; \frac{1}{det(1-qM)}. \tag{1}$$

If G is a group of $n\times n$ matrices then the right hand side of (1) may be viewed as the "*graded character*" of G as it acts on the polynomial ring $\mathbf{R}=\mathbb{C}[x_1,x_2,\ldots,x_n]$.

We will be dealing here with the case $G=SL_n(\mathbb{C})$ acting on the polynomials in the set of variables

$$\{x_{i,j} : 1\leq i\neq j\leq n \;\cup \{x_{i,i} : 1\leq i\leq n\}$$

modulo the relation

$$x_{11}+x_{22}+\cdots+x_{nn}=0. \tag{2}$$

where the action is defined by setting for any $g\in SL_2(\mathbb{C})$

$$T_g P(X) \;=\; P(gXg^{-1}) \quad \text{with} \quad X=\|x_{ij}\|_{i,j=1}^n. \tag{3}$$

Note that since $det(1-qM)$ is invariant under conjugation we see that if $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of M then

$$\frac{1}{det(1-qM)} \;=\; \prod_{i=1}^n \frac{1}{1-q\lambda_i} \tag{4}$$

In the particular case of the action in (3), if g is diagonal with eigenvalues $x_1,x_2,\ldots,x_n$ then

$$gXg^{-1} \;=\; \|x_i x_{ij} x_j^{-1}\|_{ij}$$

and the graded character of its action on the polynomials in x_{ij} is given by (4) with M the diagonal matrix with eigenvalues $x_i x_j^{-1}$ for $1\leq i,j\leq n$. Denoting the graded character of this action by $\chi_n(x,q)$ and taking accoutnt of the relation in (2), we derive from (4) that

$$\chi_n(x,q) \;=\; \frac{1}{(1-q)^{n-1}} \prod_{i\neq j} \frac{1}{1-qx_i/x_j} \tag{5}$$

Since traces are not affected by conjugation, we see that n polynomials

$$E_1 = traceX\ ,\ E_2 = traceX^2\ ,\ \ldots\ ,\ E_n = traceX^n$$

are all invariant under this action. Now from a Theorem of B. Kostant it follows that the ring of polynomials in the $x_{i,j}$ is free over the ring of polynomials in $E_1, E_2, \ldots, E_n$. Our goal here is to compute the character of the action of $SL_n(\mathbb{C})$ on the quotient ring

$$\mathbb{C}[\,x_{ij} : 1 \leq i,j \leq n]/(E_1, E_2, \ldots, E_n) \tag{6}$$

or equivalently on the space $\mathbf{H_n}$ of " SL_n-*Harmonic*" polynomials. That is the space polynomials in the x_{ij} that are killed by the differential operators obtained from the E_i upon replacement of each x_{ij} by $\partial_{x_{ij}}$. Denoting the graded character of this action by $\chi^{\mathbf{H_n}}(x;q)$, and taking account of the the relation in (2), it follows from the theorem of Kostant that

$$\chi_n(x;q) \;=\; \frac{\chi^{\mathbf{H_n}}(x;q)}{(1-q^2)\cdots(1-q^n)}$$

and thus (5) gives

$$\chi^{\mathbf{H_n}}(x;q) \;=\; \frac{(1-q^2)\cdots(1-q^n)}{(1-q)^{n-1}} \prod_{i\neq j} \frac{1}{1-qx_i/x_j} \tag{7}$$

Our goal here is to show the following remarkable

Theorem

$$\chi^{\mathbf{H_n}}(x;q) \;=\; \sum_{b\geq 0} \sum_{\substack{\lambda\vdash nb \\ la(\lambda)<n}} S_\lambda(x) K_{\lambda,b^n}(q) \tag{8}$$

where $K_{\lambda,b^n}(q)$ is the so-called Kotska Foulkes polynomial.

To begin note that, by setting $x_i = qx_i$ and $y_j = 1/x_j$, in the Cauchy formula

$$\prod_{i,j=1}^{n} \frac{1}{1-x_iy_j} \;=\; \sum_{l(\lambda)\leq n} S_\lambda(x)S_\lambda(y)$$

we derive that

$$\frac{1}{(1-q)^n} \prod_{i\neq j} \frac{1}{1-qx_i/x_j} \;=\; \sum_{l(\lambda)\leq n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \tag{9}$$

where $S(1/x) = S(1/x_1, 1/x_2, \ldots, 1/x_n)$. Recalling that $x_1, x_2, \ldots, x_n$ in (7) are the eigenvalues of a matrix with determinant 1, all our calculations involving our characters can be carried out modulo the relation

$$x_1x_2\ldots x_n \;=\; 1 \tag{10}$$

This implies that, if λ has k columns of length n and μ is the partition obtained by removing these columns then

$$S_\lambda(x) \;=\; (x_1x_2\ldots x_n)^k S_\mu(x) \cong S_\mu(x),$$

where here and after the symbol "$\cong$" represents "*congruence*" modulo $x_1x_2\ldots x_n$. Note that denoting by $n(\lambda)$ the number of columns of length n in λ we derive that

$$\begin{aligned}
\sum_{l(\lambda)\leq n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \;&=\; \sum_{k\geq 0} \sum_{\substack{l(\lambda)\leq n \\ n(\lambda)=k}} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \\
&=\; \sum_{k\geq 0} q^{nk} \sum_{l(\mu)<n} q^{|\mu|} S_\mu(x) S_\mu(1/x)
\end{aligned}$$

This gives

$$\sum_{l(\lambda)\leq n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \;=\; \frac{1}{1-q^n} \sum_{l(\lambda)<n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \tag{11}$$

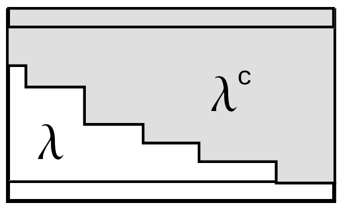
From the bideterminantal formula for Schur funtions it follows that

$$S_\lambda(1/x) = \frac{det\|x_i^{-(\lambda_j+n-j)}\|_{i,j=1}^n}{det\|x_i^{-(n-j)}\|_{i,j=1}^n} = \frac{1}{(x_1x_2\cdots x_n)^{\lambda_1}} \frac{det\|x_i^{\lambda_1-\lambda_j+j-1}\|_{i,j=1}^n}{det\|x_i^{j-1}\|_{i,j=1}^n} = \frac{1}{(x_1x_2\cdots x_n)^{\lambda_1}}\, S_{\lambda^c}(x) \tag{12}$$

where for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ we have set

$$\lambda^c \;=\; (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-2}, \ldots, \lambda_1 - \lambda_2)$$

To visualize the relation between λ and λ^c we should note that λ^c is what we obtain by removing λ from a rectancular box of height n and width λ_1, (see figure).



Note that both λ and λ^c have length $< n$ and a row of length λ_1. Thus the pair λ, λ^c may be obtained by cutting the rectangular partition λ_1^n by an EAST-SOUTH path in the $n-2 \times \lambda_1$ rectangle. We may thus derive from (11) and (12) that

$$\sum_{l(\lambda)\leq n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \cong \frac{1}{1-q^n} \sum_{l(\lambda)<n} q^{|\lambda|} S_\lambda(x) S_{\lambda^c}(x) \tag{13}$$

Now it follows from the Littlewood Richardson rule that the product $S_\lambda(x)S_{\lambda^c}(x)$ will produce one and only one Schur function indexed by the rectangular partition λ_1^n. This shows that when we reduce modulo $x_1x_2\cdots x_n = 1$ the rignt hand side of (13) the constant term of the resulting series is none other than

$$\frac{1}{1-q^n} \sum_{l(\lambda)<n} q^{|\lambda|} \;=\; \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{14}$$

keeping this in mind note that by combining (7), (9) and (13) we derive that

$$\begin{aligned}
\chi^{\mathbf{H_n}}(x;q) \;&=\; \frac{(1-q^2)\cdots(1-q^n)}{(1-q)^{n-1}} \prod_{i\neq j} \frac{1}{1-qx_i/x_j} \\
&=\; \frac{(1-q^2)\cdots(1-q^n)}{(1-q)^{n-1}}\, (1-q)^n \sum_{l(\lambda)\leq n} q^{|\lambda|} S_\lambda(x) S_\lambda(1/x) \\
&=\; \frac{(1-q^2)\cdots(1-q^n)}{(1-q)^{n-1}}\, (1-q)^n \frac{1}{1-q^n} \sum_{l(\lambda)<n} q^{|\lambda|} S_\lambda(x) S_{\lambda^c}(x)
\end{aligned}$$

In summary we have

$$\chi^{\mathbf{H_n}}(x;q) \;=\; (1-q)(1-q^2)\cdots(1-q^{n-1}) \sum_{l(\lambda)<n} q^{|\lambda|} S_\lambda(x) S_{\lambda^c}(x) \tag{15}$$

Note that by grouping terms according to the size of λ_1 we may also write (15) in the form.

$$\chi^{\mathbf{H_n}}(x;q) \quad = \quad (1-q)(1-q^2)\cdots(1-q^{n-1})\sum_{b\geq 0}\sum_{\substack{\lambda\subseteq b^n \\ l(\lambda)<n}} q^{|\lambda|}S_\lambda(x)S_{b^n/\lambda}(x) \tag{16}$$

Our effort now is to get the righthand sides sides of (7) and (16) as close as possible.

To this end, let us recall that one of the classical formulas for the Kostka-Foulkes polynomial may be written in the form

$$K_{\lambda,\mu}(q) \quad = \quad S_\lambda(z_1,z_2,\ldots,z_n)\prod_{i\neq j}\frac{1-z_j/z_i}{1-qz_j/z_i}\Big|_{z_1^{\mu_1}z_2^{\mu_2}\cdots z_n^{\mu_n}}$$

Using this in (8) we get that

$$\chi^{\mathbf{H_n}}(x;q) \quad = \quad \sum_{b\geq 0}\sum_{\substack{\lambda\vdash nb \\ la(\lambda)<n}} S_\lambda(x)S_\lambda(z)\prod_{i\neq j}\frac{1-z_j/z_i}{1-qz_j/z_i}\Big|_{z_1^b z_2^b\cdots z_n^b} \tag{17}$$

Unfortunarely so far we can only verify the equality of (16) and (17) only for $n=2$. In fact we can easily see that for $n=2$ the right hand side of (16) becomes

$$\begin{aligned}
(1-q)\sum_{b\geq 0}\sum_{\lambda\subseteq b^2, l(\lambda)\leq 1} q^{|\lambda|}S_\lambda(x)S_{b^2/\lambda}(x) \quad &= \quad (1-q)\sum_{b\geq 0}q^b S_b(x)S_b(x) \\
&= \quad (1-q)\sum_{b\geq 0}q^b\sum_{d=0}^{b}S_{b+b-d,d}(x) \\
&\cong (1-q)\sum_{b\geq 0}q^b\sum_{d=0}^{b}S_{b-d+b-d}(x) \\
&= \quad (1-q)\sum_{b\geq 0}q^b\sum_{d=0}^{b}S_{2d}(x) \quad = \quad (1-q)\sum_{d\geq 0}S_{2d}(x)\sum_{b\geq d}q^b \quad = \quad \sum_{d\geq 0}q^d S_{2d}(x)
\end{aligned}$$

On the other hand the right hand side of (8) for $n=2$ becomes

$$\sum_{b\geq 0}\sum_{\substack{\lambda\vdash 2b \\ la(\lambda)=1}} S_\lambda(x)K_{2b,b^2}(q) \quad = \quad \sum_{b\geq 0}S_{2b}(x)q^b$$

since for any b we have $K_{2b,b^2}(q) = q^b$